A Short History of Length.

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PPPL
Science on Saturday
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To Measure a Path: Pedestrian Exercise?

Waywiser, c. 1875 (Atwater Kent Museum, Philadelphia).
The Waywiser Works on Curved Surfaces

Taking measure of the world’s roundness and becoming way wiser.
Carl Friedrich Gauss studies geodesy, surveys the Kingdom of Hanover, and discovers the notion of intrinsic curvature (1827).
Geometry with Negative Curvature

Escher’s “Circle Limit III” (1959).
The three most important curves in the history of length:

Circle  Line  Lemniscate of Bernoulli
Once Upon A Time: *To Measure Was to Count.*

Jacob Köbel (1460-1533): Ask 16 men leaving the church service to line up heel to toe—thus establish the *surveyor’s rod* (16.5 feet).
Counting Out, Counting In

Counting out

0 2 3 1

-1 6 1 3 1 2
Counting Out, Counting In

Counting out

and counting in
“But what has been said once can always be repeated.”

Zeno

And counting further in
1. A *point* is that which has no part.
2. A *line* is breadthless length...
Rules of Straightedge and Compass

- Draw the line through two given points.
- Draw the circle with given center and radius.
- Locate points of intersection of given lines/circles.
From Two Points to Graph Paper
The rational line.
From Ruler to Circle

The rational circle.
Fifteen examples of $a^2 + b^2 = c^2$. 
The Pythagorean Theorem

\[ 3^2 + 4^2 = 5^2 \]

Length of hypotenuse = \( c = \sqrt{a^2 + b^2} \)
The Diagonal of a Square

Yale 7289: \( \frac{1}{30} \left( 42 + \frac{25}{60} + \frac{35}{3600} \right) = 1.4142130 \cdots \approx \sqrt{2} \)
The Pentagon and the Golden Ratio $\phi = \frac{1 + \sqrt{5}}{2}$
The Way to the Waywiser

Constructible regular \( n \)-gons: \( n = 3, \ n = 5 \).
Constructible regular $n$-gons: $n = 3, \ n = 5$. 
Constructible regular $n$-gons: $n = 3, 5, 15$. 
Constructible regular $n$-gons: $n = 3, 5, 15, 30$. 
The Way to the Waywiser

The constructible Babylonian wayiser \((n = 60)\).
A Chiliagon or a Circle?

\[ \pi = \text{Area}(\bigcirc) = \frac{\text{Length}(\bigcirc)}{2} \]
Pieces of $\pi$

$2 = p_2 < \pi < P_2 = 4$
2 = p_2 < p_3 < \pi < P_3 < P_2 = 4
Pieces of $\pi$

$2 = p_2 < p_3 < p_4 < \pi < P_4 < P_3 < P_2 = 4$
Pieces of $\pi$

\[ \pi = p_\infty = P_\infty = 3.14159\ldots \]
By Whatever Means $\pi$

**Theorem** (Archimedes)

We may compute $\pi$ as a *geometric-harmonic mean*:

$$\pi = p_\infty = P_\infty = M_{GH}(2, 4)$$
Impossible Problems of Antiquity (in order of impossibility)

1. Duplicate a cube (construct $\sqrt[3]{2}$)
2. Trisect an angle (construct solution to cubic)
3. Square the circle (construct $\pi$)
SEVENTEEN CENTURIES PASS . . .
As the 16th Century Opens ...

Luca Pacioli publishes *Divina Proportione* and Leonardo battles *morbus cyclometricus*. 
As the 16th Century Closes ...

Kepler nests Platonic solids and spheres in *Mysterium Cosmographicum*. 
The Nautilus as Equiangular Spiral

The spiral cuts each ray at the same angle $\alpha$. 
The Discrete Spiral of Harriot (1590)

Length(spiral) = $|OV| + |VP| = |OP| \sec \alpha$. 
But How Long is the Rainbow?

“The ratios between straight and curved lines are not known, and I believe cannot be discovered by human minds.”

René Descartes
Descartes uses $x, y$ coordinates—analytic geometry is born!
The Pythagorean Theorem Revisited

Distance Formula:  \( d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \)
The parabola $y = 1 - \frac{x^2}{2}$
The Infinitesimal Pythagorean Theorem and Arc Length

\[ ds^2 = dx^2 + dy^2 = (1 + \left(\frac{dy}{dx}\right)^2)\,dx^2 \]

Length = \[ \int_a^b ds = \int_a^b \sqrt{1 + y'^2} \, dx \]
How Far the Journey?

distance travelled = speed \times time
How Far the Journey?

\[ \text{average distance travelled} = \sqrt{\text{speed} \times \text{time}} \]
A Successor to Descartes and Leibniz

Jakob Bernoulli (1654–1705)

“Against my father’s will, I turn to the stars.”
Jakob Bernoulli’s Challenge

What is the shape of the Rectangular Elastica?
"I cannot longer deny to the public the golden theorem."
A New Kind of Pi

Theorem (Jakob Bernoulli)

Lemniscate and elastica of equal width have equal length:

\[ L(\infty) = 2\varpi = L(S) \]
The Arithmetic-Geometric Mean

Theorem (Gauss, 1799)

The length of the circle divided by the length of the inscribed lemniscate is the arithmetic-geometric mean of 1 and $\sqrt{2}$:

$$\frac{L(\bigcirc)}{L(\infty)} = \frac{\pi}{\varpi} = M(\sqrt{2}, 1).$$
Count Giulio Fagnano (1682–1766)
Theorem (Fagnano, 1718)

The lemniscate’s first quadrant can be bisected, trisected and quinquesected by ruler and compass:

\[ r_{\frac{1}{2}} = \sqrt{\sqrt{2} - 2} \]

\[ r_{\frac{1}{3}} = \sqrt{\frac{2}{1 + \sqrt{2\sqrt{3} - 3}} - 1} \]

\[ r_{\frac{1}{5}} = \sqrt[5]{\frac{1 - \sqrt{-13 + 6\sqrt{5} + 2\sqrt{85} - 38\sqrt{5}}}{1 + \sqrt{-13 + 6\sqrt{5} + 2\sqrt{85} - 38\sqrt{5}}}} \]
Count Fagnano’s Gift
Count Fagnano’s Gift
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Count Fagnano’s Gift
Count Fagnano’s Gift
Count Fagnano’s Gift
At the Church of Santa Maria Magdalena in Senigallia

Fagnano with lemniscate in hand.
On December 23rd, 1751, Euler received Fagnano’s *Produzioni*.

In many places divided and measured out is the glory of truth in God.
Which Sines are Constructible?

\[ r = \sin \frac{j\pi}{n} = \text{constructible?} \]
Which Sines are Constructible?

\[ r = \sin \frac{j \pi}{n} = \text{constructible?} \]

**Theorem** (Gauss, 1801; Wantzel, 1837)

The regular \( n \)-gon may be constructed by ruler and compass for precisely the integers \( n = 2^j p_1 p_2 \ldots p_k \) (power of two times product of distinct Fermat primes \( p = 2^{2^m} + 1 \)).
A New Kind of Sine

$$r = \sin\text{lemn} \frac{j\omega}{n} = \text{constructible?}$$
A New Kind of Sine

\[ r = \sin \text{lemn} \frac{j\omega}{n} = \text{constructible?} \]

**Theorem** (Abel, 1827)

The lemniscate may be evenly subdivided by ruler and compass for precisely the same integers \( n = 2^j p_1 p_2 \ldots p_k \).
Counting Out in the Complex Plane
Counting In
The Lemniscate Waywiser
The Spiric Sections of Perseus
Zooming in on $\infty$
Zooming in on $\infty$: $M = 1 \times$
Zooming in on $\infty$: $M = 2\times$
Zooming in on $\infty$: $M = 10 \times$
Zooming in on $\infty$: $M = 50 \times$
Zooming in on $\infty$: $M = 250\times$
Zooming in on $\infty$: $M = 1250 \times$
THANK YOU!
The rational circle.
“It’s true that mathematics requires mental exertion. But there’s no evidence that being able to prove

\[(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2\]

leads to more credible political opinions or social analysis.”
Stereographic Projection

Peter Paul Rubens (from d’Aguilon’s *Optics*, 1613).