AST-565 Homework Set 10

Problem 1: Envelope equations for periodically-focused beam at low intensity (Davidson & Qin, Problem 5.8)

Consider the low intensity kinetic beam equilibria in a periodic quadrupole focusing field described by the Vlasov equation

$$\frac{\partial f_b}{\partial s} + x' \frac{\partial f_b}{\partial x} + y' \frac{\partial f_b}{\partial y} - \kappa_q(s) x \frac{\partial f_b}{\partial x'} + \kappa_q(s) y \frac{\partial f_b}{\partial y'} = 0 \,.$$

(a) By taking the moment of the Vlasov equation, derive the following dynamical equations for the evolution of $\langle x^2 \rangle$, $\langle x x' \rangle$ and $\langle x'^2 \rangle$

$$\frac{d}{ds}\langle x^2 \rangle - 2\langle x x' \rangle = 0 \,,$$

$$\frac{d}{ds}\langle x x' \rangle - \langle x'^2 \rangle + \kappa_q(s)\langle x^2 \rangle = 0 \,,$$

$$\frac{d}{ds}\langle x'^2 \rangle + 2\kappa_q(s)\langle x x' \rangle = 0 \,,$$

where $\langle \chi \rangle(s) \equiv \dfrac{1}{N_b} \displaystyle\int dx dy dx' dy' \chi f_b$.

(b) Make use of the results in (a) and the definition of emittance $\varepsilon_x^2(s) \equiv 4[\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2]$ and the definition of the RMS beam radius $\langle x^2 \rangle = a_b^2(s)$ to show that

$$\frac{d^2}{ds^2} a_b(s) + \kappa_q(s) a_b(s) = \frac{\varepsilon_x^2}{4 a_b^3(s)} \,.$$

(c) Derive the following dynamic equation for emittance,

$$\frac{1}{4}\frac{d}{ds}\varepsilon_x^2(s) = \langle x^2 \rangle \frac{d}{ds}\langle x'^2 \rangle + \langle x'^2 \rangle \frac{d}{ds}\langle x^2 \rangle - 2\langle x x' \rangle \frac{d}{ds}\langle x x' \rangle \,,$$

and prove that that $(d \,/\, ds)\varepsilon_x^2(s) = 0$.

Problem 2: Global force balance for a thermal equilibrium beam in a smooth focusing field (Davidson & Qin, Problem 5.5)

For a thermal equilibrium beam in a smooth focusing field, consider the radial force balance equation

$$\hat{T}_b \frac{\partial n_b}{\partial r} = \frac{e_b}{\gamma_b^2} n_b E_r^s - \gamma_b m_b \omega_{\beta\perp}^2 n_b r \ ,$$

where the radial self-electric field is given in terms of the density profile $n_b(r)$ by

$$E_r^s(r) = \frac{4\pi e_b}{r} \int_0^r dr r n_b(r) \ .$$

Show that the radial force balance equation gives the following global force balance:

$$\gamma_b m_b \omega_{\beta\perp}^2 R_b^2 = \frac{N_b e_b^2}{\gamma_b^2} + 2\hat{T}_b \ ,$$

where $N_b = 2\pi \int_0^\infty dr r n_b(r)$ and $R_b^2 = N_b^{-1} 2\pi \int_0^\infty dr r r^2 n_b(r)$.