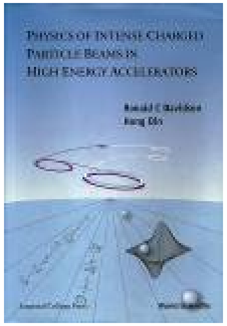
High Intensity Beam Physics

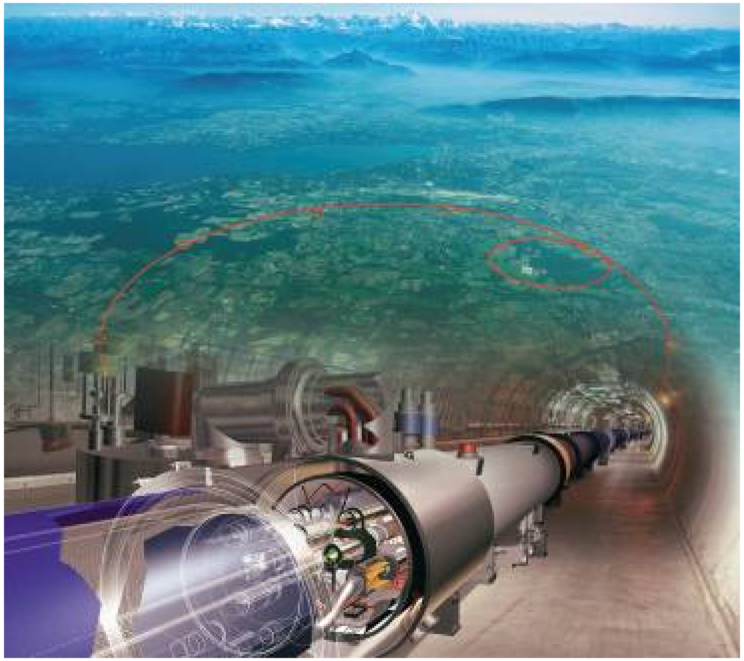
- Introduction to beam physics and accelerator physics.
- Particle and envelope dynamics.
- Vlasov-Maxwell theory for high intensity beams.
- Beam equilibrium and stability

Textbook:

An Introduction to the Physics of Intense Charged Particle Beams in High Energy Accelerators, R. C. Davidson and H. Qin, World Scientific (2001)



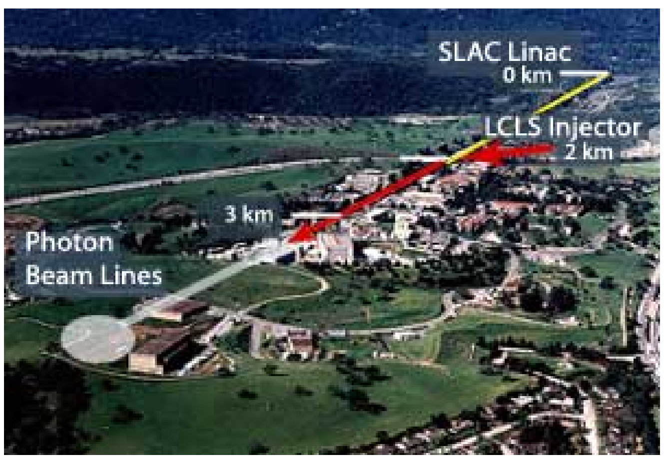
Beam physics application --- particle physics

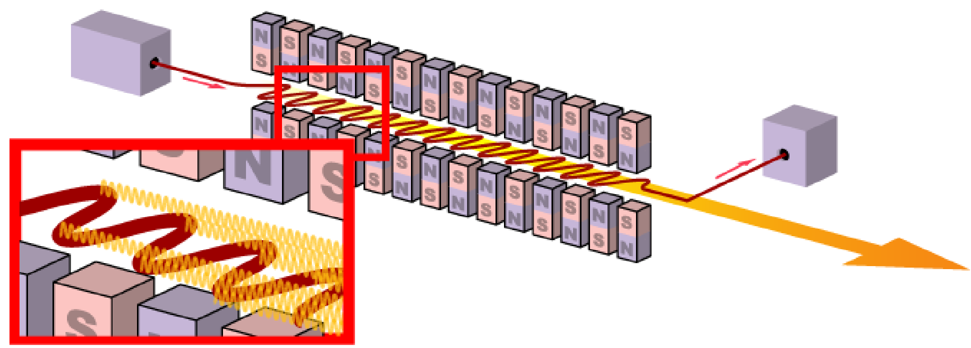


Large Hadron Collider

Beam physics application --- modern high intensity accelerators

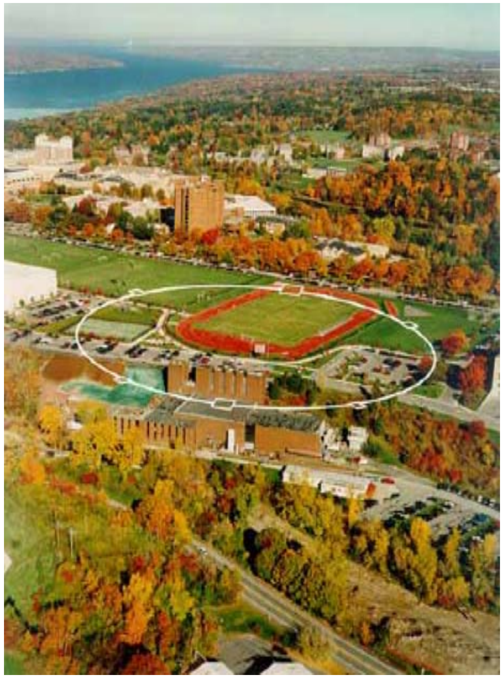
Spallation Neutron Source

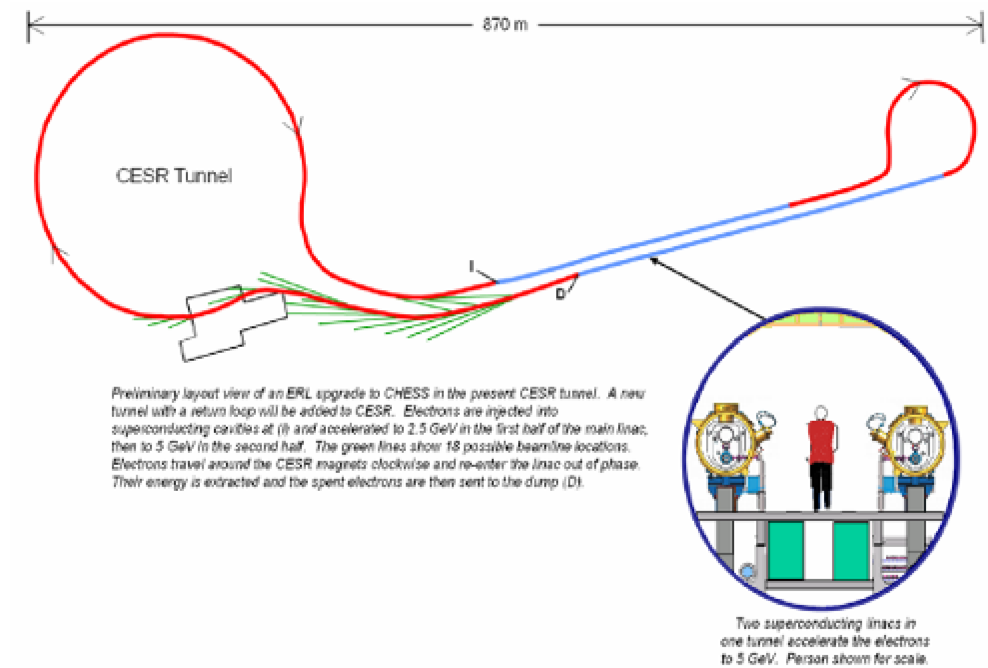




Stanford Linear Coherent Light Source 0.8 - 8 KeV

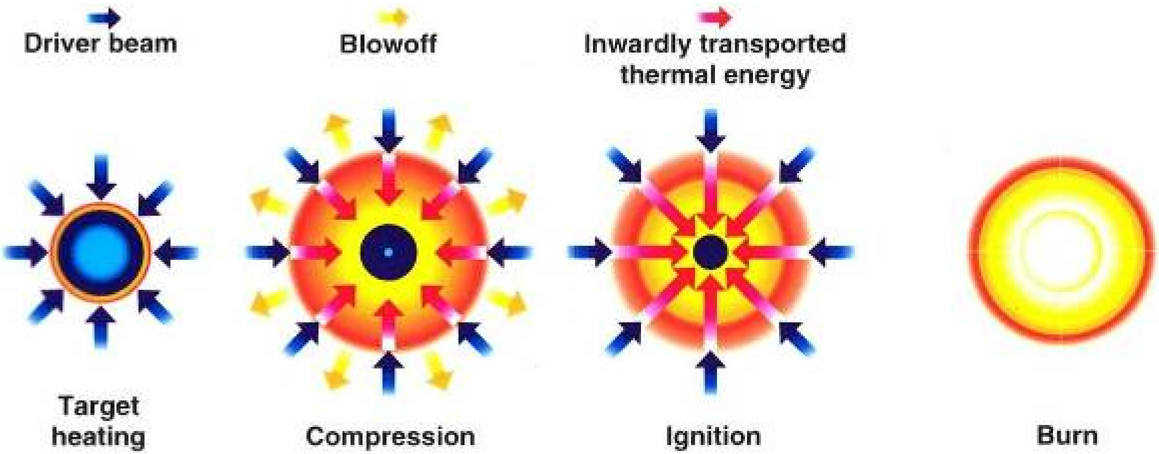
Beam physics application --- light source

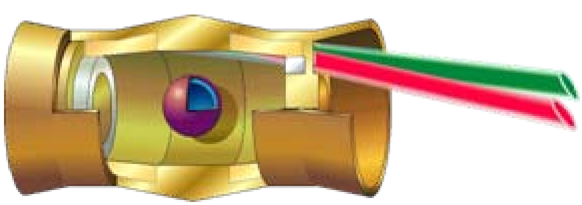


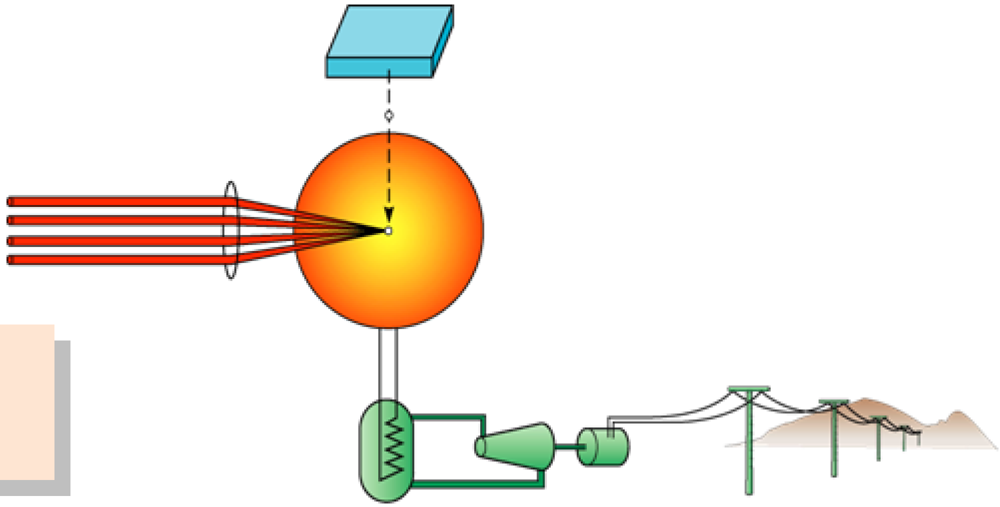


Cornell Energy Recovery Linac, 5GeV electron, 10KeV

Beam physics application --- heavy ion fusion

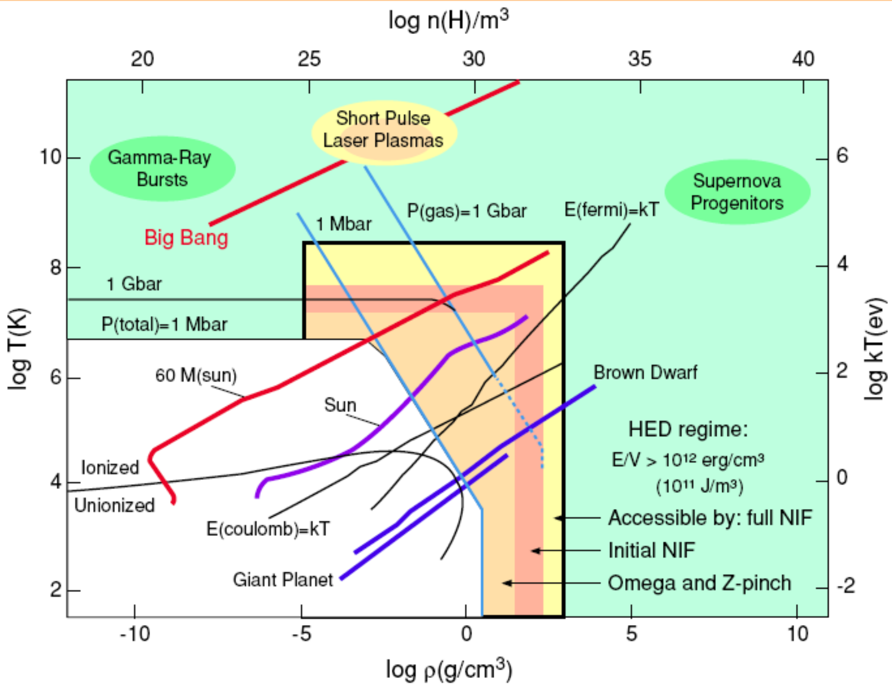


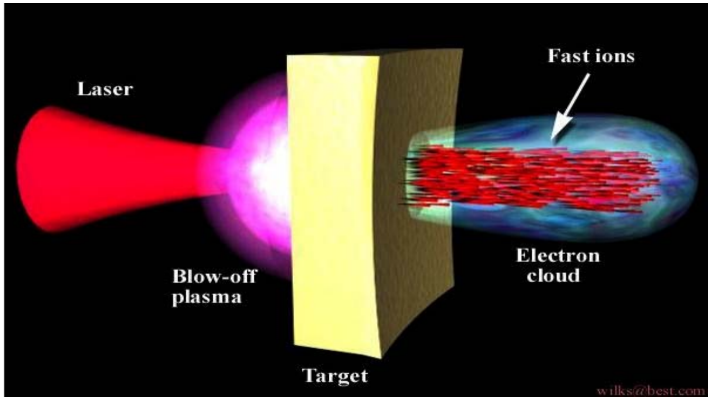




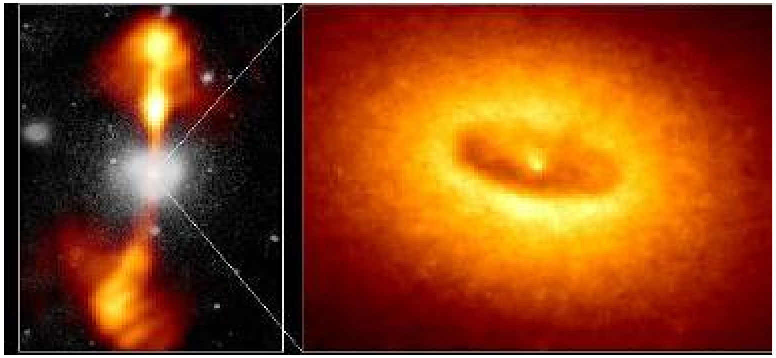
Beams strike “hohlraum,” producing x-ray bath for fusion capsules.

Beam physics application --- high energy density physics

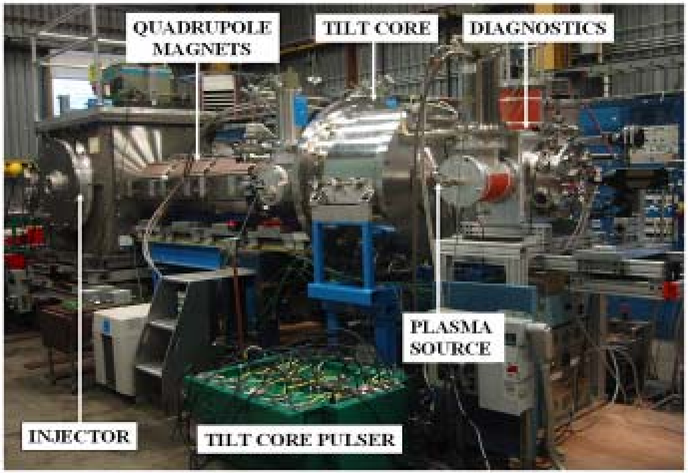




Gamma ray bursters experiment



Photoionized plasmas in an accreting massive black hole



Neutralized drift compression experiment for ion beam driven HEDP

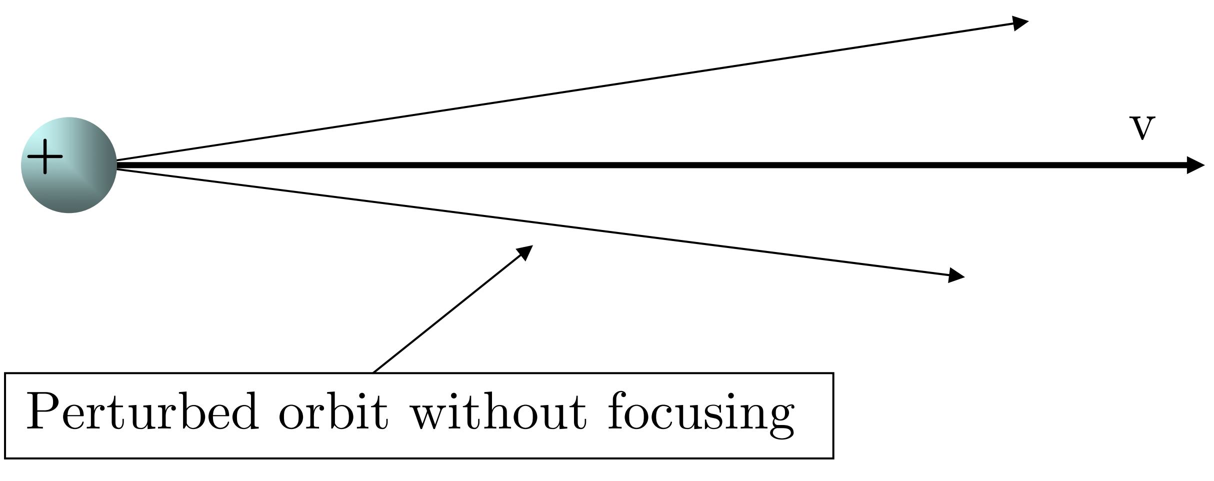
- Accelerators
    - Induction accelerators (betatron, induction linac)
    - RF accelerators (cyclotron,synchrotorn ☑)
- Collider and storage rings ☑

- Acceleration
- Stability ☑
- Focusing ☑
- Cooling

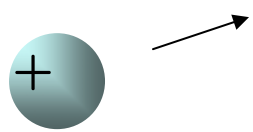
Modern accelerators are high intensity!

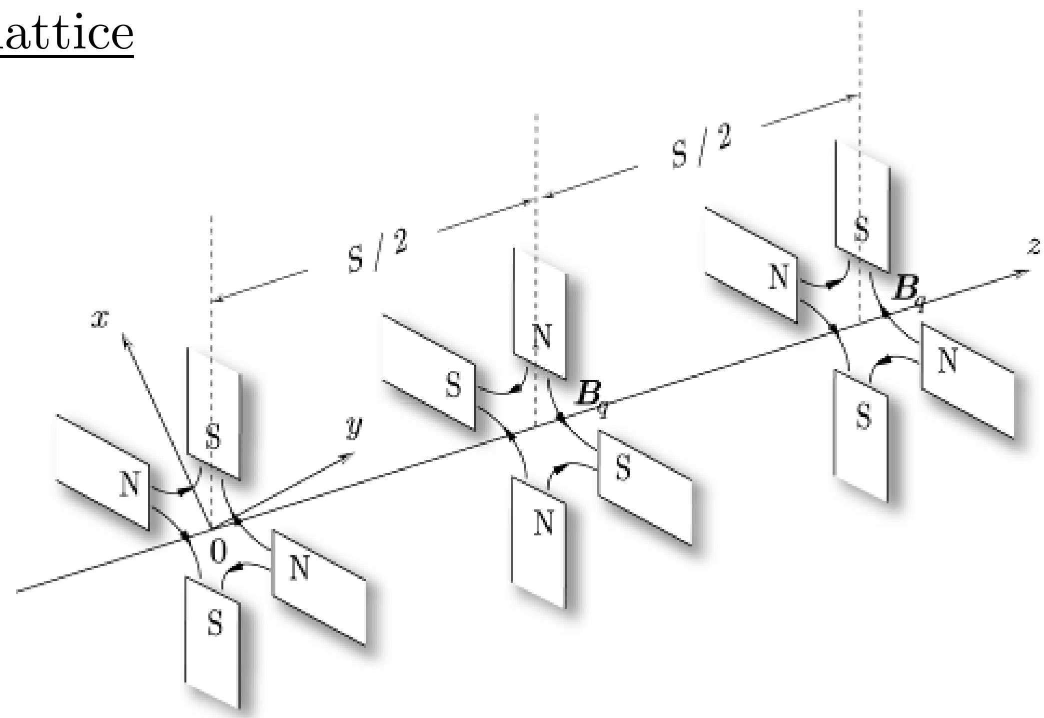
Transverse dynamics of a single particle

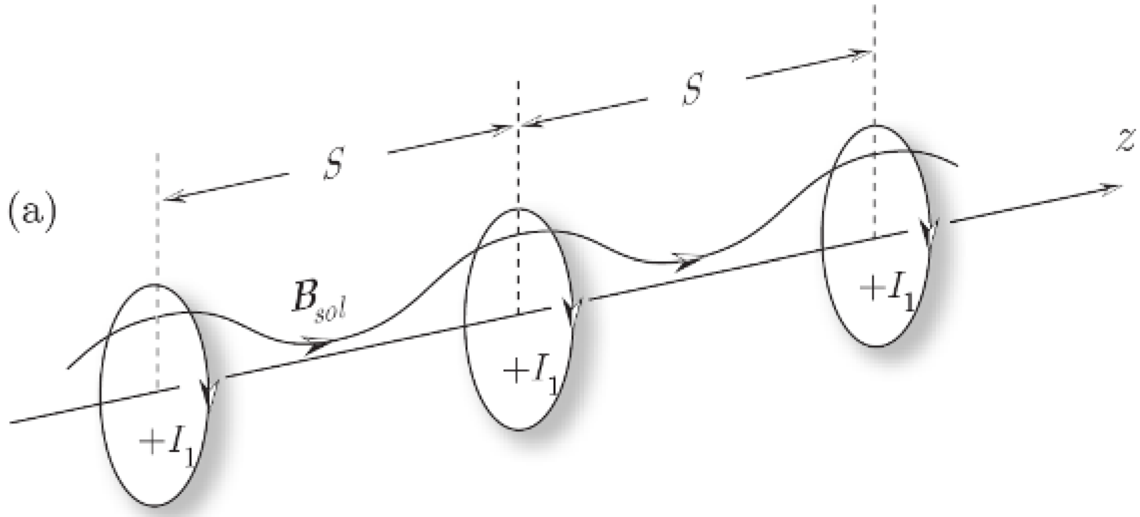
Why need transverse focusing?

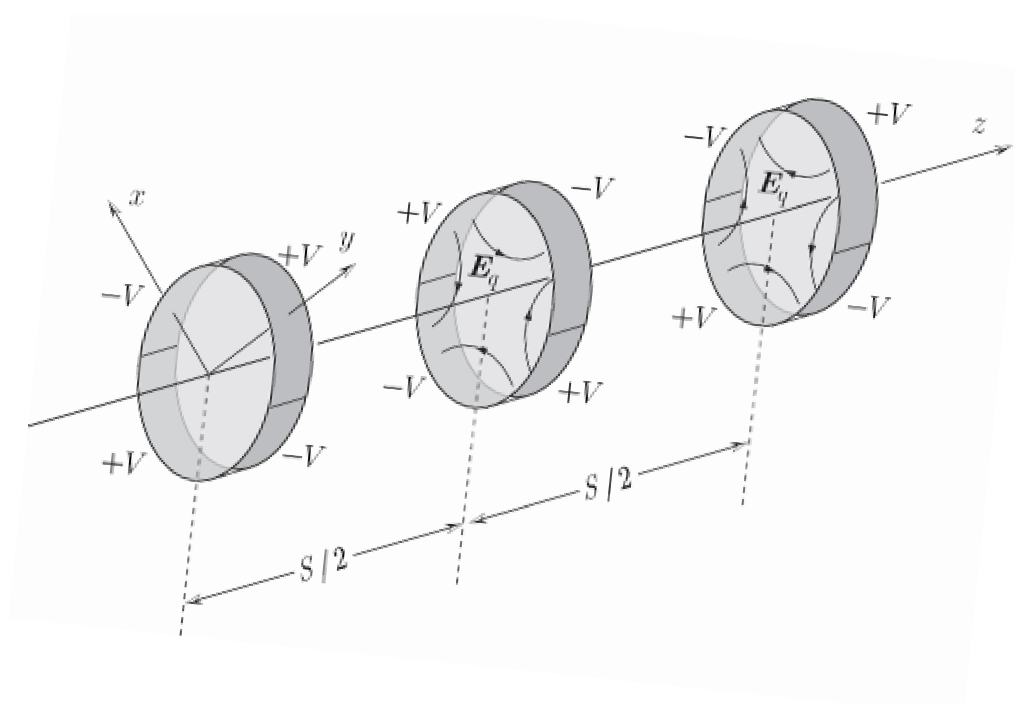


Quadrupole FODO lattice






(a)
S
S
z
B_{sol}
$+I_1$
$+I_1$
$+I_1$


x
y
z
$-V$
$+V$
$+V$
$-V$
$+V$
$-V$
E_q
$-V$
$+V$
$-V$
$+V$
E_q
$+V$
$-V$
$S/2$
$S/2$

Relativistic Hamiltonian in EM field

$$H = (m_b^2 c^4 + c^2 \mathbf{p}^2)^{1/2} + e_b \varphi(\mathbf{x}, t)$$
$$\mathbf{p} = \gamma m \mathbf{v}$$
$$\gamma = (1 + \mathbf{p}^2 / m_b^2 c^2)^{1/2}$$

$$\frac{1}{\gamma^2} + \beta^2 = 1$$
$$\gamma^2 = 1 + \gamma^2 \beta^2$$

$$H = \sqrt{m_b^2 c^4 + c^2 \left(\mathbf{P} - \frac{e_b}{c} \mathbf{A} \right)^2} + e_b \varphi(\mathbf{x}, t)$$
$$\mathbf{P} = \mathbf{p} + \frac{e_b}{c} \mathbf{A}$$

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \mathbf{P}}, \quad \frac{d\mathbf{P}}{dt} = -\frac{\partial H}{\partial \mathbf{x}}$$

$$\frac{d\mathbf{x}}{dt} = \frac{\mathbf{p}}{m_b \gamma}, \quad \frac{d\mathbf{p}}{dt} = e_b \left(\frac{\mathbf{v}}{c} \times \mathbf{B} + \mathbf{E} \right)$$

Why has it to be the canonical momentum $\mathbf{P}$?

$$\chi = \frac{e}{c}A + p = (\frac{e}{c}\boldsymbol{A} + m\gamma\boldsymbol{v})\cdot d\boldsymbol{x} - \left[\gamma mc^2 + \frac{e}{c}\phi\right]dt$$

$A \equiv (-\phi, \boldsymbol{A})$, four vector potential, 1-form

$p \equiv (-\gamma mc^2, \boldsymbol{p})$, 1-form momentum

$\boxed{\times dt^{-1}}$

$$L = \left(\frac{e}{c}\boldsymbol{A} + \gamma m\boldsymbol{v}\right)\cdot\boldsymbol{v} - \left(\gamma mc^2 + \frac{e}{c}\phi\right)$$

γ depends on $\boldsymbol{v}$

$$\boldsymbol{P} \equiv \frac{\partial L}{\partial \boldsymbol{v}} = \frac{e}{c}\boldsymbol{A} + \gamma m\boldsymbol{v}$$

$$H = \boldsymbol{P}\cdot\boldsymbol{v} - L = \gamma mc^2 + \frac{e}{c}\phi$$

Orbit Equations in the Paraxial Approximation

$$p_x^2,\ p_y^2,\ (p_z - p_b)^2 \ll p_b^2$$
$$p_b = \gamma_b m_b \beta_b c$$
$$\gamma_b = (1 + p_b^2 / m_b^2 c^2)^{1/2}$$

$$\gamma m_b c^2 = [m_b^2 c^4 + c^2(p_x^2 + p_y^2) + c^2(p_b + p_z - p_b)^2]^{1/2}$$

2nd order: $\left[p_x^2,\ p_y^2,\ (p_z - p_b)^2\right] / \gamma_b m_b^2 c^2$

$$H = \gamma_b m_b c^2 + V_b(p_z - p_b) + \frac{1}{2\gamma_b m_b}(p_x^2 + p_y^2) + \frac{1}{2\gamma_b^3 m_b}(p_z - p_b)^2 + e_b \varphi(\mathbf{x}, t)$$

$$\delta p_z = p_z - p_b$$
$$\delta P_z = P_z - p_b$$

$$\frac{dx}{dt} = \frac{p_x}{\gamma_b m_b} = v_x$$

$$\frac{dy}{dt} = \frac{p_y}{\gamma_b m_b} = v_y$$

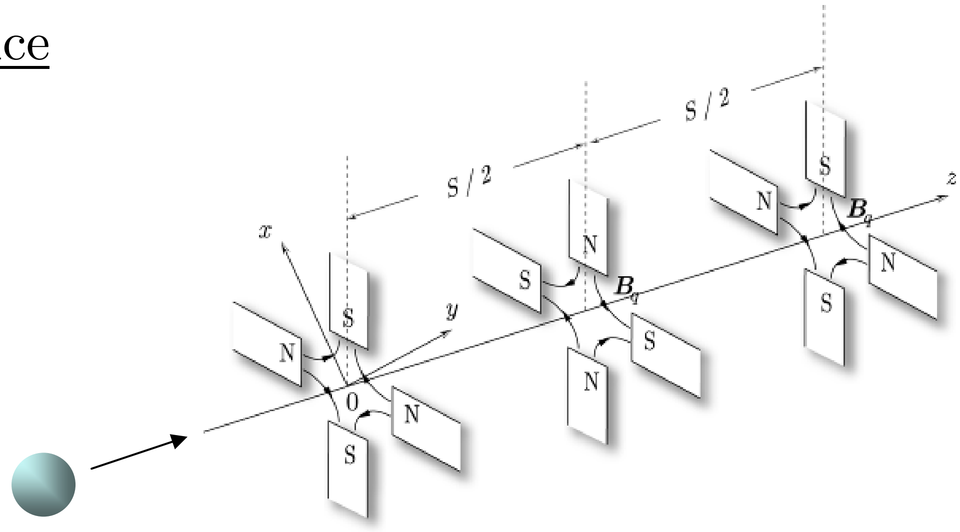
$$\frac{dz}{dt} = V_b + \frac{\delta p_z}{\gamma_b^3 m_b} = v_z$$

$$\gamma_b m_b \frac{d^2 x}{dt^2} = e_b E_x + \frac{1}{c}\left(v_y B_z - v_z B_y\right)$$

$$\gamma_b m_b \frac{d^2 y}{dt^2} = e_b E_y - \frac{1}{c}\left(v_x B_z - v_z B_x\right)$$

$$\gamma_b^3 m_b \frac{d^2 z}{dt^2} = e_b E_z + \frac{1}{c}\left(v_x B_y - v_y B_x\right)$$

Quadrupole FODO lattice



$$\nabla \times \mathbf{B}_q = 0,\ \nabla \cdot \mathbf{B}_q = 0$$
$$(\,|\,x\,|,\ |\,y\,| \ll S)$$

$$\mathbf{B}_q(\mathbf{x}) = B_q'(z)(y\mathbf{e}_x + x\mathbf{e}_y)$$

$$B_q'(z) \equiv \left.\frac{\partial B_x^q}{\partial y}\right|_{(x,y)=(0,0)} = \left.\frac{\partial B_y^q}{\partial x}\right|_{(x,y)=(0,0)}$$

Transverse dynamics in quadrupole field (no space-charge effect)

$$x''(s) + \kappa_q(s)x(s) = 0$$
$$y''(s) - \kappa_q(s)y(s) = 0$$

$$s = z_0 + \beta_b ct$$

$$\kappa_{xq}(s) = -\kappa_{yq}(s) = \kappa_q(s)$$
$$\kappa_q(s) \equiv \frac{e_b B_q'(s)}{\gamma_b m_b \beta_b c^2}$$

$$x(s) = A_x w_x(s)\cos[\psi_x(s) + \varphi_{x0}]$$

$$y(s) = A_y w_y(s)\sin[\psi_y(s) + \varphi_{y0}]$$

Const.

envelope

phase

Courant-Snyder theory

$$\boxed{\begin{aligned} x'(s) &= A_x w_x' \cos(\psi_x + \varphi_{x0}) - A_x w_x \psi_x' \sin(\psi_x + \varphi_{x0}) \\ x''(s) &= A_x [w_x'' - w_x (\psi_x')^2] \cos(\psi_x + \varphi_{x0}) \\ &\quad \underbrace{- A_x [w_x' \psi_x' + (w_x \psi_x')'] \sin(\psi_x + \varphi_{x0})} \end{aligned}}$$

Let: $w_x' \psi_x' + (w_x \psi_x')' = 0 \longrightarrow \boxed{\psi_x(s) = \int_{s_0}^{s} \frac{ds}{w_x^2(s)}}$

$$\boxed{x''(s) + \kappa_q(s) x(s) = 0} \longrightarrow \boxed{w_x'' - w_x (\psi_x')^2 + \kappa_q(s) w_x = 0}$$

Envelop Eq. $\boxed{w_x''(s) + \kappa_x(s) w_x(s) = w_x^{-3}(s)}$

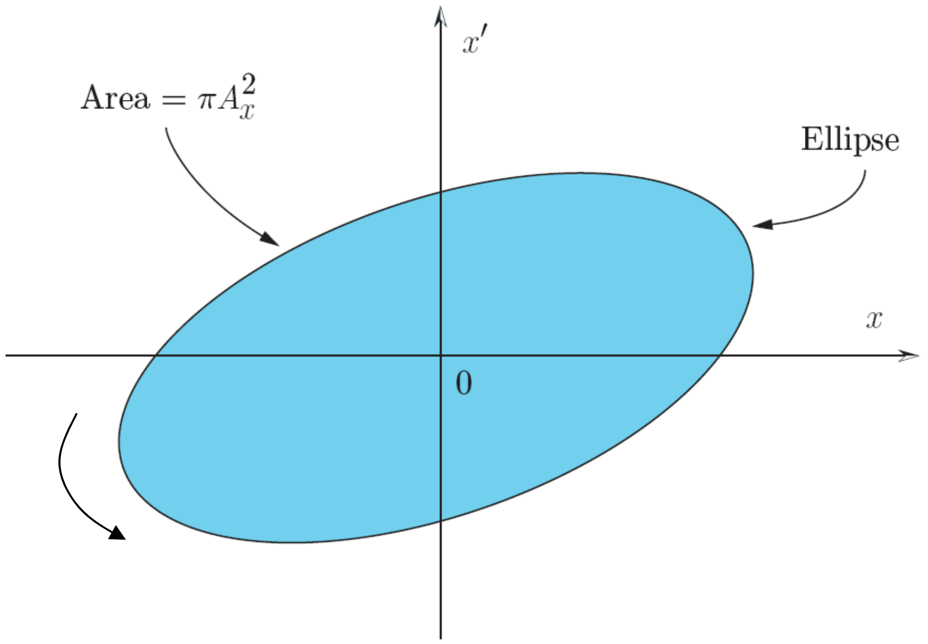
Courant-Snyder Invariant

$$A_x^2 = \frac{x^2}{w_x^2} + \left(w_x x' - w_x' x\right)^2 = const.$$

$$x(s) = A_x w_x(s)\cos[\psi_x(s) + \varphi_{x0}]$$

$$\frac{x}{w_x} = A\cos[\psi_x(s) + \varphi_{x0}]$$

$$\left(\frac{x}{w_x}\right)' = A\sin[\psi_x(s) + \varphi_{x0}]\psi_x'(s)$$



What's the Courant-Snyder theory about?

Given a focusing lattice, solve once:

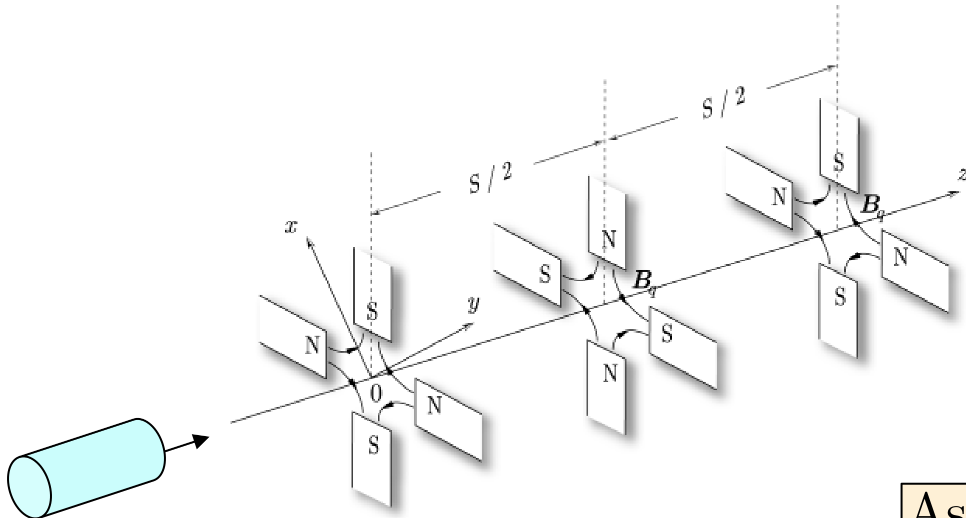
$$w_x''(s) + \kappa_x(s) w_x(s) = w_x^{-3}(s),$$

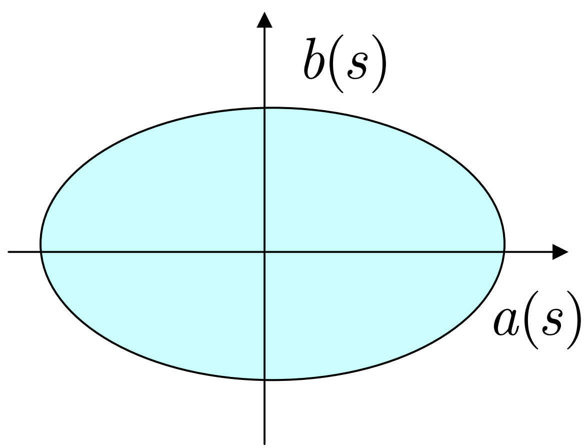
then every particle's orbit is known.

Birkhoff (1908)

Courant (1958)

Transverse dynamics in quadrupole field (with space-charge effect)





Assume beam has a uniform density inside

$$\frac{x^2}{a^2(s)} + \frac{y^2}{b^2(s)} = 1.$$

$$N_b = \int dx dy n_b(x, y, s) = \hat{n}_b \pi a b = const.$$

Poisson's equation for self-field

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi(x,y,s)=\begin{cases}-\frac{2K_b}{ab}\,, & 0\le \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}<1\,,\\ 0\,, & \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}>1\,.\end{cases}$$

$$\psi = e_b\varphi^s \,/\, \gamma_b^3 m_b \beta_b^2 c^2$$

$$\boxed{a, b \ll S}$$

Perveance $K_b = \dfrac{2N_b e_b^2}{\gamma_b^3 m_b \beta_b^2 c^2}$

$$\psi(x,y,s)=-\frac{K_b}{a+b}\left[\frac{1}{a}x^2+\frac{1}{b}y^2\right],\ \ 0\le \frac{x^2}{a^2}+\frac{y^2}{b^2}<1$$

Transverse dynamics in quadrupole field (no space-charge effect)

$$-\frac{\partial\psi}{\partial x} = \frac{2K_b}{a(a+b)}x$$
$$-\frac{\partial\psi}{\partial y} = \frac{2K_b}{b(a+b)}y$$

$$\longrightarrow$$

$$x''(s) + \kappa_x(s)x(s) = 0$$
$$y''(s) + \kappa_y(s)y(s) = 0$$

$$\kappa_x(s) = \kappa_q(s) - \frac{2K_b}{a(s)[a(s)+b(s)]}$$
$$\kappa_y(s) = -\kappa_q(s) - \frac{2K_b}{b(s)[a(s)+b(s)]}$$

Const.

$$x(s) = A_x w_x(s) \cos[\psi_x(s) + \varphi_{x0}]$$

phase

$$w_x''(s) + \kappa_x(s) w_x(s) = w_x^{-3}(s)$$

$$a(s) = A_{x\,\max} w_x(s)$$

emittance: $\varepsilon_x = A_{x\,\max}^2$

Envelope Eq.

$$\frac{d^2}{ds^2} a(s) + \kappa_q(s) a(s) - \frac{2K_b}{a(s) + b(s)} = \frac{\varepsilon_x^2}{a^3(s)}$$

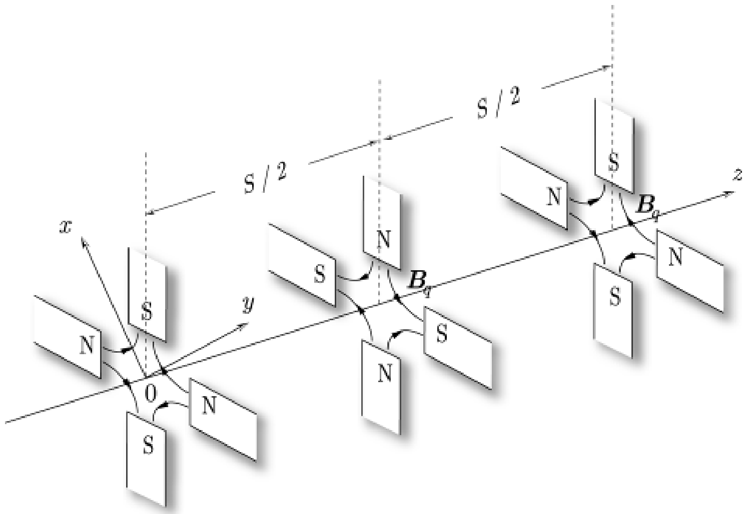
$$\frac{d^2}{ds^2}a(s) + \kappa_q(s)a(s) - \frac{2K_b}{a(s)+b(s)} = \frac{\varepsilon_x^2}{a^3(s)}$$
$$\frac{d^2}{ds^2}b(s) - \kappa_q(s)b(s) - \frac{2K_b}{a(s)+b(s)} = \frac{\varepsilon_y^2}{b^3(s)}$$

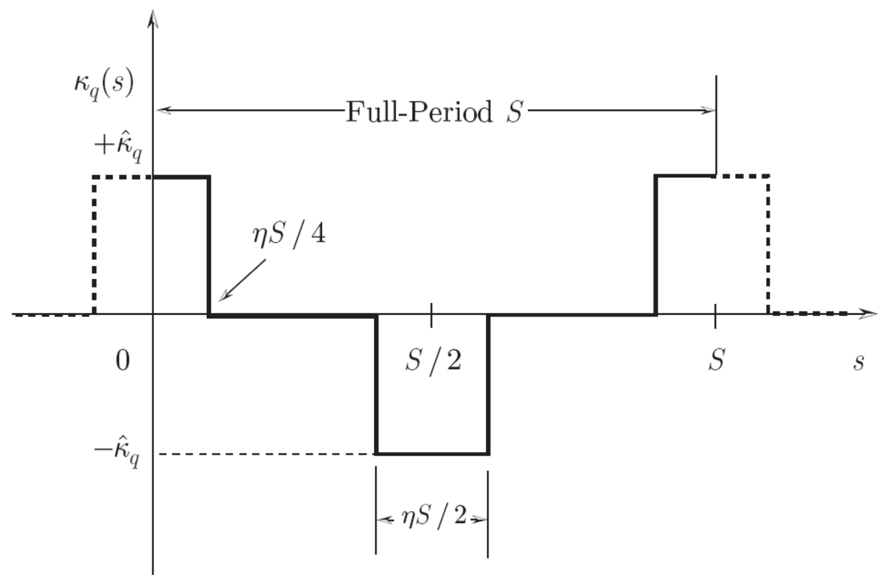
Phase advance

$$\sigma_x = \int_{s_0}^{s_0+S}\frac{ds}{w_x^2(s)} = \varepsilon_x\int_{s_0}^{s_0+S}\frac{ds}{a^2(s)}$$
$$\sigma_y = \int_{s_0}^{s_0+S}\frac{ds}{w_y^2(s)} = \varepsilon_y\int_{s_0}^{s_0+S}\frac{ds}{b^2(s)}.$$

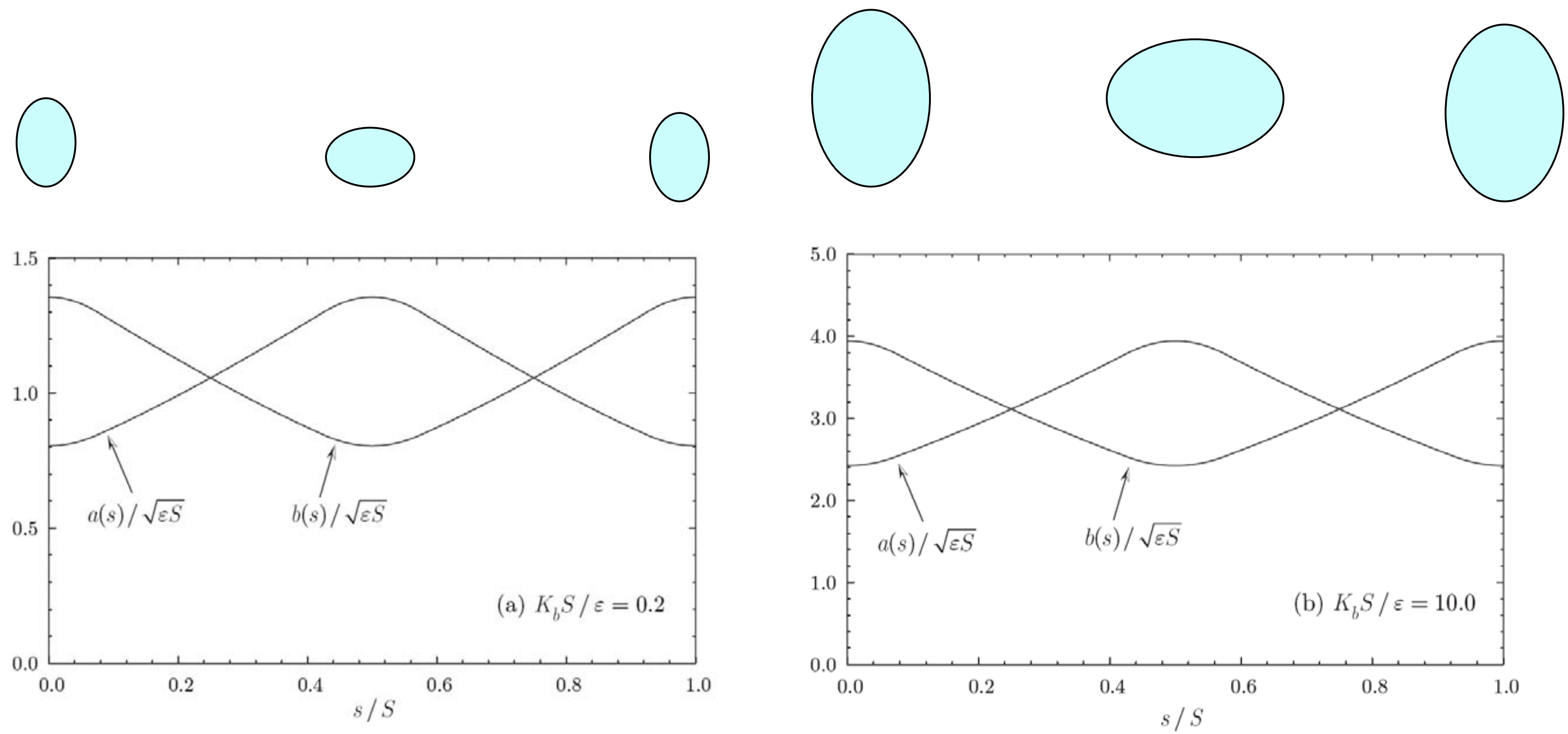
$$\boxed{\psi_x(s) = \int_{s_0}^{s}\frac{ds}{w_x^2(s)}}$$

FODO Lattice





Numerical solutions of envelope eqs.



<u>Smooth focusing model for periodic focusing lattice</u>

$$a(s) = \overline{a}(s) + \delta a(s) \qquad \delta a(s+S) = \delta a(s) \qquad \langle \cdots \rangle = \frac{1}{S}\int_{s_0}^{s_0+S} ds \cdots .$$

$$b(s) = \overline{b}(s) + \delta b(s) \qquad \langle \delta a \rangle = 0$$

slow fast

$$\boxed{\mid \delta a \mid \ll \overline{a}} \longleftrightarrow \boxed{\delta = \mid \hat{\kappa}_q \mid S^2 \ll 1}$$

$$\langle \kappa_q(s) \rangle = 0$$

$$\boxed{\begin{aligned} &\frac{d^2}{ds^2}\overline{a} + \frac{d^2}{ds^2}\delta a + \kappa_q(s)\overline{a} + \kappa_q(s)\delta a \\ &- \frac{2K_b}{\overline{a}+\overline{b}}\left(1 - \frac{\delta a + \delta b}{\overline{a}+\overline{b}}\right) = \frac{\varepsilon^2}{\overline{a}^3}\left(1 - \frac{3\delta a}{\overline{a}}\right) \end{aligned}}$$

$$\Big\downarrow \langle \cdots \rangle$$

slow

$$\boxed{\frac{d^2}{ds^2}\overline{a}(s) + \langle \kappa_q(s)\delta a(s)\rangle - \frac{2K_b}{\overline{a}(s)+\overline{b}(s)} = \frac{\varepsilon^2}{\overline{a}^3(s)}}$$

fast

$$\boxed{\begin{aligned}&\frac{d^2}{ds^2}\delta a(s) + \kappa_q(s)\overline{a} + \kappa_q(s)\delta a(s) - \langle \kappa_q(s)\delta a(s)\rangle \\ &+ \frac{2K_b}{(\overline{a}+\overline{b})^2}[\delta a(s)+\delta b(s)] = -\frac{3\varepsilon^2}{\overline{a}^4}\delta a(s)\end{aligned}}$$

$$\boxed{\text{For } b,\ \overline{a}\to\overline{b},\ \delta a\to\delta b,\ \kappa_q(s)\to-\kappa_q(s)}$$

$$\boxed{\varepsilon_x = \varepsilon_y = \varepsilon} \longrightarrow \boxed{\begin{gathered}\overline{a}(s) = \overline{b}(s) \equiv r_b(s)\\ \delta a(s) = -\delta b(s)\end{gathered}}$$

slow

$$\boxed{\frac{d^2}{ds^2}\overline{a}(s) + \langle \kappa_q(s)\delta a(s)\rangle - \frac{K_b}{\overline{a}(s)} = \frac{\varepsilon^2}{\overline{a}^3(s)}}$$

$$\langle \kappa_q(s)\delta a(s)\rangle \equiv \kappa_{sf}\overline{a}$$

fast

$$\boxed{\frac{d^2}{ds^2}\delta a(s) + \kappa_q(s)\overline{a} + \kappa_q(s)\delta a(s) - \langle \kappa_q(s)\delta a(s)\rangle = -\frac{3\varepsilon^2}{\overline{a}^4}\delta a(s)}$$

$$\boxed{\frac{d^2}{ds^2}\delta a(s) + \kappa_q(s)\overline{a} = 0}$$

$$\left\langle \kappa_q(s)\delta a(s)\right\rangle = \left\{ \left\langle \left[\int_{s_0}^{s} ds\kappa_q(s)\right]^2 \right\rangle - \left\langle \int_{s_0}^{s} ds\kappa_q(s)\right\rangle^2 \right\} \overline{a}$$

Homework 3.14

$$\kappa_{sf} = \left\langle \left[\int_{s_0}^{s} ds\kappa_q(s)\right]^2 \right\rangle - \left\langle \int_{s_0}^{s} ds\kappa_q(s)\right\rangle^2$$

Smooth focusing approximation

$$\frac{d^2}{ds^2}\overline{a}(s) + [\kappa_{sf} - \frac{K_b}{\overline{a}^2(s)}]\overline{a}(s) = \frac{\varepsilon^2}{\overline{a}^3(s)}$$

Equilibrium beam size

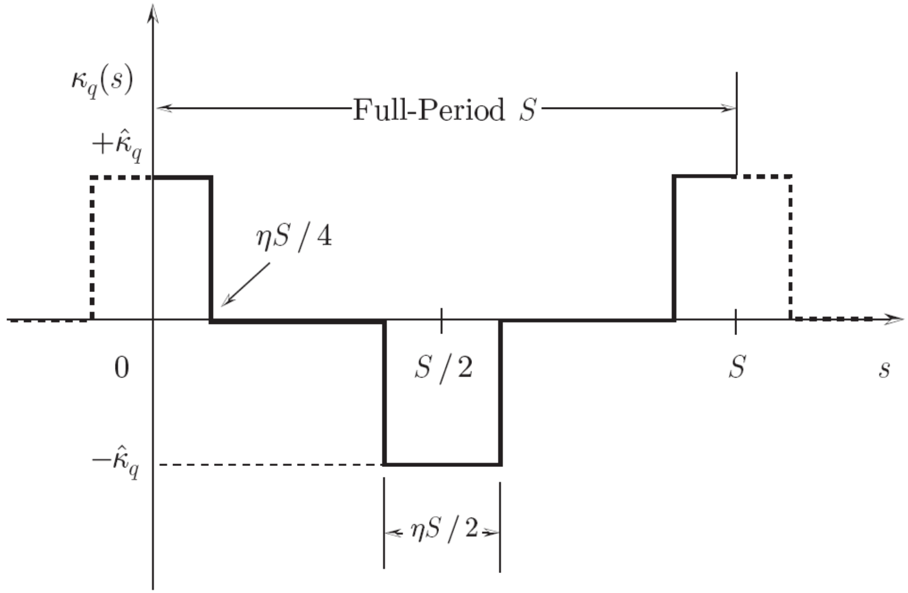
$$\boxed{\overline{a}_0^2 = \frac{K_b}{2\kappa_{sf}} + [(\frac{K_b}{2\kappa_{sf}})^2 + \frac{\varepsilon^2}{\kappa_{sf}}]^{1/2}}$$

Phase advance

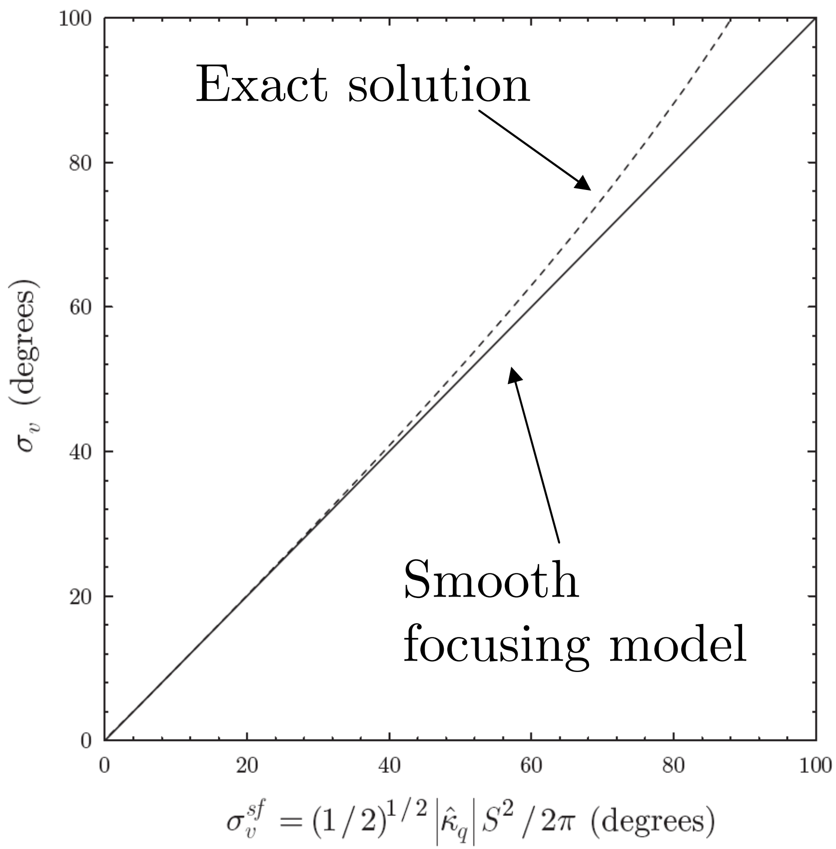
$$\boxed{\sigma^{sf} \equiv \varepsilon S \,/\, \overline{a}_0^2}$$

Vacuum phase advance

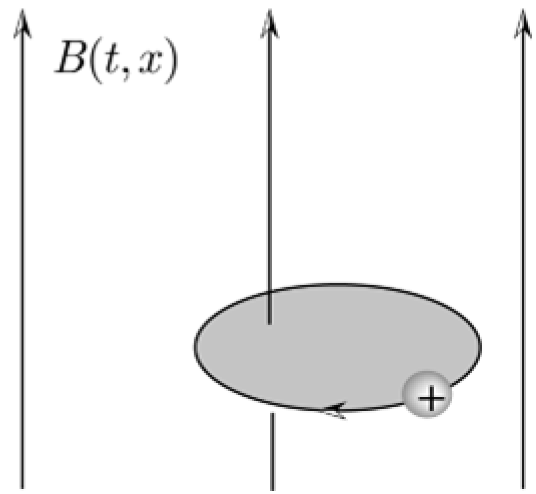
$$\boxed{\sigma_v^{sf} = \sqrt{\kappa_{sf}} S}$$



$$\kappa_{sf} = \frac{1}{16}\eta^2\hat{\kappa}_q^2 S^2\left(1 - \frac{2}{3}\eta\right)$$



Gyromotion in time-dependent field



There is an exact invariant of magnetic moment similar to the Courant-Snyder invariant.

An Exact Magnetic Moment Invariant of Charged Particle Gyromotion, H. Qin and R. C. Davidson, Physical Review Letters **96**, 085003 (2006).

Theorem 1. For an arbitrary function $\kappa(t)$ and $w_1,\ w_2$ satisfying

$$\ddot{w}_1 + \kappa w_1 = \frac{\varepsilon_1}{w_1^3},$$

$$\ddot{w}_2 + \kappa w_2 = \frac{\varepsilon_2}{w_2^3},$$

where ε_1 and ε_2 are real constants, the quantity

$$I = \varepsilon_1\left(\frac{w_2}{w_1}\right)^2 + \varepsilon_2\left(\frac{w_1}{w_2}\right)^2 + \left(w_2\,\dot{w}_1 - \dot{w}_2 w_1\right)^2$$

is an invariant.