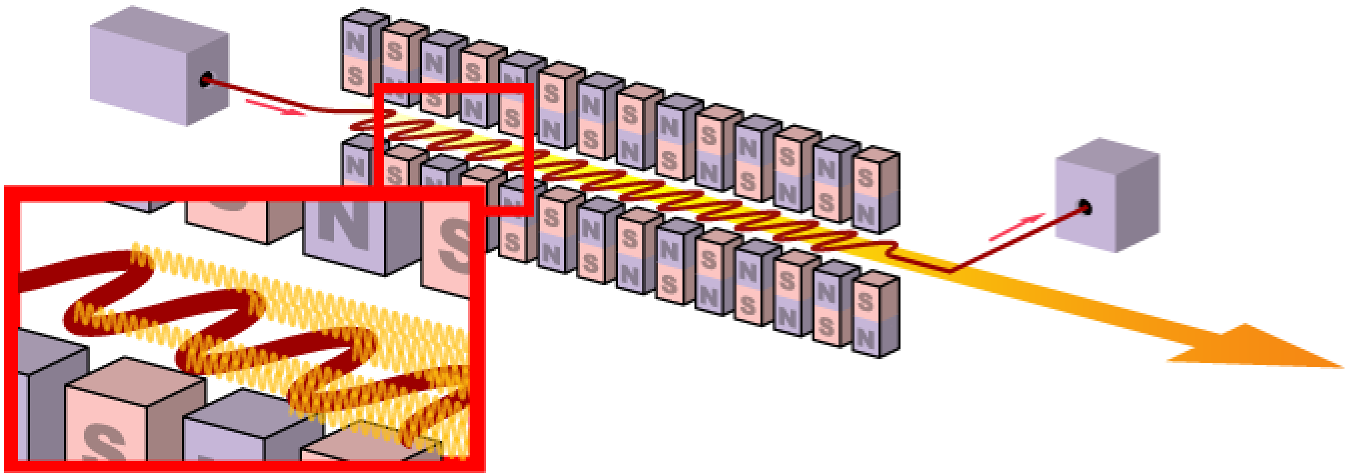
# Free Electron laser (FEL)

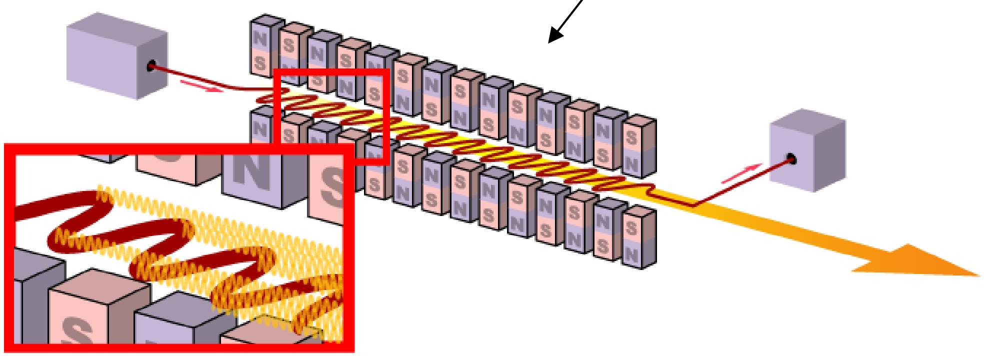


<u>Applications:</u>

- ❑ Imaging
- ❑ Medical
- ❑ Defense

$$\underset{\sim}{B}{}_w^0(\underset{\sim}{x}) = -B_w\left[\cos(k_0 z)\hat{\underset{\sim}{e}}_x + \sin(k_0 z)\hat{\underset{\sim}{e}}_y\right]$$

$$\underset{\sim}{A}{}_w^0(\underset{\sim}{x}) = \left(\frac{B_w}{k_0}\right)\left[\cos(k_0 z)\hat{\underset{\sim}{e}}_x + \sin(k_0 z)\hat{\underset{\sim}{e}}_y\right]$$



$$\frac{1}{\gamma^2}+\beta^2=1$$

$$\gamma^2=1+\gamma^2\beta^2$$

$$H=\gamma mc^2=\sqrt{m^2c^4+c^2\left(\mathbf{P}-\frac{e}{c}\mathbf{A}\right)^2}+e\phi$$

$$\mathbf{P}=\mathbf{p}+\frac{e}{c}\mathbf{A}$$

$$\mathbf{p}=m\gamma\mathbf{v}$$

$$a_w = \frac{eB_w}{m_e c^2 k_0}$$

$$\mathbf{P}_x=\mathbf{P}_y=const=0 \longrightarrow \begin{aligned} v'_x(t') &= \frac{ca_w}{\gamma}\cos\left[k_0 z'(t')\right] \\ v'_y(t') &= \frac{ca_w}{\gamma}\sin\left[k_0 z'(t')\right] \end{aligned}$$

Free streaming in z: $z'(t') = z + v_z(t' - t)$

$$\gamma = \left(1 + \frac{p_z^2}{m_e^2 c^2} + a_w^2\right)^{1/2}$$

$$v_x'(t') = \frac{c a_w}{\gamma} \cos[k_0 z'(t')]$$

$$v_y'(t') = \frac{c a_w}{\gamma} \sin[k_0 z'(t')]$$

$$\left.\begin{matrix} v_x'(t') \\ v_y'(t') \end{matrix}\right\} \sim e^{\pm i k_0 v_z \tau}$$

$$z'(t') = z + v_z (t'-t)$$

$$\eta(\omega) = \frac{1}{T} \frac{d^2 I}{d\omega d\Omega} \quad \left\{ \sim e^{\pm i k_0 v_z \tau} \right. \tag{7.58}$$

$$z'(\tau) = z + v_z \tau$$

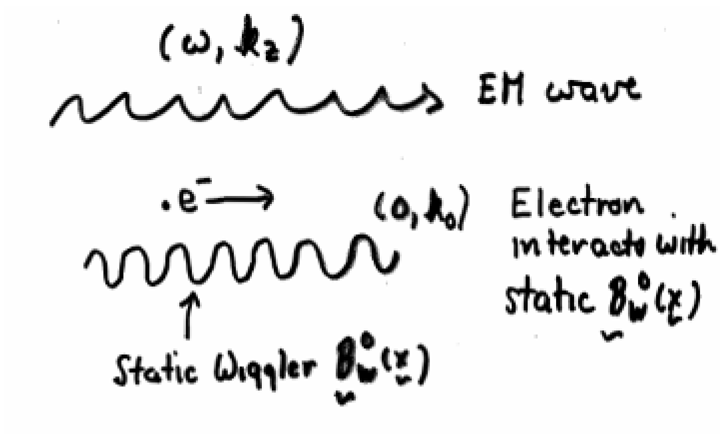
$$= \frac{e^2\omega^2}{4\pi^2 c^3 T} \left| \int_0^T d\tau \left[ v_x'(\tau)\hat{e}_x + v_y'(\tau)\hat{e}_y \right] \exp\left( i k_z z - i\omega\tau + i k_z v_z \tau \right) \right|^2 .$$

Resonant factor: $\exp\left\{ -i\left[ \omega - (k_z \pm k_0) v_z \right] \tau \right\}$

$$\eta(\omega) = \frac{a_w^2 e^2 \omega^2 T}{\gamma^2 8\pi^2 c} \frac{\sin^2[(\omega - k_z v_z - k_0 v_z)T/2]}{[(\omega - k_z v_z - k_0 v_z)T/2]^2}$$

## Compton (Thompson) scattering

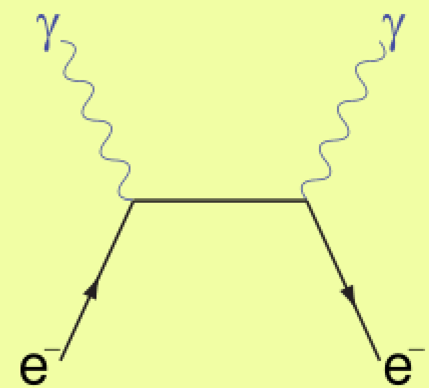
strong maximum at $\omega = (k_z + k_0)v_z$




In beam frame

$$\omega' = \gamma_z(\omega - k_z v_z)$$

$$\omega_0' = \gamma_z k_0 v_z$$




$$\hbar\omega_0' = h\omega'$$

$$\hat{\omega} = (\hat{k}_z + k_0) v_{ze}$$

$$\hat{\omega} = c\hat{k}_z \, .$$

In vacuum

$$\hat{\gamma}_e = \Big(1 + \underbrace{\frac{\hat{p}_z^2}{m_e^2 c^2}}_{\hat{\gamma}_e^2 \beta_{ze}^2} + \underbrace{\frac{p_x^2 + p_y^2}{m_e^2 c^2}}_{a_w^2}\Big)^{1/2}$$

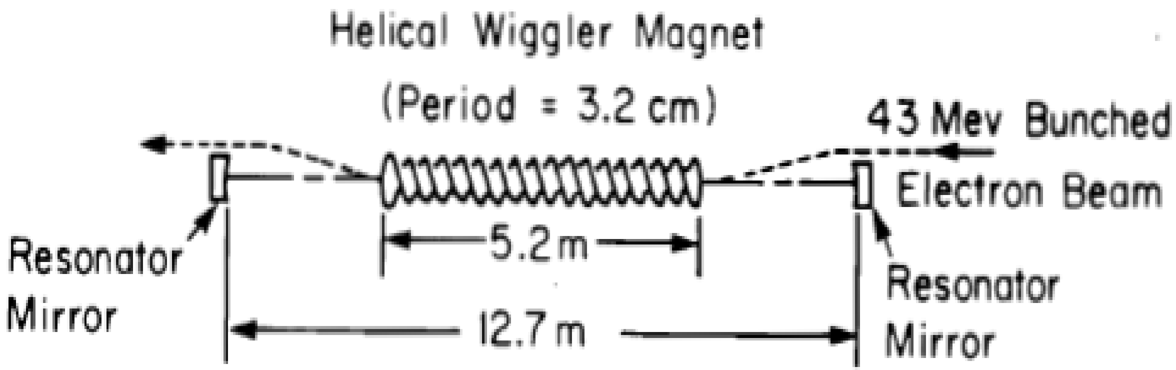
$$\hat{\gamma}_e^2 = \beta_{ze}^2 \hat{\gamma}_e^2 + 1 + a_w^2$$

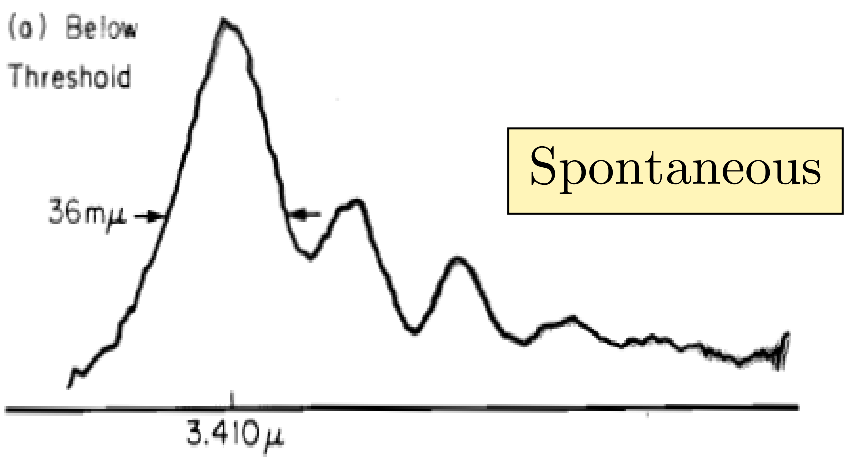
$$1 - \beta_{ze}^2 = \frac{1 + a_w^2}{\hat{\gamma}_e^2}$$

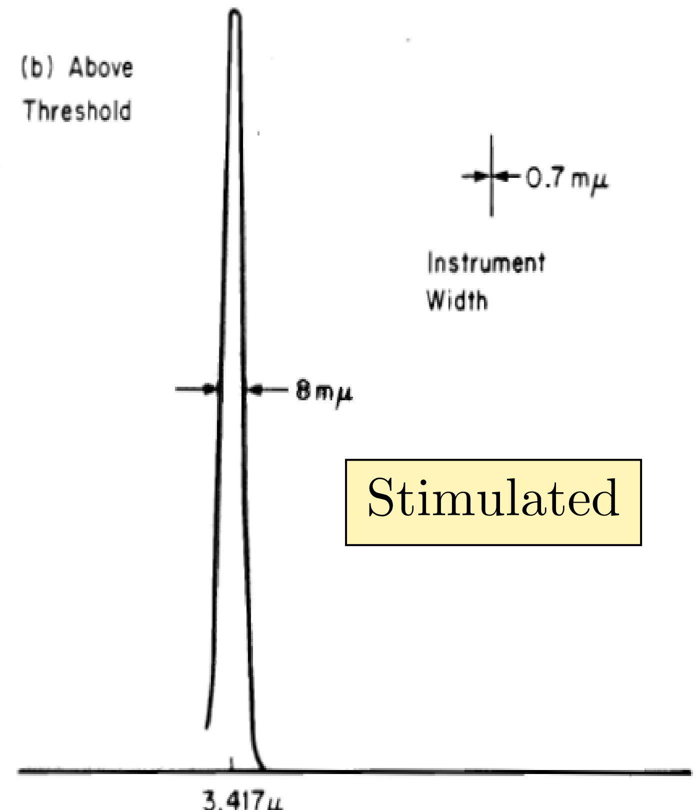
$$\hat{k}_z = \frac{\beta_{ze}}{1 - \beta_{ze}} k_0 = \frac{\beta_{ze}(1 + \beta_{ze})}{1 - \beta_{ze}^2} k_0$$

$$\hat{k}_z = \frac{\hat{\gamma}_e^2 (1 + \beta_{ze}) \beta_{ze}}{(1 + a_w^2)} k_0$$

Deacon, Stanford, PRL 1977



## Free electron laser

- Kinetic dispersion relation (Sec. 7.6) ☑
- Cold-beam stability properties (sec. 7.7) ☑
- Warm-beam Compton regime (Sec. 7.8) ☑
- Nonlinear evolution (Sec. 7.9)
- Sideband instability (Sec. 7.10) ☑
- Harmonic generation in planar wiggler (Sec. 7.11)
- High-gain FEL amplifier experiments (Sec. 7.12) ☑

# 1D Kinetic dispersion relation (Sec. 7.6)

$$\delta \underset{\sim}{E}(\underset{\sim}{x},t) = -\frac{1}{c}\frac{\partial}{\partial t}\delta A_x(z,t)\hat{\underset{\sim}{e}}_x - \frac{1}{c}\frac{\partial}{\partial t}\delta A_y(z,t)\hat{\underset{\sim}{e}}_y - \frac{\partial}{\partial z}\delta\phi(z,t)\hat{\underset{\sim}{e}}_z$$

$$\delta \underset{\sim}{B}(\underset{\sim}{x},t) = -\frac{\partial}{\partial z}\delta A_y(z,t)\hat{\underset{\sim}{e}}_x + \frac{\partial}{\partial z}\delta A_x(z,t)\hat{\underset{\sim}{e}}_y$$

$$P_x = p_x - \frac{e}{c}A^0_{xw}(z) - \frac{e}{c}\delta A_x(z,t) = \text{const.},$$

$$P_y = p_y - \frac{e}{c}A^0_{yw}(z) - \frac{e}{c}\delta A_y(z,t) = \text{const.},$$

Maxwell

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial z^2}\right)\delta A_x = -\frac{4\pi e}{c}\int d^3p\, v_x\left(f_e - f_e^0\right)$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial z^2}\right)\delta A_y = -\frac{4\pi e}{c}\int d^3p\, v_y\left(f_e - f_e^0\right)$$

$$\frac{\partial^2}{\partial z^2}\delta\phi = 4\pi e\int d^3p\left(f_e - f_e^0\right)\ ,$$

Vlasov

$$\left\{\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z} - e\left(\delta\underset{\sim}{E} + \frac{\underset{\sim}{v}\times(\underset{\sim}{B}_w^0 + \delta\underset{\sim}{B})}{c}\right)\cdot\frac{\partial}{\partial\underset{\sim}{p}}\right\}f_e(z,\underset{\sim}{p},t) = 0$$

$$f_e(z,\underset{\sim}{p},t) = \hat{n}_e \delta(P_x)\delta(P_y) F_e(z,p_z,t)$$

Vlasov

$$\left\{\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} - \frac{\partial}{\partial z}\hat{H}(z,p_z,t)\frac{\partial}{\partial p_z}\right\} F_e(z,p_z,t) = 0$$

$$\hat{H}(z,p_z,t) = \gamma_T(z,p_z,t) m_e c^2 - e\delta\phi(z,t)$$

$$\gamma_T(z,p_z,t) = \left[1 + \frac{p_z^2}{m_e^2c^2} + \frac{e^2}{m_e^2c^4}\left(A_{xw}^0 + \delta A_x\right)^2 + \frac{e^2}{m_e^2c^4}\left(A_{yw}^0 + \delta A_y\right)^2\right]^{1/2}$$

$$F_z = -\frac{\partial}{\partial z}\hat{H}$$

$$= -\frac{e^2}{2\gamma_T m_e c^2}\frac{\partial}{\partial z}\left[2A_{xw}^0\delta A_x + 2A_{yw}^0\delta A_y + \overbrace{(\delta A_x)^2 + (\delta A_y)^2}^{\text{quadratic}}\right] + e\frac{\partial}{\partial z}\delta\phi \ . \qquad (7.78)$$

## Maxwell

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)\delta A_x = -\frac{\omega_{pe}^2}{c^2}\left[\left(A_{xw}^0 + \delta A_x\right)\int \frac{dp_z}{\gamma_T} F_e - A_{xw}^0 \int \frac{dp_z}{\gamma} F_e^0\right] , \tag{7.75}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right)\delta A_y = -\frac{\omega_{pe}^2}{c^2}\left[\left(A_{yw}^0 + \delta A_y\right)\int \frac{dp_z}{\gamma_T} F_e - A_{yw}^0 \int \frac{dp_z}{\gamma} F_e^0\right] , \tag{7.76}$$

$$\frac{\partial^2}{\partial z^2}\delta\phi = 4\pi e\hat{n}_e \int dp_z \left(F_e - F_e^0\right) . \tag{7.77}$$

$$\delta\Phi = \frac{e}{m_e c^2}\delta\phi ,$$

$$\delta A_\pm = \frac{e}{m_e c^2}\left(\delta A_x \pm i\delta A_y\right) ,$$

$$A_w^\pm = \frac{e}{m_e c^2}\left(A_{xw}^0 \pm iA_{yw}^0\right) = a_w \exp(\pm ik_0 z)$$

# Linearized Vlasov-Maxwell

$$\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z}\right)\delta F_e = m_e c^2 \left\{\frac{a_w}{2\gamma} \frac{\partial}{\partial z} \left[\exp(ik_0 z)\delta A_- + \exp(-ik_0 z)\delta A_+\right] - \frac{\partial}{\partial z}\delta\Phi\right\} \frac{\partial}{\partial p_z} F_e^0(p_z)$$

$$\delta F_z = -\frac{m_e c^2 a_w}{2\gamma_T} \frac{\partial}{\partial z} \left[\exp(ik_0 z)\delta A_- + \exp(-ik_0 z)\delta A_+\right] + m_e c^2 \frac{\partial}{\partial z}\delta\Phi$$

$$\gamma = (1 + p_z^2/m_e^2c^2 + a_w^2)^{1/2}$$

$$\frac{1}{\gamma_T} = \frac{1}{\gamma} - \frac{a_w}{2\gamma^3}\left[\exp(ik_0 z)\delta A_- + \exp(-ik_0 z)\delta A_+\right]$$

$$\frac{\partial^2}{\partial z^2}\,\delta\phi = \frac{\omega_{pe}^2}{c^2}\int dp_z\,\delta F_e$$

$$\left(c^2\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} - \omega_{pe}^2\int\frac{dp_z}{\gamma}F_e^0\right)\delta A_+ = \omega_{pe}^2\left\{a_w\exp(ik_0z)\int\frac{dp_z}{\gamma}\,\delta F_e - \frac{1}{2}a_w^2\exp(ik_0z)\left[\exp(ik_0z)\delta A_- + \exp(-ik_0z)\delta A_+\right]\int\frac{dp_z}{\gamma^3}F_e^0\right\}$$

$$\left(c^2\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} - \omega_{pe}^2\int\frac{dp_z}{\gamma}F_e^0\right)\delta A_- = \omega_{pe}^2\left\{a_w\exp(-ik_0z)\int\frac{dp_z}{\gamma}\,\delta F_e - \frac{1}{2}a_w^2\exp(-ik_0z)\left[\exp(ik_0z)\delta A_- + \exp(-ik_0z)\delta A_+\right]\int\frac{dp_z}{\gamma^3}F_e^0\right\}$$

$$\frac{1}{\gamma_T} = \frac{1}{\gamma} - \frac{a_w}{2\gamma^3}\left[\exp(ik_0z)\delta A_- + \exp(-ik_0z)\delta A_+\right]$$

# Discretized coupling due to wiggler

$$\delta A_- \sim \sum_{k_z} \delta A_-(k_z)\exp(ikz - i\omega t)$$

$$\delta A_-(k_z) \quad \begin{cases} \delta f\,(k_z + k_0) \\ \delta\phi(k_z + k_0) \\ \delta A_+(k_z + 2k_0) \end{cases}$$

$$\delta F_e(k_z + k_0, p_z) = m_e c^2 \frac{(k_z + k_0)\partial F_e^0/\partial p_z}{[\omega - (k_z + k_0)v_z]}$$

$$\times \left\{ \delta\Phi(k_z + k_0) - \frac{a_w}{2\gamma}\left[\delta A_-(k_z) + \delta A_+(k_z + 2k_0)\right]\right\}$$

Maxwell

$$\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} \begin{pmatrix} \delta A_-(k_z) \\ \delta A_+(k_z+2k_0) \\ \delta \Phi(k_z+k_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\det\{D_{ij}\} = 0$$

DR

$$c^2(k_z+k_0)^2 D^L(k_z+k_0,\omega) D^T(k_z,\omega) D^T(k_z+2k_0,\omega)$$

$$= \frac{1}{2} a_w^2 \left[D^T(k_z,\omega) + D^T(k_z+2k_0,\omega)\right]$$

$$\times\left\{\left[\chi^{(1)}(k_z+k_0,\omega)\right]^2 - c^2(k_z+k_0)^2 D^L(k_z+k_0,\omega)\left[\alpha_3\omega_{pe}^2 + \chi^{(2)}(k_z+k_0,\omega)\right]\right\}$$

$$\alpha_3 = \int dp_z F_e^0(p_z)/\gamma^3$$

Susceptibilities:

$$\chi^{(n)}(k_z + k_0, \omega) = m_e c^2 \omega_{pe}^2 \int \frac{dp_z}{\gamma^n} \frac{(k_z + k_0)\partial F_e^0/\partial p_z}{\omega - (k_z + k_0)v_z}$$

Longitudinal and transverse dielectric functions:

$$D^L(k_z + k_0, \omega) = 1 + \frac{m_e \omega_{pe}^2}{(k_z + k_0)^2} \int dp_z \frac{(k_z + k_0)\partial F_e^0/\partial p_z}{\omega - (k_z + k_0)v_z}$$

$$D^T(k_z, \omega) = \omega^2 - c^2 k_z^2 - \omega_{pe}^2 \int \frac{dp_z}{\gamma} F_e^0(p_z) \;,$$

$$D^T(k_z + 2k_0, \omega) = \omega^2 - c^2(k_z + 2k_0)^2 - \omega_{pe}^2 \int \frac{dp_z}{\gamma} F_e^0(p_z)$$

# Polarization relation

$$\frac{\delta\Phi(k_z + k_0)}{\delta A_-(k_z)} = \frac{a_w}{2c^2(k_z + k_0)^2}$$

$$\times\ \chi^{(1)}(k_z + k_0, \omega)\ \frac{[D^T(k_z + 2k_0, \omega) + D^T(k_z, \omega)]}{D^L(k_z + k_0, \omega) D^T(k_z + 2k_0, \omega)}$$

$$\frac{\delta A_+(k_z + 2k_0)}{\delta A_-(k_z)} = \frac{D^T(k_z, \omega)}{D^T(k_z + 2k_0, \omega)}$$

Coupling due to the wiggler

DR

$$c^2(k_z+k_0)^2 D^L(k_z+k_0,\omega) D^T(k_z,\omega) D^T(k_z+2k_0,\omega)$$

$$= \frac{1}{2} a_w^2 \left[D^T(k_z,\omega) + D^T(k_z+2k_0,\omega)\right] \tag{7.95}$$

$$\times\left\{\left[\chi^{(1)}(k_z+k_0,\omega)\right]^2 - c^2(k_z+k_0)^2 D^L(k_z+k_0,\omega)\left[\alpha_3\omega_{pe}^2 + \chi^{(2)}(k_z+k_0,\omega)\right]\right\}$$

Beam intensity

$\chi^{(1)} \longrightarrow 0$

Compton regime

$$D^T(k_z,\omega) D^T(k_z+2k_0,\omega)$$

$$= -\frac{1}{2} a_w^2 \left[D^T(k_z,\omega) + D^T(k_z+2k_0,\omega)\right]\left[\alpha_3\omega_{pe}^2 + \chi^{(2)}(k_z+k_0,\omega)\right] .$$

# Compton regime

$$\frac{\delta\Phi(k_z + k_0)}{\delta A_-(k_z)} = \frac{a_w}{2c^2(k_z + k_0)^2}$$

$$\times\ \chi^{(1)}(k_z + k_0, \omega)\ \frac{[D^T(k_z + 2k_0, \omega) + D^T(k_z, \omega)]}{D^L(k_z + k_0, \omega)D^T(k_z + 2k_0, \omega)}$$

$\longrightarrow$ $\delta\phi \to 0$

# Scattering of EM wave

| | Elastic, low energy<br>Classical | Inelastic, high energy<br>Quantum |
|---|---|---|
| Molecule<br>phono | Rayleigh | Raman, Brillouin |
| Atom<br>electron | Thompson | Compton<br>X-ray |

# Cold Beam Stability Properties (Sec. 7.7)

$$f_e(z,\mathbf{p},t) = \hat{n}_e \delta(P_x)\delta(P_y) F_e(z,p_z,t)$$

$$F_e^0(p_z) = \delta\left(p_z - \hat{\gamma}_e m_e V_{ze}\right)$$

$$\hat{\gamma}_e = \left(1 + \frac{\hat{\gamma}_e^2 V_{ze}^2}{c^2} + a_w^2\right)^{1/2}$$

$$\hat{\gamma}_e^2 = 1 + \hat{\gamma}_e^2 V_{ze}^2/c^2 + a_w^2 \Rightarrow \hat{\gamma}_e^2(1-\beta_{ze}^2) = 1 + a_w^2$$

$$\Rightarrow \hat{\gamma}_e^2 = \gamma_z^2(1+a_w^2)$$

$$\Rightarrow \beta_{ze}^2 = 1 - \frac{1+a_w^2}{\hat{\gamma}_e^2}$$

$$\frac{1}{\gamma^2} + \beta^2 = 1$$

$$\gamma^2 = 1 + \gamma^2\beta^2$$

$$\hat{\gamma}_e^2 = \gamma_z^2(1+a_w^2)$$

Be careful about the details

$$\chi^{(n)}(k_z+k_0,\omega) = m_e c^2 \omega_{pe}^2 \int \frac{dp_z}{\gamma^n} \frac{(k_z+k_0)\partial F_e^0/\partial p_z}{\omega-(k_z+k_0)v_z}$$

$$F_e^0(p_z) = \delta\left(p_z - \hat{\gamma}_e m_e V_{ze}\right)$$

$$\chi^{(0)}(k_z+k_0,\omega) = -\frac{\omega_{pe}^2}{\hat{\gamma}_e\gamma_z^2}\frac{c^2(k_z+k_0)^2}{[\omega-(k_z+k_0)V_{ze}]^2}\ ,$$

$$\chi^{(1)}(k_z+k_0,\omega) = \frac{\omega_{pe}^2}{\hat{\gamma}_e^2}\frac{[\omega(k_z+k_0)V_{ze}-c^2(k_z+k_0)^2]}{[\omega-(k_z+k_0)V_{ze}]^2}\ ,$$

$$\chi^{(2)}(k_z+k_0,\omega) = \frac{\omega_{pe}^2}{\hat{\gamma}_e^3}\frac{[2\omega(k_z+k_0)V_{ze}-c^2(k_z+k_0)^2(1+\beta_{ze}^2)]}{[\omega-(k_z+k_0)V_{ze}]^2}$$

# Full Dispersion Relation

Unstalbe coupling by wiggler

$\delta\phi(k_z + k_0)$ $\delta A_-(k_z)$ $\delta A_+(k_z + 2k_0)$

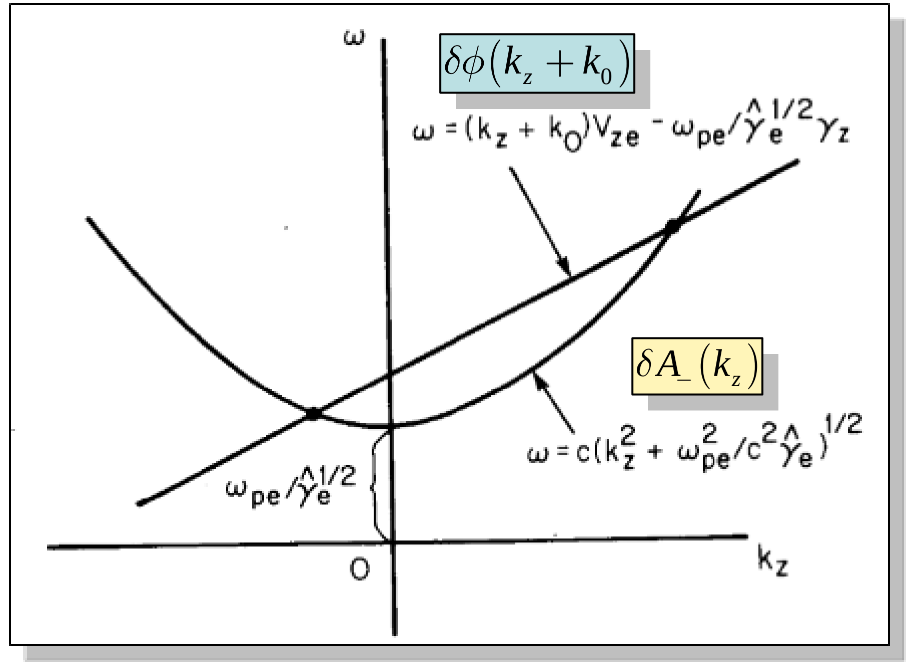
$$\left\{[\omega-(k_z+k_0)V_{ze}]^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e\gamma_z^2}\right\}\left[\omega^2-c^2k_z^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right]\left[\omega^2-c^2(k_z+2k_0)^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right]$$

$$=-\frac{a_w^2}{\hat{\gamma}_e^3}\omega_{pe}^2\left[\omega^2-c^2(k_z+k_0)^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right]\left[\omega^2-c^2(k_z+k_0)^2-c^2k_0^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right]$$

Coupling due to the wiggler

$$\hat{\omega} = (\hat{k}_z + k_0)V_{ze} - \frac{\omega_{pe}}{\hat{\gamma}_e^{1/2}\gamma_z}$$

$$\hat{\omega} = \left(c^2\hat{k}_z^2 + \frac{\omega_{pe}^2}{\hat{\gamma}_e}\right)^{1/2} .$$



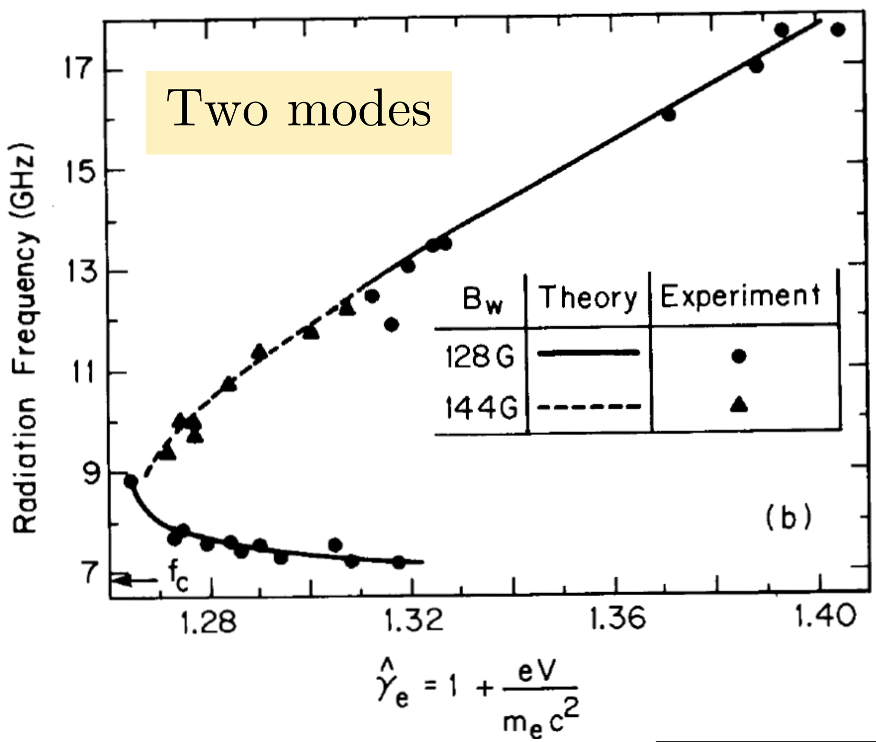

$$\boxed{\hat{k}_z^{\pm} = \frac{k_0}{(1-\beta_{ze}^2)}\left[\beta_{ze}^2 - \beta_{ze}\frac{\omega_{pe}}{\hat{\gamma}_e^{1/2}\gamma_z c k_0} \pm \left(\beta_{ze}^2 - 2\beta_{ze}\frac{\omega_{pe}}{\hat{\gamma}_e^{1/2}\gamma_z c k_0}\right)^{1/2}\right]}$$

$$\boxed{\omega_{pe}/ck_0 \ll \hat{\gamma}_e^{1/2}\gamma_z\beta_{ze}}$$

$$\boxed{\hat{k}_z^{+} = k_0\beta_{ze}/(1-\beta_{ze})} \qquad \boxed{\hat{k}_z^{-} = -k_0\beta_{ze}/(1+\beta_{ze})}$$

$$\boxed{\beta_{ze} \to -\beta_{ze}}$$

# Tunable FEL experiments (Fajans, et al, PRL 1984)

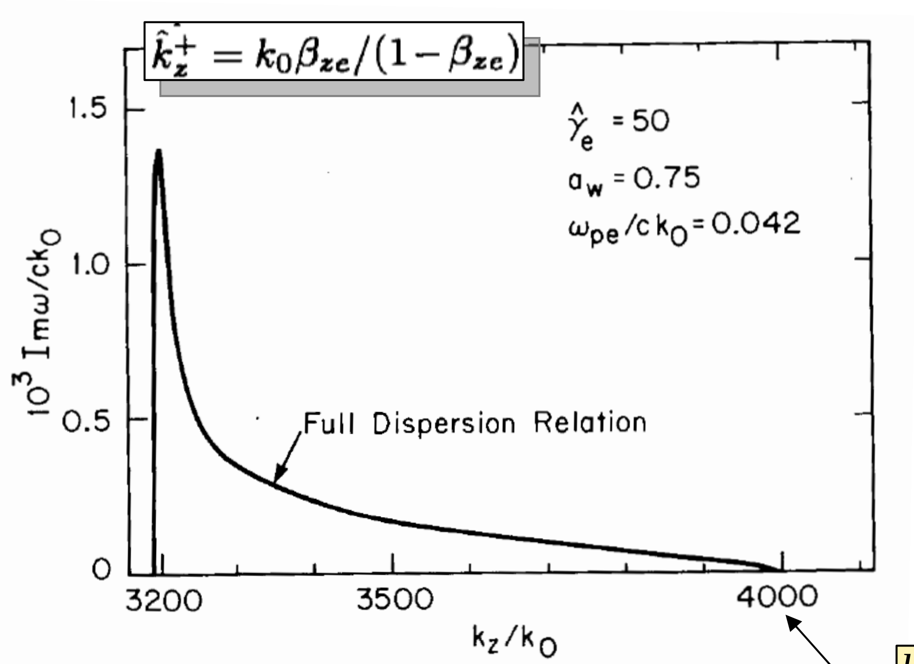



Finite geometry effect

$$\hat{\omega}^{\pm} = \gamma_z^2 \beta_{ze} c k_0 \left\{ 1 \pm \beta_{ze} \left[ 1 - \left( \frac{k_\perp}{\gamma_z \beta_{ze} k_0} \right)^2 \right]^{1/2} \right\}$$

Numerically solution of the full DR

$$\hat{k}_z^+ = k_0\beta_{ze}/(1-\beta_{ze})$$




$$k_z \longrightarrow \infty$$

$$[\omega - (k_z + k_0)V_{ze}]^2 = \frac{\omega_{pe}^2}{\hat{\gamma}_e\gamma_z^2} - a_w^2\frac{\omega_{pe}^2}{\hat{\gamma}_e^3} = \frac{\omega_{pe}^2}{\hat{\gamma}_e^3}$$

## Compton regime approximation

FDR

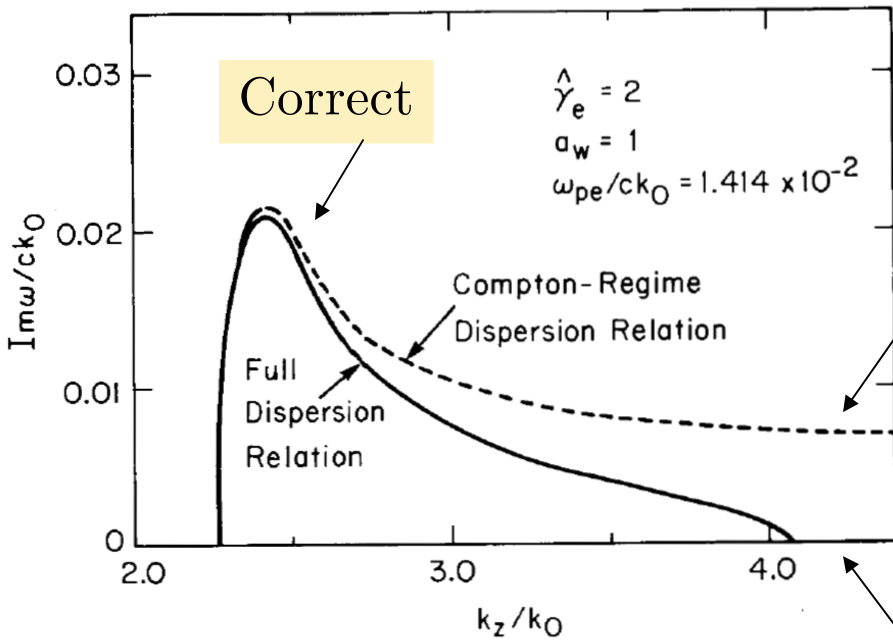
$$\left\{[\omega-(k_z+k_0)V_{ze}]^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e\gamma_z^2}\right\}\left[\omega^2-c^2k_z^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right]\left[\omega^2-c^2(k_z+2k_0)^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right]$$
$$=-\frac{a_w^2}{\hat{\gamma}_e^3}\omega_{pe}^2\left[\omega^2-c^2(k_z+k_0)^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right]\left[\omega^2-c^2(k_z+k_0)^2-c^2k_0^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right]$$

$$\frac{\omega_{pe}^2}{\hat{\gamma}_e\gamma_z^2}\ll|\omega-(k_z+k_0)V_{ze}|^2$$
$$\frac{\omega_{pe}^2}{\hat{\gamma}_e}\ll|\omega^2-c^2(k_z+k_0)^2|$$

CDR

$$\left[\omega-(k_z+k_0)V_{ze}\right]^2\left[\omega^2-c^2k_z^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right]\left[\omega^2-c^2(k_z+2k_0)^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right]$$
$$=-\frac{a_w^2}{\hat{\gamma}_e^3}\omega_{pe}^2\left[\omega^2-c^2(k_z+k_0)^2\right]\left[\omega^2-c^2(k_z+k_0)^2-c^2k_0^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right]\quad(7.105)$$

# Compton regime approximation incorrect for long wavelength




$$[\omega - (k_z + k_0)V_{ze}]^2 = -a_w^2 \frac{\omega_{pe}^2}{\hat{\gamma}_e^3}$$

Incorrect

$$[\omega - (k_z + k_0)V_{ze}]^2 = \frac{\omega_{pe}^2}{\hat{\gamma}_e \gamma_z^2} - a_w^2 \frac{\omega_{pe}^2}{\hat{\gamma}_e^3} = \frac{\omega_{pe}^2}{\hat{\gamma}_e^3}$$

## Compton regime approximation – growth rate

CDR

$$\left[\omega-(k_z+k_0)V_{ze}\right]^2\left[\omega^2-c^2k_z^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right]\left[\omega^2-c^2(k_z+2k_0)^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right]$$
$$=-\frac{a_w^2}{\hat{\gamma}_e^3}\omega_{pe}^2\left[\omega^2-c^2(k_z+k_0)^2\right]\left[\omega^2-c^2(k_z+k_0)^2-c^2k_0^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right] \quad (7.105)$$

$$\omega\simeq(k_z+k_0)V_{ze}$$

$$\omega^2\simeq c^2k_z^2+\omega_{pe}^2/\hat{\gamma}_e$$

$$[\omega-(k_z+k_0)V_{ze}]^2\left(\omega^2-c^2k_z^2-\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right)=\frac{1}{2}\frac{a_w^2\omega_{pe}^2}{\hat{\gamma}_e^3\gamma_z^2}c^2(k_z+k_0)^2$$

$$k_z=\hat{k}_z^+\simeq k_0\beta_{ze}/(1-\beta_{ze})$$

$$\omega-(k_z+k_0)V_{ze}=\left[\frac{a_w^2\omega_{pe}^2ck_0}{4\hat{\gamma}_e^3}\frac{(1+\beta_{ze})}{\beta_{ze}}\right]^{1/3}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$\frac{\omega_{pe}^2}{\hat{\gamma}_e \gamma_z^2} \ll |\omega - (k_z + k_0)V_{ze}|^2$$

$$\frac{\omega_{pe}^2}{\hat{\gamma}_e} \ll |\omega^2 - c^2(k_z + k_0)^2|$$

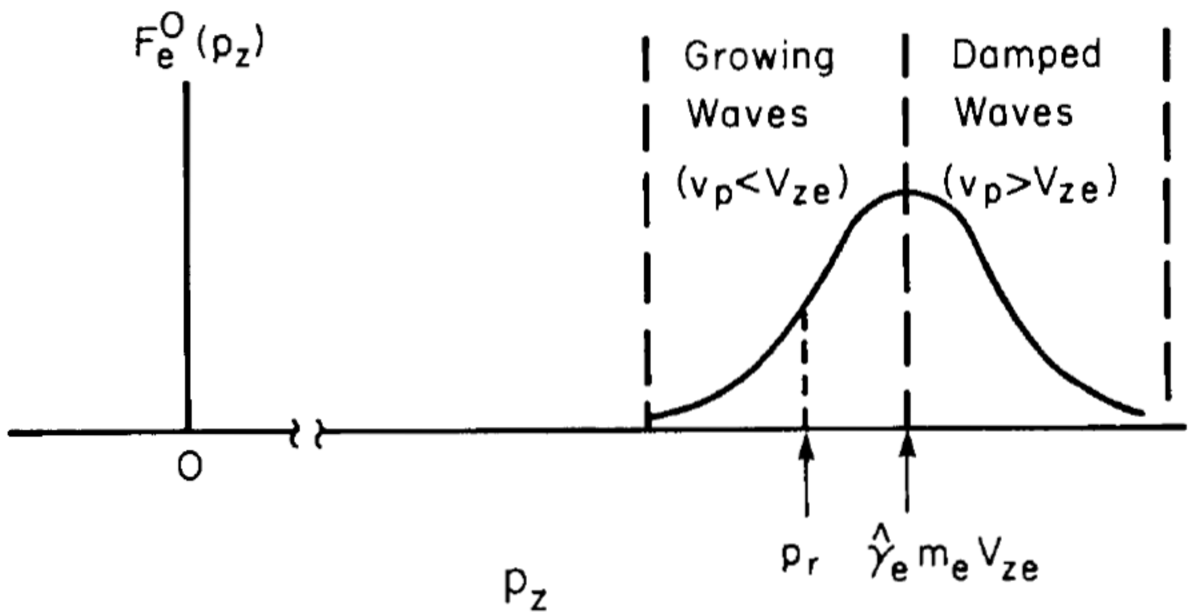
$$\frac{\omega_{pe}}{ck_0} \ll \frac{1}{4} \frac{a_w^2 \hat{\gamma}_e^{3/2}}{(1 + a_w^2)^{3/2}} \frac{(1 + \beta_{ze})}{\beta_{ze}}$$

Compton approximation is valid for tenuous, relativistic beams

Raman regime: high intensity, lower frequency

Warm beam effect (Compton regime)

Landau damping




$$v_p \equiv \omega_r/(k_z + k_0)$$

## Compton regime

$$D^T(k_z,\omega)D^T(k_z+2k_0,\omega)$$
$$= -\frac{1}{2}a_w^2\left[D^T(k_z,\omega)+D^T(k_z+2k_0,\omega)\right]\left[\alpha_3\omega_{pe}^2+\chi^{(2)}(k_z+k_0,\omega)\right] .$$

$$\boxed{D^T(k_z,\omega)\simeq 0}$$

$$D^T(k_z,\omega) = -\tfrac{1}{2}a_w^2\left[\alpha_3\omega_{pe}^2+\chi^{(2)}(k_z+k_0,\omega)\right]$$

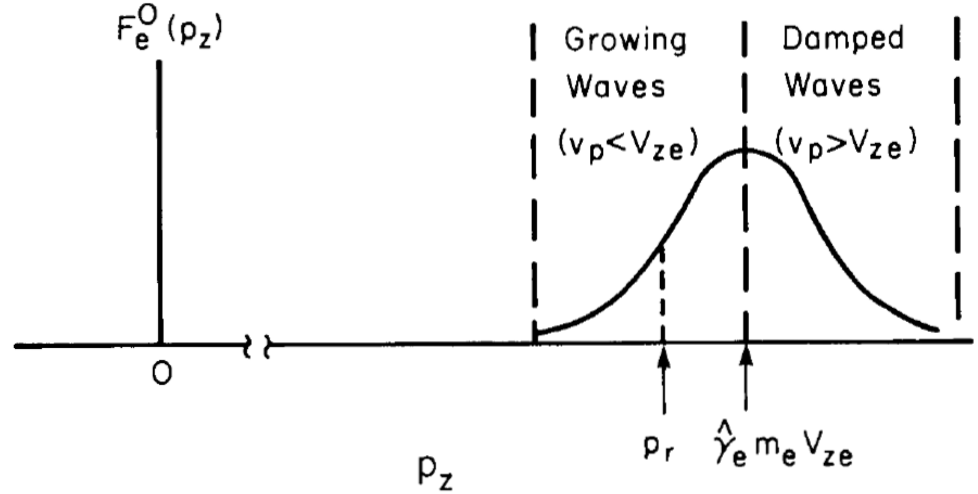
$$0 = D^c(k_z,\omega) \equiv \omega^2 - c^2k_z^2 - \omega_{pe}^2\int\frac{dp_z}{\gamma}\left(1-\frac{a_w^2}{2\gamma^2}\right)F_e^0(p_z)$$
$$+\tfrac{1}{2}a_w^2 m_e c^2\omega_{pe}^2\int\frac{dp_z}{\gamma^2}\frac{(k_z+k_0)\partial F_e^0/\partial p_z}{\omega-(k_z+k_0)v_z} .$$

$$|\omega_i| \ll |\omega_r|,\ |(k_z + k_0)\Delta v_z|$$

$$D^c(k_z, \omega_r + i\omega_i) = 0$$

$$0 = D_r^c(k_z,\omega_r) + i\left[\omega_i \frac{\partial D_r^c}{\partial \omega_r} + D_i^c(k_z,\omega_r)\right] + \cdots$$

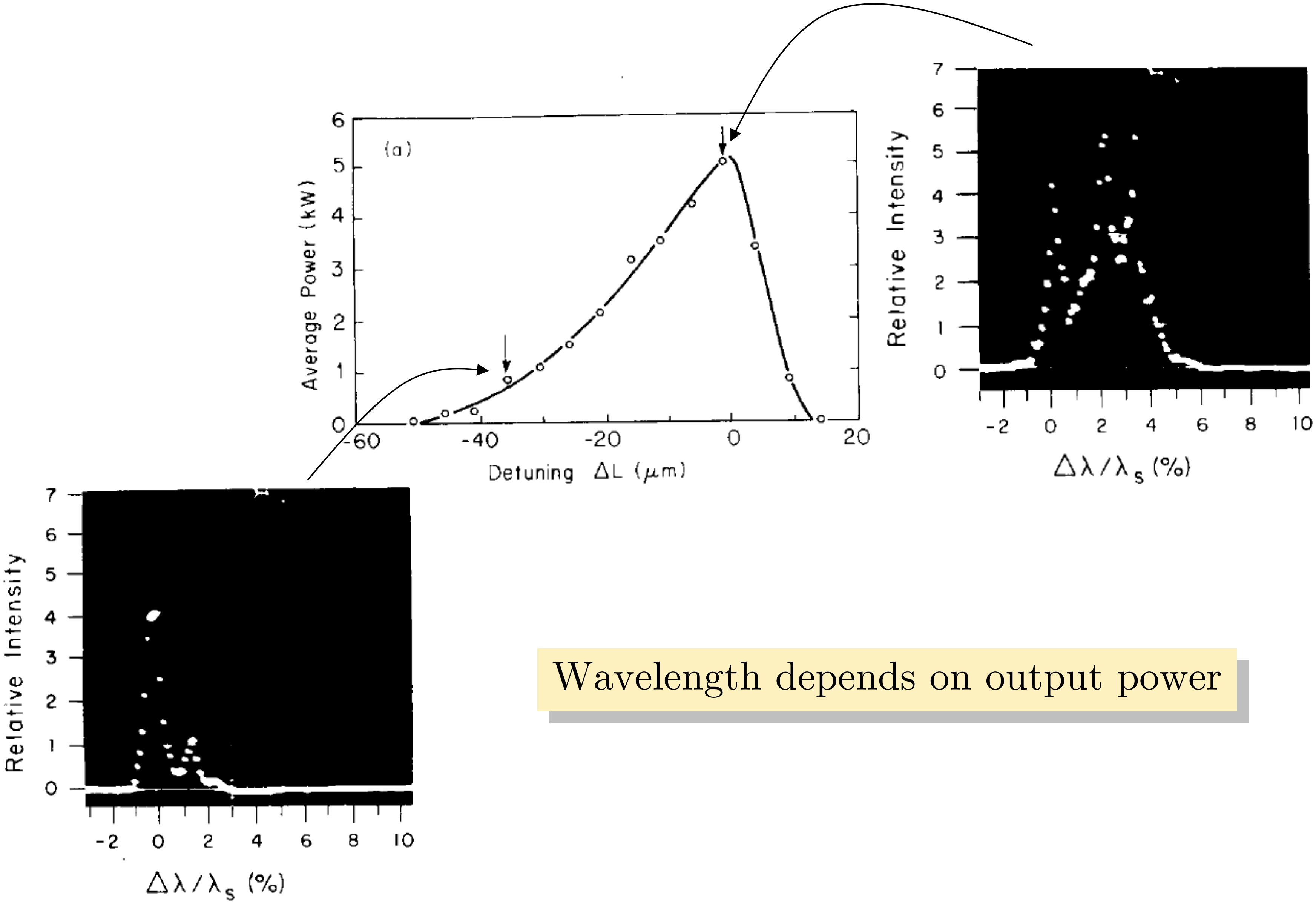
$$\mathrm{Im}\,\omega = \frac{\pi}{4}\frac{a_w^2 m_e^2 c^2 \omega_{pe}^2}{(1+a_w^2)}\frac{1}{|k_z+k_0|}\left[\frac{\gamma}{v_z}\frac{\partial}{\partial p_z}F_e^0(p_z)\right]_{v_z=v_p}$$



$$F_e^0(p_z) = \frac{1}{\sqrt{\pi}\Delta} \exp\left[-\frac{(p_z - \hat{\gamma}_e m_e V_{ze})^2}{\Delta^2}\right]$$

$$[\mathrm{Im}\,\omega]_{\mathrm{max}} = ck_0 \left(\frac{\pi}{8}\right)^{1/2} \frac{a_w^2}{(1+a_w^2)} \frac{\omega_{pe}^2}{c^2 k_0^2} \frac{ck_0}{\omega_r} \frac{\hat{\gamma}_e m_e^2 c^2}{\Delta^2} \exp(-0.5)$$

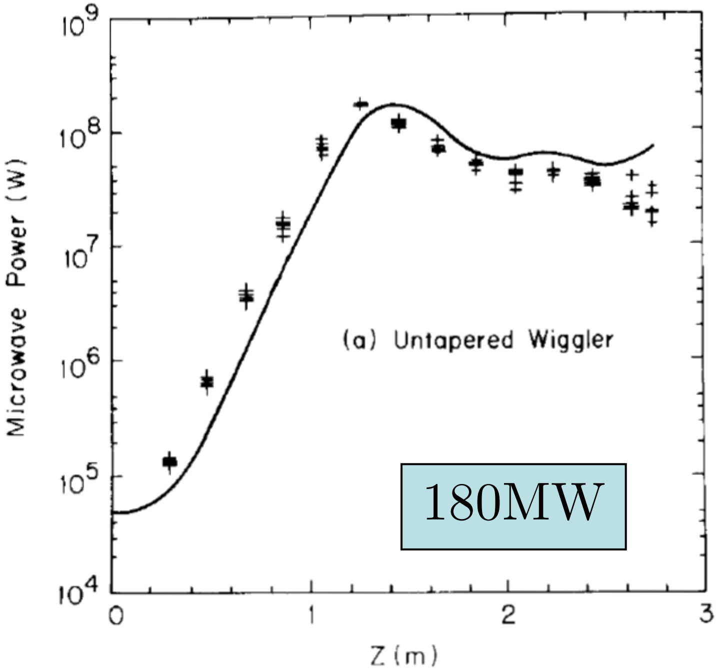
## Sideband instability (coherent nonlinear theory, Sec. 7.10)

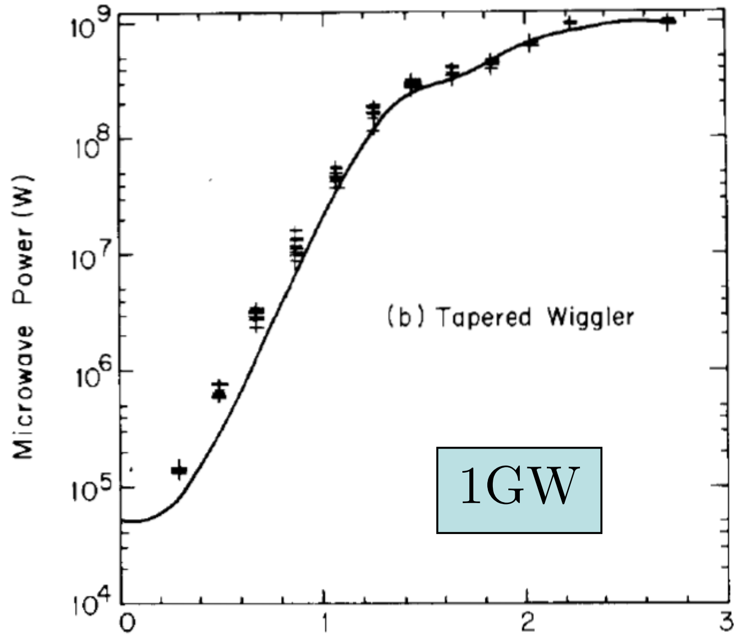



Wavelength depends on output power

# High-Gain FEL amplifier experiments (Orzechowski, PRL, 1985)

Beam: 3GW, 0.85KA, B=3.72kG, 3.45GHZ



$$\beta_{ze}^2 = 1 - \frac{1 + a_w^2(z)/2}{\hat{\gamma}_e^2(z)}$$

$$\beta_{ze}(z) = const.$$

$$\frac{1}{2}\frac{a_w}{1 + a_w^2/2}\frac{d}{dz}a_w = \frac{1}{\hat{\gamma}_e}\frac{d}{dz}\hat{\gamma}_e$$