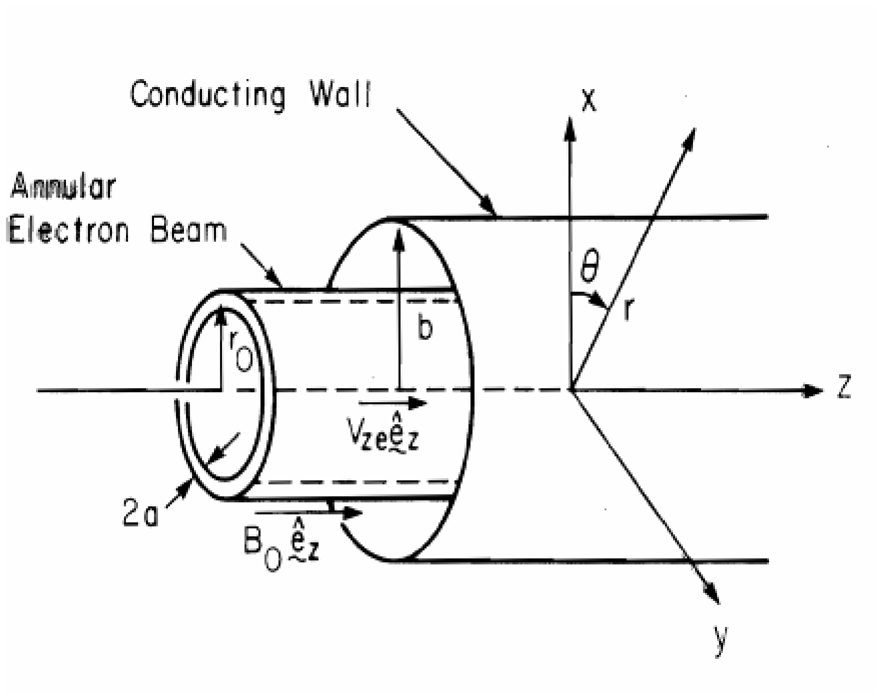
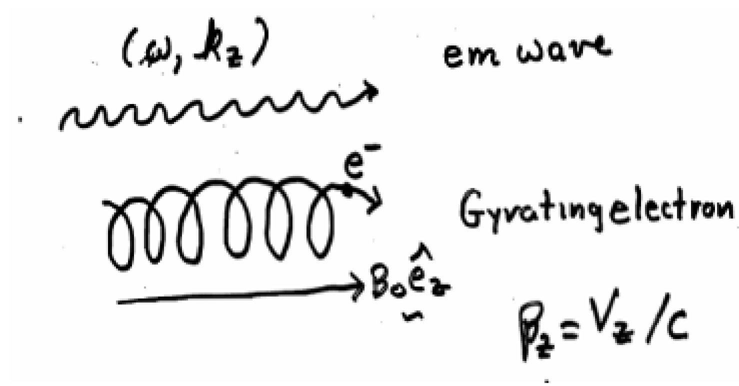
Cyclotron Maser Instability





Cyclotron resonance

$$w - k_z V_z = \frac{\omega_{ce}}{\hat{\gamma}}$$

Resonant Wave-Particle Interaction

Consider motion of a single electron in an applied magnetic field $\underset{\sim}{B}^0(\underset{\sim}{x})$ and a right-circularly polarized electromagnetic wave (ω, k_z) with $\underset{\sim}{\delta E} = (\delta E_x, \delta E_y, 0)$ and $\underset{\sim}{\delta B} = (\delta B_x, \delta B_y, 0)$ propagating in the z-direction. The oscillatory field components are proportional to

$$e^{\pm i(k_z z - \omega t)}$$

The motion of a single electron ("primed" orbits) is described by

$$\frac{d}{dt'}\underset{\sim}{p}' = -e\left[\underset{\sim}{\delta E} + \frac{1}{c}\underset{\sim}{v}' \times (\underset{\sim}{B}^0 + \underset{\sim}{\delta B})\right]$$

and the change in particle energy is given by

$$\frac{d}{dt'}\gamma' m_e c^2 = -e\,\underset{\sim}{v}'_\perp \cdot \underset{\sim}{\delta E}$$

Resonant Wave-Particle Interaction

In absence of em wave field, "zero-order" motion is determined from

$$\frac{d}{dt'}(\underset{\sim}{p}')^0 = -\frac{e}{c}(\underset{\sim}{v}')^0 \times \underset{\sim}{B}^0(\underset{\sim}{x}')$$

For special case where $\underset{\sim}{B}^0(\underset{\sim}{x}) = B_0 \hat{\underset{\sim}{e}}_z$ (corresponding to a uniform magnetic field) the electron moves with constant axial velocity $V_{ze} = \text{const}$. The motion perpendicular to $B_0 \hat{\underset{\sim}{e}}_z$ corresponds to a circular gyration with speed $V_{\perp e} = \text{const}$ and frequency

$$\Omega_\perp = \frac{\omega_{ce}}{\hat{\gamma}_e}$$

where $\hat{\gamma}_e \equiv (1 - V_{\perp e}^2/c^2 - V_{ze}^2/c^2)^{-1/2}$

Resonant Wave-Particle Interaction

Correct to first-order (in the em wave amplitude), the change in electron energy is given by

$$\frac{d}{dt'}\gamma' m_e c^2 = -e(\underset{\sim}{v}_\perp')^0 \cdot \delta \underset{\sim}{E}(\underset{\sim}{x}',t')$$

The time dependence of $\delta \underset{\sim}{E}$ in this expression is proportional to

$$e^{\pm i(k_z z' - \omega t')} = e^{\pm i k_z z_0} e^{\mp i(\omega - k_z V_{ze})t'}$$

where $z' = z_0 + V_{ze} t'$. The time dependence of $(\underset{\sim}{v}_\perp')^0$ is proportional to

$$e^{\pm i \Omega_\perp t'} \quad \text{where } \Omega_\perp \equiv \omega_{ce}/\hat{\gamma}_e$$

Resonant Wave-Particle Interaction

The change in electron energy is

$$\frac{d}{dt'}\gamma' m_e c^2 = -e\underbrace{(\mathbf{v}_\perp')^0}_{e^{\pm i\Omega_\perp t'}} \cdot \underbrace{\delta \mathbf{E}(\mathbf{x}',t')}_{e^{\mp i(\omega - k_z V_{ze})t'}}$$

As expected, systematic energy transfer between electron and em wave whenever "resonance" condition

$$\omega - k_z V_{ze} = \Omega_\perp$$

is satisfied. For $\mathbf{B}^0 = B_0 \hat{\mathbf{e}}_z$, $\Omega_\perp = \omega_{ce}/\hat{\gamma}_e$ and this becomes

$$\omega - k_z V_{ze} = \omega_{ce}/\hat{\gamma}_e$$

Cyclotron Maser Instability for Tenuous Beam (Sec. 7.2)

Consider tenuous electron beam with negligible self fields

$$\boxed{s_e = \hat{\gamma}_e \frac{\omega_{pe}^2}{\omega_{ce}^2} << 1}$$

The beam propagates parallel to a uniform magnetic field $B_0 \hat{\underset{\sim}{e}}_z$, and neglecting spatial variations in the radial direction, is described by the equilibrium distribution function

$$\boxed{f_e^0 = \hat{n}_e F_e(p_\perp^2, p_z).}$$

$p_\perp^2$ and p_z are constants of the motion in $B_0 \hat{\underset{\sim}{e}}_z$

Examine transverse electromagnetic stability properties for perturbations with $\delta \underset{\sim}{E} = (\delta E_x, \delta E_y, 0)$ and $\underset{\sim}{k} = (0, 0, k_z)$.

Problem 7.1 (Assigned)

Consider transverse electromagnetic wave perturbations with $\delta\underset{\sim}{E}\cdot\hat{\underset{\sim}{e}}_z = 0 = \delta\underset{\sim}{B}\cdot\hat{\underset{\sim}{e}}_z$ propagating parallel to a uniform magnetic field $B_0\hat{\underset{\sim}{e}}_z$ through a uniform electron plasma with equilibrium distribution function $f_e^0(\underset{\sim}{x},\underset{\sim}{p}) = \hat{n}_e F_e(p_\perp^2, p_z)$. Express perturbed quantities as

$$\boxed{\delta\psi(z,t) = \sum_{k_z=-\infty}^{\infty} \delta\psi(k_z)\exp(ik_z z - i\omega t)_k ,} \qquad (7.1.1)$$

where $\text{Im}\,\omega > 0$ corresponds to instability.

(a) Show that the linearized Maxwell equations can be expressed as

$$\boxed{\left(\omega^2 - c^2k_z^2\right)\left[\delta E_x(k_z) \pm i\delta E_y(k_z)\right] = 4\pi e i\omega\int d^3p\,(v_x \pm iv_y)\,\delta f_e(k_z,\underset{\sim}{p}) .}$$

(7.1.2)

where $\delta f_e(z,\underset{\sim}{p},t) = \sum_{k_z} \delta f_e(k_z,\underset{\sim}{p})\exp(ik_z z - i\omega t)$

Cyclotron Maser Instability

$$\delta \underset{\sim}{E} = \delta E_x \hat{\underset{\sim}{e}}_x + \delta E_y \hat{\underset{\sim}{e}}_y \; ; \; \delta \underset{\sim}{B} = \delta B_x \hat{\underset{\sim}{e}}_x + \delta B_y \hat{\underset{\sim}{e}}_y \quad \text{or}$$

$$\delta \underset{\sim}{E} = (\delta E_x, \delta E_y, 0) \; ; \; \delta \underset{\sim}{B} = (\delta B_x, \delta B_y, 0)$$

All perturbations vary like

$$\boxed{\delta\psi(z,t) \sim e^{i(k_z z - \omega t)} \qquad \mathrm{Im}\,\omega > 0}$$

Maxwell's Equations $\quad \frac{\partial}{\partial t} \to -i\omega \quad \frac{\partial}{\partial \underset{\sim}{x}} \to \hat{\underset{\sim}{e}}_z \, i k_z$

$$\boxed{\begin{aligned} &\frac{\omega}{c} \delta \hat{\underset{\sim}{B}} = k_z \hat{\underset{\sim}{e}}_z \times \delta \hat{\underset{\sim}{E}} \\ &i k_z \hat{\underset{\sim}{e}}_z \times \delta \hat{\underset{\sim}{B}} = -\frac{4\pi e}{c} \int d^3p \, \underset{\sim}{v} \, \delta \hat{f}_e(k_z, \underset{\sim}{p}) - i \frac{\omega}{c} \delta \hat{\underset{\sim}{E}} \end{aligned}}$$

$$\frac{\omega}{c} \hat{\underset{\sim}{e}}_z \times \delta \hat{\underset{\sim}{B}} = k_z \hat{\underset{\sim}{e}}_z \times (\hat{\underset{\sim}{e}}_z \times \delta \hat{\underset{\sim}{E}}) = -k_z \delta \hat{\underset{\sim}{E}}$$

$$\Rightarrow \boxed{(\omega^2 - c^2 k_z^2) \delta \hat{\underset{\sim}{E}} = 4\pi e i \omega \int d^3p \, (v_x \hat{\underset{\sim}{e}}_x + v_y \hat{\underset{\sim}{e}}_y) \delta \hat{f}_e(k_z, \underset{\sim}{p})}$$

Problem 7.1

$\hat{\gamma}_e \omega_{pe}^2/\omega_{ce}^2 << 1$

(b) Neglect equilibrium self-field effects and make use of the method of characteristics (Sec. 2.2) to integrate the linearized Vlasov equation. Show that the amplitude of the perturbed distribution function $\delta f_e(k_z,\underset{\sim}{p})$ can be expressed as

$$\begin{aligned}
\delta f_e(k_z,\underset{\sim}{p}) = \frac{e\hat{n}_e}{\omega}\int_{-\infty}^{t} & dt' \exp\left[ik_z(z'-z) - i\omega(t'-t)\right] \\
& \times \left[v_x'(t')\delta E_x(k_z) + v_y'(t')\delta E_y(k_z)\right] \qquad (7.1.3) \\
& \times \left[\frac{\gamma m\omega}{p_\perp}\frac{\partial}{\partial p_z} + k_z\left(\frac{\partial}{\partial p_\perp} - \frac{p_z}{p_\perp}\frac{\partial}{\partial p_\perp}\right)\right] F_e(p_\perp^2, p_z) \; .
\end{aligned}$$

where $\underset{\sim}{v} = \underset{\sim}{p}/\gamma m_e$, $\gamma = (1+ p^2/m_e^2c^2)^{1/2}$

Linearized Vlasov Equation

Write $f_e^0(\underset{\sim}{p}) = \hat{n}_e F_e(p_\perp^2, p_z)$

$$\boxed{\begin{aligned}&\left\{\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z} - \frac{e}{c}\underset{\sim}{v}\times B_0\hat{\underset{\sim}{e}}_z\cdot\frac{\partial}{\partial \underset{\sim}{p}}\right\}\delta f_e(z,\underset{\sim}{p},t)\\ &\qquad = \hat{n}_e e\left(\delta\underset{\sim}{E} + \frac{1}{c}\underset{\sim}{v}\times\delta\underset{\sim}{B}\right)\cdot\frac{\partial}{\partial\underset{\sim}{p}}F_e(p_\perp^2,p_z)\end{aligned}}$$

- $$\frac{\partial}{\partial\underset{\sim}{p}}F_e(p_\perp^2,p_z) = \left(\underset{\sim}{p}_\perp\frac{1}{p_\perp}\frac{\partial}{\partial p_\perp} + \hat{\underset{\sim}{e}}_z\frac{\partial}{\partial p_z}\right)F_e(p_\perp^2,p_z)$$

where

$$\delta\underset{\sim}{E}\cdot\frac{\partial}{\partial\underset{\sim}{p}}F_e = \underset{\sim}{p}_\perp\cdot\delta\underset{\sim}{E}\,\frac{1}{p_\perp}\frac{\partial}{\partial p_\perp}F_e(p_\perp^2,p_z)$$

and

$$\underset{\sim}{p}_\perp\cdot\underset{\sim}{v}\times\delta\underset{\sim}{B} = v_z\,\underset{\sim}{p}_\perp\cdot\hat{\underset{\sim}{e}}_z\times\delta\underset{\sim}{B} = -\frac{ck_z}{\omega}v_z\,\underset{\sim}{p}_\perp\cdot\delta\underset{\sim}{E}$$

$$\hat{\underset{\sim}{e}}_z\cdot\underset{\sim}{v}\times\delta\underset{\sim}{B} = -\underset{\sim}{v}_\perp\cdot\hat{\underset{\sim}{e}}_z\times\delta\underset{\sim}{B} = \frac{ck_z}{\omega}\underset{\sim}{v}_\perp\cdot\delta\underset{\sim}{E}$$

Therefore

$$\boxed{\begin{aligned}&\hat{n}_e e\left(\delta\underset{\sim}{E} + \frac{1}{c}\underset{\sim}{v}\times\delta\underset{\sim}{B}\right)\cdot\frac{\partial}{\partial\underset{\sim}{p}}F_e(p_\perp^2,p_z)\\ &= \hat{n}_e e\left\{\delta\underset{\sim}{E}\cdot\underset{\sim}{p}_\perp\frac{1}{p_\perp}\frac{\partial}{\partial p_\perp} + \frac{1}{c}\underset{\sim}{v}_\perp\cdot(\hat{\underset{\sim}{e}}_z\times\delta\underset{\sim}{B})\left(\frac{p_z}{p_\perp}\frac{\partial}{\partial p_\perp} - \frac{\partial}{\partial p_z}\right)\right\}F_e(p_\perp^2,p_z)\end{aligned}}$$

$\underset{\sim}{E}^0(\underset{\sim}{x}) = 0$

$\underset{\sim}{B}^0(\underset{\sim}{x}) = B_0 \hat{\underset{\sim}{e}}_z$

Problem 7.1

The orbits are (relativistically)

where

$$\boxed{\begin{aligned} z'(t') &= z + \frac{p_z}{\gamma m_e}(t' - t) , \\ v'_x(t') &= \frac{p_\perp}{\gamma m_e} \cos\left[\phi + \frac{\omega_{ce}}{\gamma}\left(t' - t\right)\right] , \\ v'_y(t') &= \frac{p_\perp}{\gamma m_e} \sin\left[\phi + \frac{\omega_{ce}}{\gamma}\left(t' - t\right)\right] , \end{aligned}} \tag{7.1.4}$$

Here, $v'_x(t' = t) = v_x = (p_\perp/\gamma m_e)\cos\phi$, $v'_y(t' = t) = v_y = (p_\perp/\gamma m_e)\sin\phi$, $\gamma = (1 + p_\perp^2/m_e^2c^2 + p_z^2/m_e^2c^2)^{1/2}$, and ϕ is the phase angle of the perpendicular momentum $\underset{\sim}{p}_\perp$.

Problem 7.1

(c) Substitute Eqs.(7.1.3) and (7.1.4) into Eq.(7.1.2) and integrate over the phase angle ϕ with $\int d^3p \cdots = \int_0^{2\pi} d\phi \int_0^{\infty} dp_\perp p_\perp \int_{-\infty}^{\infty} dp_z \cdots$.

Show that

$$\boxed{D^T_\pm(k_z,\omega)\left[\delta E_x(k_z) \pm i\delta E_y(k_z)\right] = 0 \ .} \qquad (7.1.5)$$

Dispersion relation is

$$\int d^3p \cdots = 2\pi \int_0^{\infty} dp_\perp p_\perp \int_{-\infty}^{\infty} dp_z \cdots$$

$$\boxed{\begin{aligned} 0 = D^T_\pm(k_z,\omega) &= \omega^2 - c^2k_z^2 + \omega_{pe}^2 \int \frac{d^3p}{\gamma} \frac{p_\perp/2}{(\gamma\omega - k_z p_z/m_e \pm \omega_{ce})} \\ &\times \left[\left(\gamma\omega - \frac{k_z p_z}{m_e}\right)\frac{\partial}{\partial p_\perp} + \frac{k_z p_\perp}{m_e}\frac{\partial}{\partial p_z}\right] F_e(p_\perp^2, p_z) \ . \end{aligned}} \qquad (7.3)$$

The lower and upper signs in Eq.(7.3) correspond to waves with right-hand circular polarization $[\delta E_x(k_z) = \cancel{\mp} + i\delta E_y(k_z)]$ and left-hand circular polarization $[\delta E_x(k_z) = \cancel{\pm} - i\delta E_y(k_z)]$, respectively.

Cyclotron Maser Instability

$$\frac{\partial \gamma}{\partial p_\perp} = \frac{p_\perp}{\gamma m_e^2 c^2} \; ; \; \frac{\partial \gamma}{\partial p_z} = \frac{p_z}{\gamma m_e^2 c^2}$$

Expressing $\int d^3p \cdots = 2\pi \int_0^\infty dp_\perp p_\perp \int_{-\infty}^{\infty} dp_z \cdots$ in the dispersion relation (7.3) and integrating by parts with respect to $p_\perp$ and p_z gives

$$\boxed{0 = D_\pm^T(k_z,\omega) = \omega^2 - c^2k_z^2 - \omega_{pe}^2 \int \frac{d^3p}{\gamma} F_e(p_\perp^2,p_z) \times \left[\frac{(\gamma\omega - k_z p_z/m_e)}{(\gamma\omega - k_z p_z/m_e \pm \omega_{ce})} - \frac{p_\perp^2}{2m_e^2c^2}\frac{(\omega^2 - c^2k_z^2)}{(\gamma\omega - k_z p_z/m_e \pm \omega_{ce})^2}\right],} \quad (7.4)$$

Assigned

which is fully equivalent to Eq.(7.3).

If $F_e(p_\perp^2,p_z)$ is an isotropic function of $p_\perp^2 + p_z^2$, and decreases monotonically with $\partial F_e/\partial(p_\perp^2 + p_z^2) \le 0$, then the solutions to Eq.(7.4) [or Eq.(7.3)] correspond to stable oscillations. On the other hand, if $F_e(p_\perp^2,p_z)$ has the nonthermal feature corresponding to an inverted population in perpendicular momentum $p_\perp$, then there is a free energy source to drive the cyclotron maser instability.

Cyclotron Maser Instability

DR

$$0 = D_{\pm}^{T}(k_z,\omega) = \omega^2 - c^2k_z^2 - \omega_{pe}^2 \int \frac{d^3p}{\gamma} F_e(p_\perp^2, p_z)$$

$$\times \left[\frac{(\gamma\omega - k_z p_z/m_e)}{(\gamma\omega - k_z p_z/m_e \pm \omega_{ce})} - \frac{p_\perp^2}{2m_e^2c^2} \frac{(\omega^2 - c^2k_z^2)}{(\gamma\omega - k_z p_z/m_e \pm \omega_{ce})^2} \right]$$

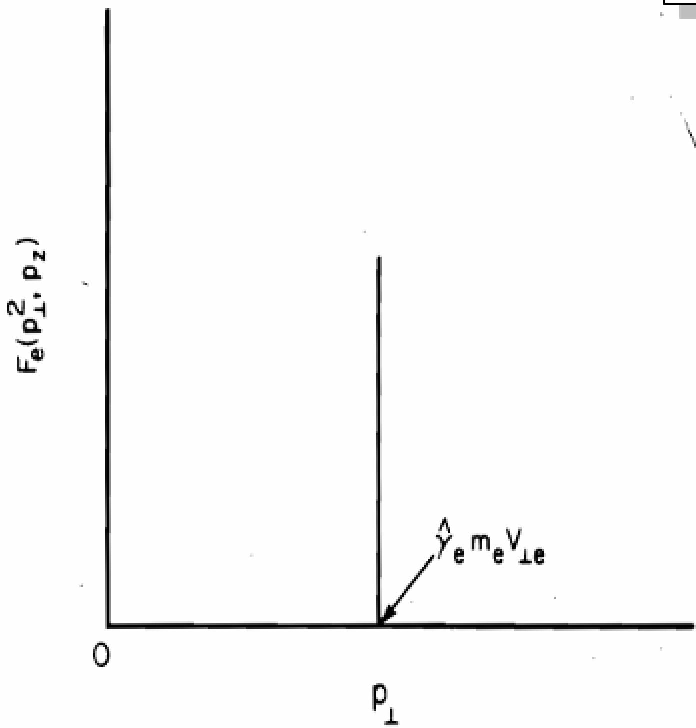
Cyclotron resonance

$$w - k_z V_z = \frac{\omega_{ce}}{\hat{\gamma}}$$

$\partial F_e/\partial(p_\perp^2 + p_z^2) \leq 0$ stable

Inverted population, unstable

Inverted population



$$\boxed{F_e(p_\perp^2, p_z) = \frac{1}{2\pi p_\perp}\,\delta\left(p_\perp - \hat{\gamma}_e m_e V_{\perp e}\right)\delta\left(p_z - \hat{\gamma}_e m_e V_{ze}\right)}$$

$$\hat{\gamma}_e = \left(1 - V_{\perp e}^2/c^2 - V_{ze}^2/c^2\right)^{-1/2}$$

$$\downarrow$$

$\boxed{\text{Cyclotron maser instability}}$

Equilibrium Distribution

$$F_e(p_\perp, p_z) = \frac{1}{2\pi p_\perp}\,\delta(p_\perp - \hat{\gamma}_e m_e V_{\perp e})\,\delta(p_z - \hat{\gamma}_e m_e V_{ze})$$

$$\int d^3p \cdots = \int_0^{2\pi} d\phi \int_0^{\infty} dp_\perp p_\perp \int_{-\infty}^{\infty} dp_z \cdots$$

Normalization

$$\int d^3p\, F_e = 1$$

Average (Directed) Motion

$$\int d^3p\, p_z F_e = \hat{\gamma}_e m_e V_{ze} \quad \text{(directed \underline{axial} motion)}$$

$$\left.\begin{aligned} \int d^3p\, p_x F_e &= \int d^3p\, p_\perp \cos\phi\, F_e = 0 \\ \int d^3p\, p_y F_e &= \int d^3p\, p_\perp \sin\phi\, F_e = 0 \end{aligned}\right\} \begin{array}{l}\underline{\text{zero}} \text{ directed} \\ \perp \text{ momentum}\end{array}$$

Average Perpendicular Speed

$$\int d^3p\, p_\perp F_e = \hat{\gamma}_e m_e V_{\perp e}$$

$$\int d^3p\, \frac{p_\perp}{\gamma m_e} F_e = V_{\perp e}$$

$$\int d^3p\, p_\perp^2 = (\hat{\gamma}_e m_e V_{\perp e})^2$$

$$F_e(p_\perp^2, p_z) = \frac{1}{2\pi p_\perp} \delta\left(p_\perp - \hat{\gamma}_e m_e V_{\perp e}\right) \delta\left(p_z - \hat{\gamma}_e m_e V_{ze}\right)$$

$$(\omega^2 - c^2 k_z^2)\left[1 + \frac{1}{2}\frac{V_{\perp e}^2}{c^2} \frac{\omega_{pe}^2/\hat{\gamma}_e}{(\omega - k_z V_{ze} - \omega_{ce}/\hat{\gamma}_e)^2}\right] = \frac{\omega_{pe}^2}{\hat{\gamma}_e}\left[\frac{\omega - k_z V_{ze}}{\omega - k_z V_{ze} - \omega_{ce}/\hat{\gamma}_e}\right].$$

$V_{\perp e} = 0$ $\qquad$ $V_{\perp e} \neq 0$

R wave

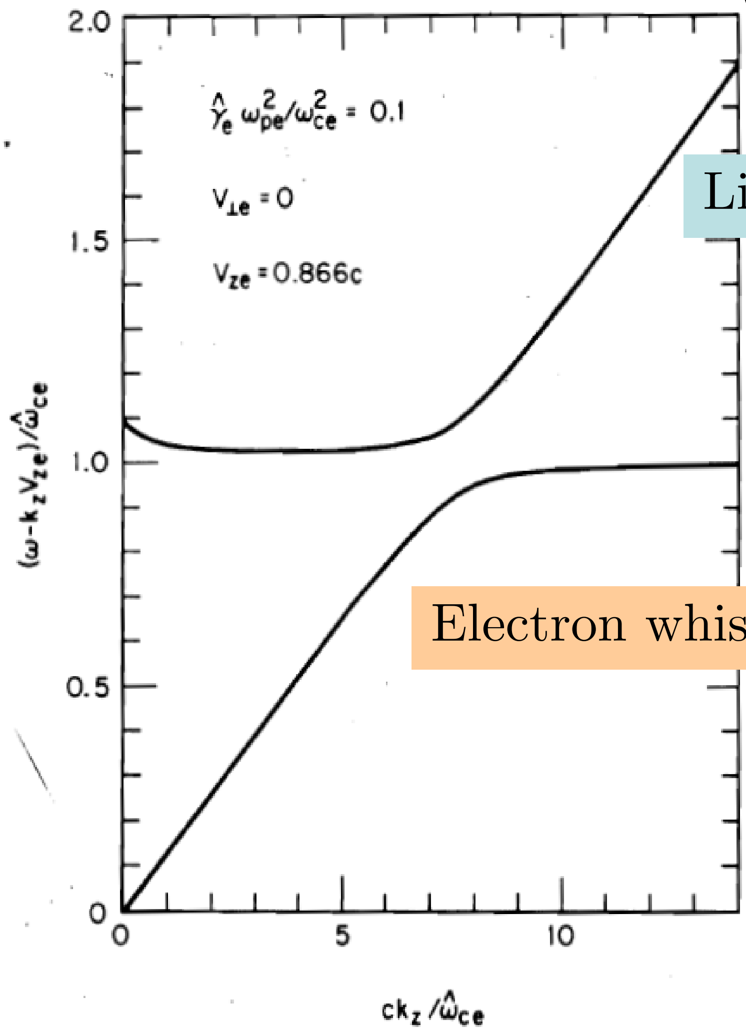
$$\left(\omega^2 - c^2 k_z^2 - \frac{\omega_{pe}^2}{\hat{\gamma}_e}\right)\left(\omega - k_z V_{ze} - \frac{\omega_{ce}}{\hat{\gamma}_e}\right) = \frac{\omega_{pe}^2}{\hat{\gamma}_e}\frac{\omega_{ce}}{\hat{\gamma}_e}$$

Cyclotron maser instability

$V_{\perp e} = 0$

R wave

$$\left(\omega^2 - c^2 k_z^2 - \frac{\omega_{pe}^2}{\hat{\gamma}_e}\right)\left(\omega - k_z V_{ze} - \frac{\omega_{ce}}{\hat{\gamma}_e}\right) = \frac{\omega_{pe}^2}{\hat{\gamma}_e}\,\frac{\omega_{ce}}{\hat{\gamma}_e}$$

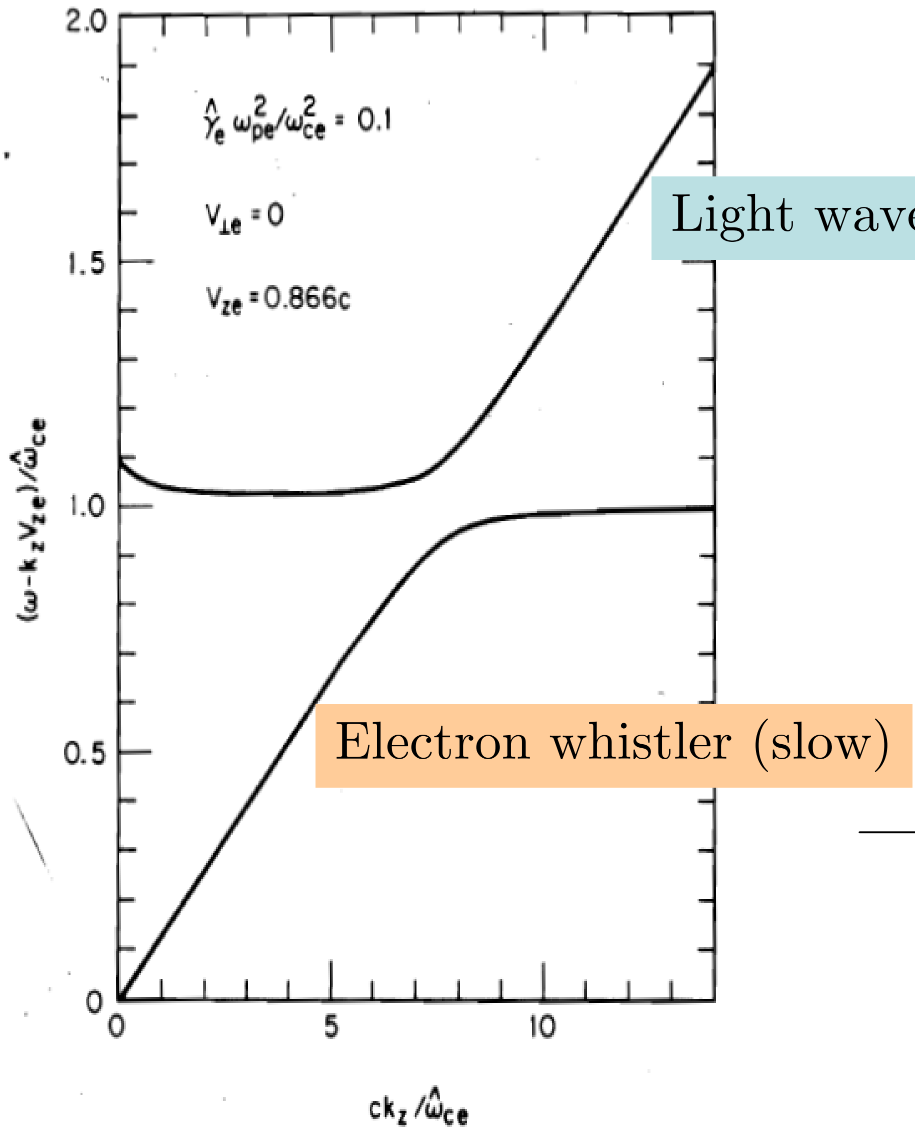


Light wave (fast)

Electron whistler (slow)

3rd branch?

$\boxed{V_{\perp e} \neq 0}$



for small k

→ Cyclotron maser instability

→ unstable for large k

Cyclotron Maser Instability

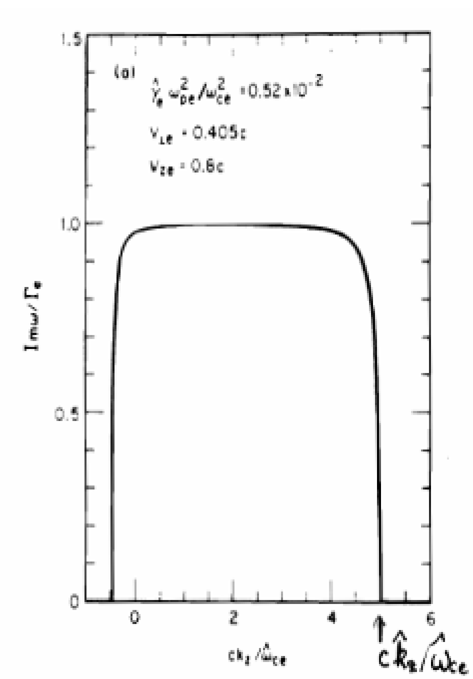
$$(\omega^2 - c^2k_z^2)\left[1 + \frac{1}{2}\frac{v_{\perp e}^2}{c^2}\frac{\omega_{pe}^2/\hat{\gamma}_e}{(\omega - k_zV_{ze} - \omega_{ce}/\hat{\gamma}_e)^2}\right] = \frac{\omega_{pe}^2}{\hat{\gamma}_e}\left[\frac{\omega - k_zV_{ze}}{\omega - k_zV_{ze} - \omega_{ce}/\hat{\gamma}_e}\right] .$$

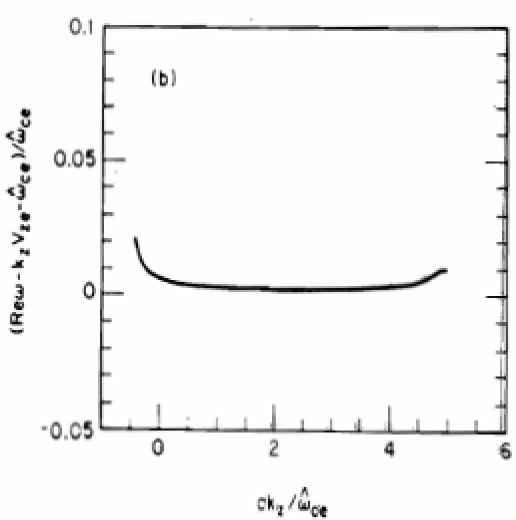
for small k

$$\left(\omega - k_zV_{ze} - \omega_{ce}/\hat{\gamma}_e\right)^2 = -\frac{1}{2}\frac{v_{\perp e}^2}{c^2}\frac{\omega_{pe}^2}{\hat{\gamma}_e}$$

$$\mathrm{Re}\,\omega = k_zV_{ze} + \omega_{ce}/\hat{\gamma}_e ,$$

$$\mathrm{Im}\,\omega = \left(\frac{1}{2}\frac{v_{\perp e}^2}{c^2}\frac{\omega_{pe}^2}{\hat{\gamma}_e}\right)^{1/2} = \Gamma_e$$



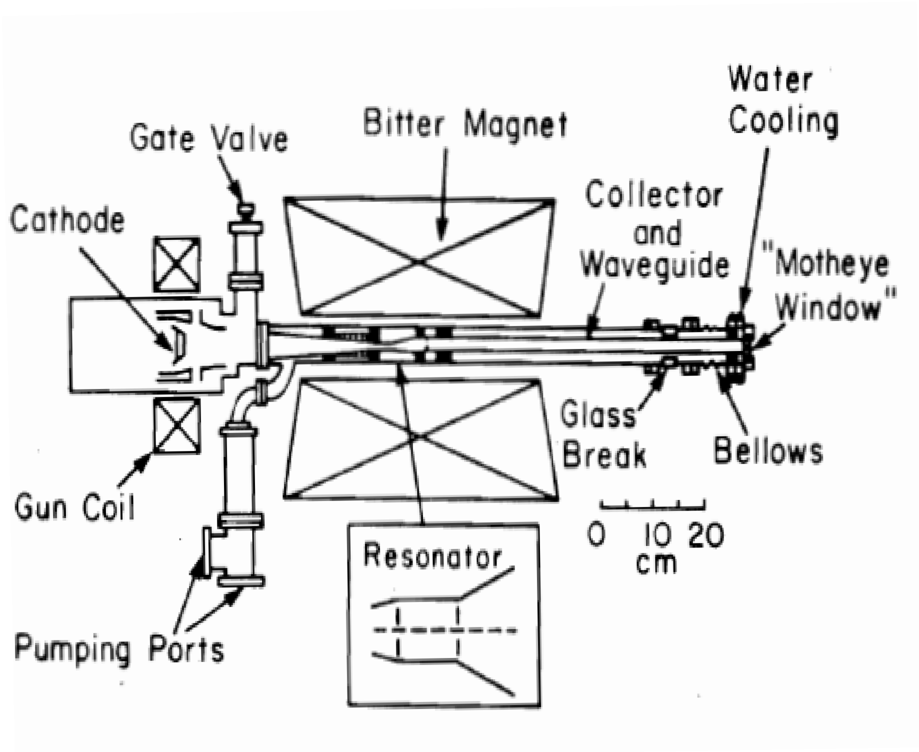


Cyclotron Maser Instability (Summary)

- Unstable fast wave for long wavelength
- Self-field effect (7.2.2)
- Finite radial geometry (7.3) ✓
- Damping by momentum spread

High-Power Gyrotron Experiments (Sec. 7.4)

Kreischer & Temkin (PRL, 1987)



Parameters

$E_b \sim 80\,keV$	$4\,Hz$ rep rate
$I_b \sim 35\,A$	$\tau_p \sim 3\,\mu s$
$V_{\perp e}/V_{ze} = 1.93$	$B_0 \sim 48-97\,kG$
$\Delta V_{\perp e}/V_{\perp e} \sim 4\%$	$\left.\begin{array}{l} L = 1.28\,cm \\ b = 0.75\,cm \end{array}\right\}$ resonator /cavity

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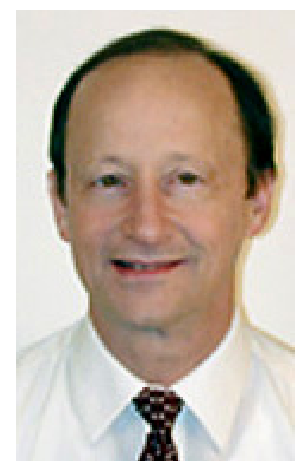
$$\delta E_\theta \sim J_\ell'(k_{\ell n} r) \text{ where } J_\ell'(k_{\ell n} b) = 0$$

$TE_{\ell n}$ excitations

P_μ up to $0.5\,MW$

f up to $243\,GHz$

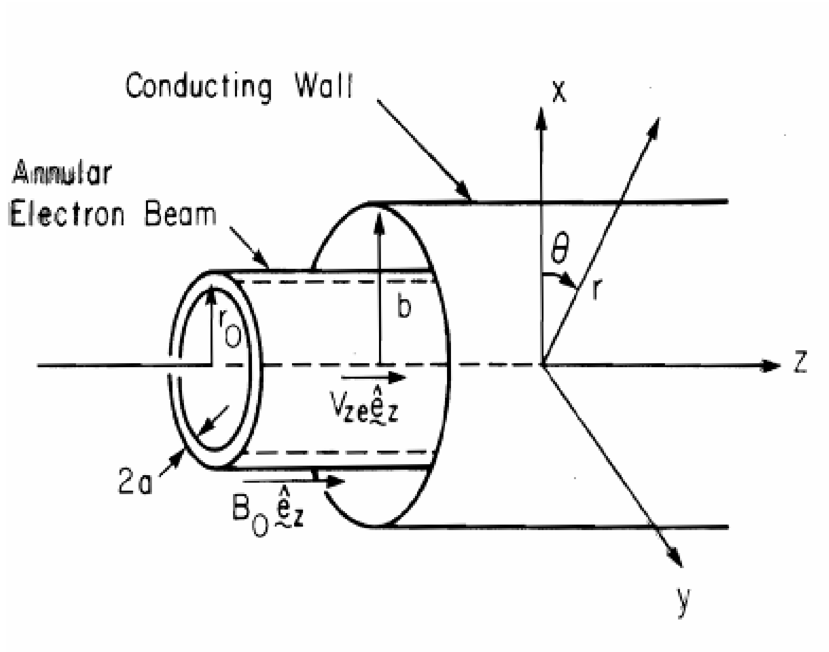
PSFC, MIT, 1.5MW by 2006



Question 1: where does the wave energy come from?

Question 2: how is the beam spinned?

Cyclotron maser instability for annular beam (Sec. 7.3)



To be analytically tractable, assume low intensity:

$$\boxed{\frac{\nu}{\hat{\gamma}_e} = \frac{N_e}{\hat{\gamma}_e}\frac{e^2}{m_e c^2} \ll 1}$$

$$f_e^0(r,\underset{\sim}{p}) = \frac{N_e\omega_{ce}}{4\pi^2\hat{\gamma}_e m_e c^2}\,\delta\left(\gamma - \hat{\gamma}_e\right)\delta\left(p_z - \hat{\gamma}_e m_e V_{ze}\right)\delta\left(P_\theta - P_0\right)$$

Integrals of single particle dynamics:

$$H = \left(m_e^2c^4 + c^2\underset{\sim}{p}^2\right)^{1/2} - m_ec^2 = (\gamma - 1)m_ec^2$$

$$P_\theta = r\left(p_\theta - \frac{e}{2c}B_0 r\right),$$

$$p_z = \gamma m_e v_z .$$

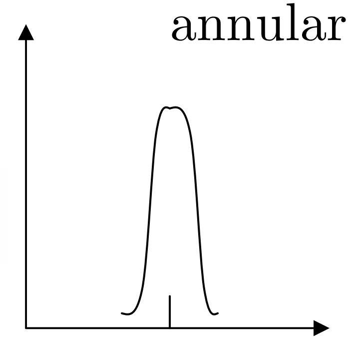
$$\hat{\gamma}_e = \left(1 - V_{\perp e}^2/c^2 - V_{ze}^2/c^2\right)^{-1/2}$$

$$P_0 = -\frac{1}{2}m_e\omega_{ce}\left(r_0^2 - \overbrace{\hat{\gamma}_e^2V_{\perp e}^2/\omega_{ce}^2}^{a^2}\right)$$

Why r ?

$$\hat{\gamma}_e V_{\perp e}/\omega_{ce} \ll r_0 \longrightarrow n_e^0(r) = \int d^3p\, f_e^0(r,\underset{\sim}{p})$$

annular

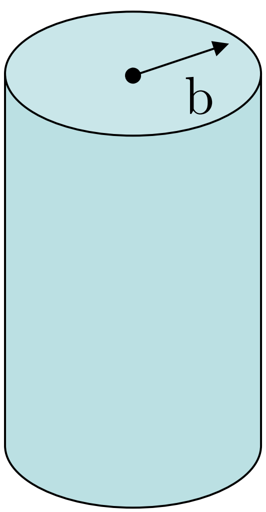


Difficulty: inhomogenous in r

Assume $\frac{\nu}{\hat{\gamma}_e} = \frac{\bar{\omega}_{pe}^2 a r_0}{\hat{\gamma}_e c^2} << 1$

look at TE_{0n} vacuum mode

$l \sim \partial/\partial\theta = 0$



$$\delta \underset{\approx}{E}(\underset{\sim}{x}, t) = \delta E_\theta(r,z,t)\hat{\underset{\sim}{e}}_\theta ,$$

$$\delta \underset{\approx}{B}(\underset{\sim}{x}, t) = \delta B_r(r,z,t)\hat{\underset{\sim}{e}}_r + \delta B_z(r,z,t)\hat{\underset{\sim}{e}}_z$$

$$\delta E_\theta(r,k_z) = a(k_z)J_1(k_{0n}r)$$ $\sim J_0'(k_{0n}r)$

DR

$$\omega^2 = c^2k_z^2 + c^2k_{0n}^2$$

k_{0n} is the n'th zero of $J_1(k_{0n}b) = 0$

No. 1 trick of physics

Treat beam as a perturbation

DR

$$\omega^2 - c^2k_z^2 - c^2k_{0n}^2 = \frac{4\nu c^2}{\hat{\gamma}_e[bJ_2(k_{0n}b)]^2}\left\{Q_1 \frac{(\omega - k_zV_{ze})}{(\omega - k_zV_{ze} - \omega_{ce}/\hat{\gamma}_e)} - H_1 \frac{v_{\perp e}^2}{c^2} \frac{(\omega^2 - c^2k_z^2)}{(\omega - k_zV_{ze} - \omega_{ce}/\hat{\gamma}_e)^2}\right\}.$$

Finte geometry

peaks when r_0 is at the maximuam of TE_{0n}

$$H_1 = \left[J_1(k_{0n}r_0)J_1'(k_{0n}a)\right]^2$$

$$Q_1 = 2H_1 + 2(k_{0n}a)\left[J_1(k_{0n}r_0)\right]^2 J_1'(k_{0n}a)J_1''(k_{0n}a)$$

$$\omega^2 - c^2k_z^2 - c^2k_{0n}^2 = \frac{4\nu c^2}{\hat{\gamma}_e[bJ_2(k_{0n}b)]^2}\left\{Q_1 \frac{(\omega - k_zV_{ze})}{(\omega - k_zV_{ze} - \omega_{ce}/\hat{\gamma}_e)} - H_1 \frac{v_{\perp e}^2}{c^2}\frac{(\omega^2 - c^2k_z^2)}{(\omega - k_zV_{ze} - \omega_{ce}/\hat{\gamma}_e)^2}\right\}.$$

DR

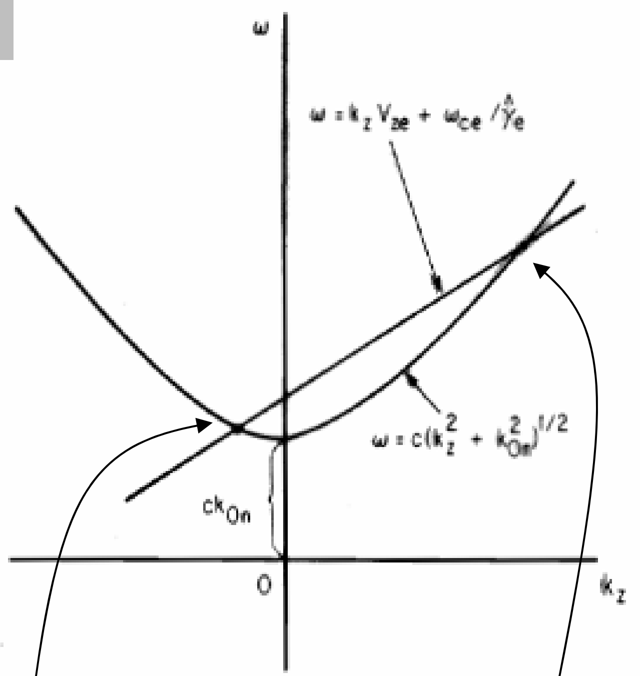
0th order: $\omega^2 - c^2k_z^2 - c^2k_{0n}^2 = 0$

1st order:

Light wave

whistler

$$\left(\omega^2 - c^2k_z^2 - c^2k_{0n}^2\right)\left(\omega - k_zV_{ze} - \omega_{ce}/\hat{\gamma}_e\right)^2 = -\frac{4\nu c^2H_1c^2k_{0n}^2}{\hat{\gamma}_e[bJ_2(k_{0n}b)]^2}\frac{v_{\perp e}^2}{c^2},$$



$$ck_{z0} = \gamma_z^2\beta_{ze}\frac{\omega_{ce}}{\hat{\gamma}_e} \pm \gamma_z\left(\gamma_z^2\frac{\omega_{ce}^2}{\hat{\gamma}_e^2} - c^2k_{0n}^2\right)^{1/2}$$

$$\text{Maximum interaction when } \gamma_z \frac{\omega_{ce}}{\hat{\gamma}_e} = ck_{0n}$$

$$\omega_0 = \gamma_z^2 \omega_{ce}/\hat{\gamma}_e ,$$

$$ck_{z0} = \gamma_z^2 \beta_{ze} \omega_{ce}/\hat{\gamma}_e .$$

$$\omega = \omega_0 + \delta\omega$$

$$k_z = k_{z0} + \delta k_z$$

$$\left(\frac{\delta\omega}{\hat{\omega}_{ce}} - \frac{V_{ze}}{c} \frac{c\delta k_z}{\hat{\omega}_{ce}} \right)^3 = -2 \frac{\nu}{\hat{\gamma}_e} \frac{V_{\perp e}^2}{c^2} \gamma_z^2 g_1$$

$$g_1 = \frac{H_1}{[k_{0n} b J_2(k_{0n} b)]^2}$$

$$\mathrm{Im}\,\delta\omega = \frac{\sqrt{3}}{2} \left(2 \frac{\nu}{\hat{\gamma}_e} \frac{V_{\perp e}^2}{c^2} \gamma_z^2 g_1 \right)^{1/3} \frac{\omega_{ce}}{\hat{\gamma}_e} ,$$

$$\mathrm{Re}\,\delta\omega = V_{ze}\delta k_z + \frac{1}{2} \left(2 \frac{\nu}{\hat{\gamma}_e} \frac{V_{\perp e}^2}{c^2} \gamma_z^2 g_1 \right)^{1/3} \frac{\omega_{ce}}{\hat{\gamma}_e}$$

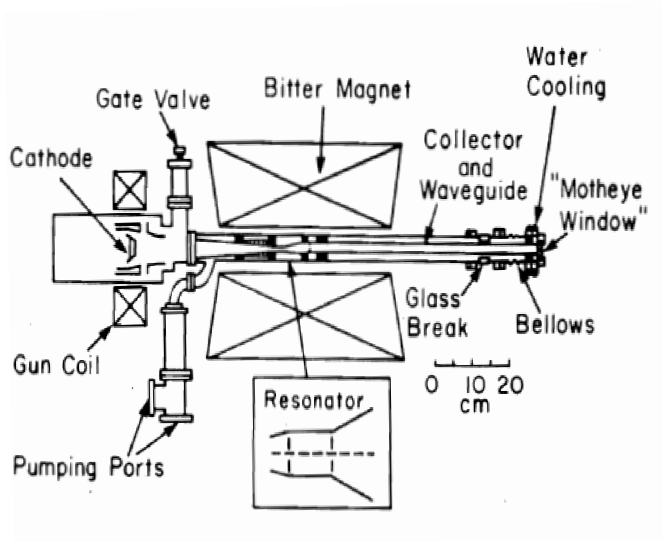
Stabilization when δk_z is large

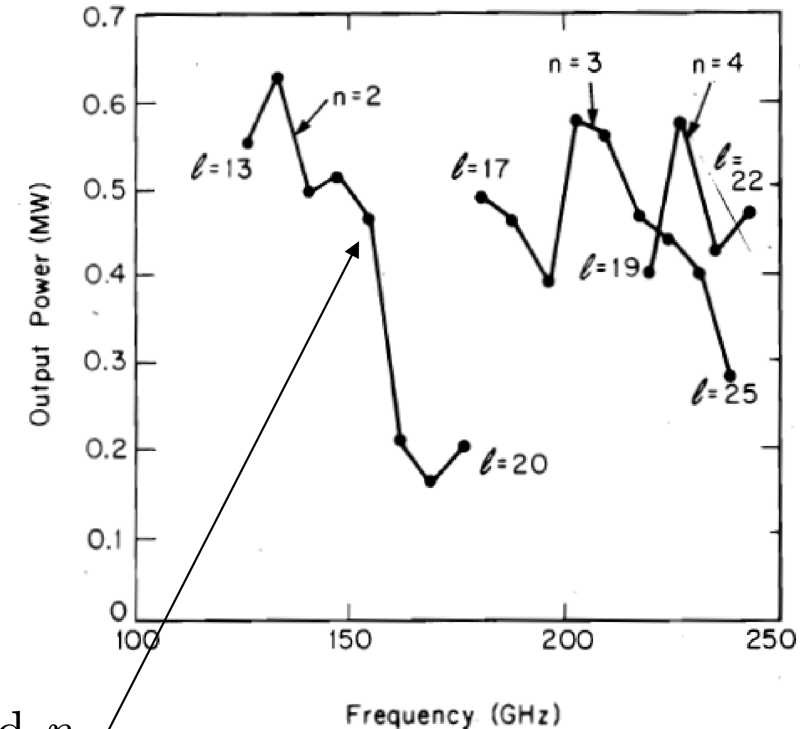
DR

$$\omega^2 - c^2k_z^2 - c^2k_{0n}^2 = \frac{4\nu c^2}{\hat{\gamma}_e[bJ_2(k_{0n}b)]^2}\left\{Q_1 \frac{(\omega - k_zV_{ze})}{(\omega - k_zV_{ze} - \omega_{ce}/\hat{\gamma}_e)} - H_1 \frac{V_{\perp e}^2}{c^2}\frac{(\omega^2 - c^2k_z^2)}{(\omega - k_zV_{ze} - \omega_{ce}/\hat{\gamma}_e)^2}\right\} .$$

$$c^2(\delta k_z)^2_{cr} = 2\gamma_z^4 \frac{\omega_{ce}^2}{\hat{\gamma}_e^2}\left(\frac{27}{2}\frac{\nu}{\hat{\gamma}_e}\frac{V_{\perp e}^2}{c^2}\gamma_z^2 g_1\right)^{1/3}$$

for $\mathrm{Im}\,\omega = 0$





select l and n

$$\boxed{\begin{aligned} \omega_0^2 &= c^2k_{z0}^2 + c^2k_{ln}^2 \approx c^2k_{ln}^2 \\ \omega_0 &= k_{z0}V_{ze} + \omega_{ce} / \hat{\gamma}_e \approx \omega_{ce} / \hat{\gamma}_e \end{aligned}}$$

$$f_{ce} \approx 2.8 \times 10^9 \frac{B_0(kG)}{\hat{\gamma}_e}$$