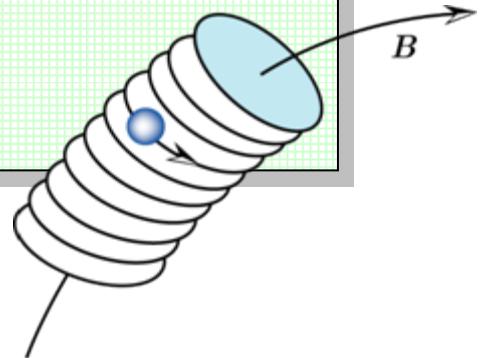


# Variational Symplectic Integrator of the Guiding Center Motion



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[www.pppl.gov/~hongqin/Gyrokinetics.php](http://www.pppl.gov/~hongqin/Gyrokinetics.php)



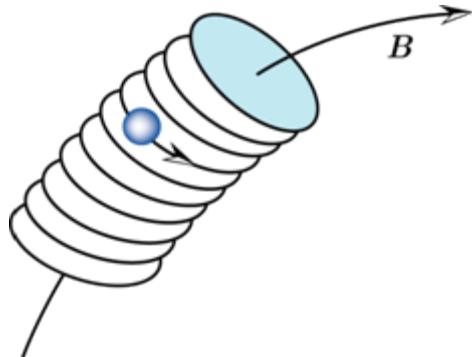
# Gyrocenter dynamics and algorithms

$$\frac{d\mathbf{X}}{dt} = u\mathbf{b} + \frac{\mu \nabla B \times \mathbf{b}}{B} + \frac{\mathbf{E} \times \mathbf{b}}{B} + \frac{u^2 \mathbf{b} \times (\mathbf{b} \cdot \nabla b)}{B}$$
$$\frac{du}{dt} = -\frac{\mu \mathbf{b} \cdot \nabla B}{B} + \mathbf{b} \cdot \mathbf{E}$$



Carl Runge  
(1856-1927)

Martin Kutta  
(1867-1944)

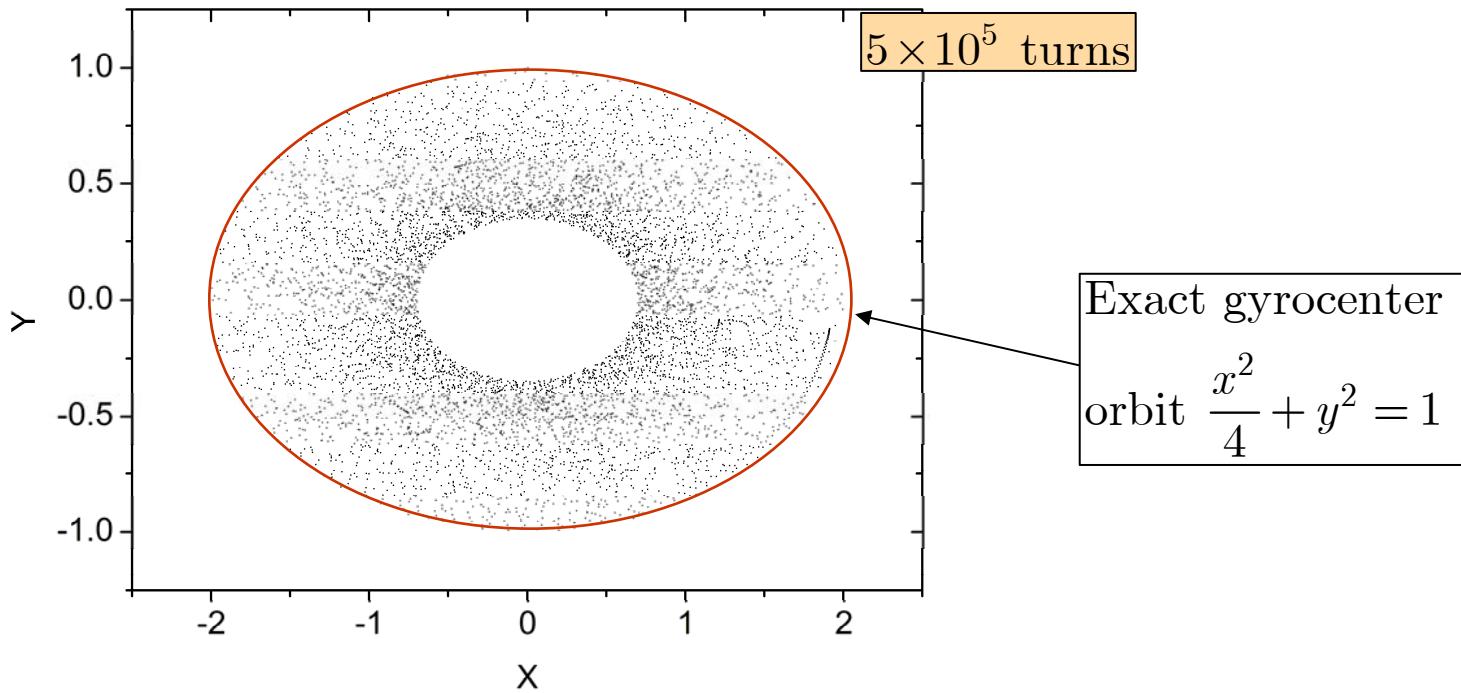
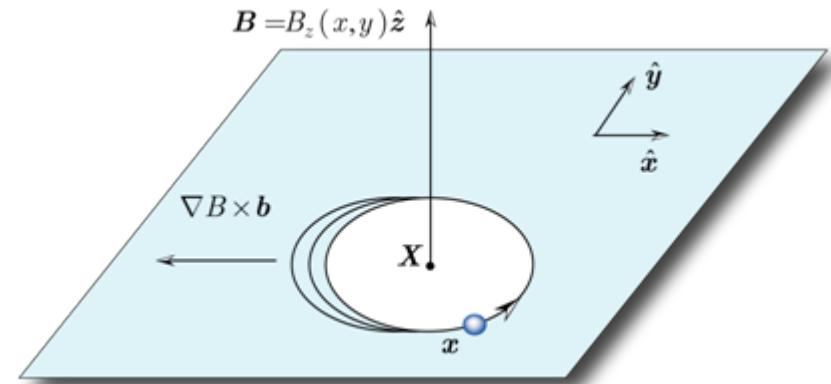


nth (4<sup>th</sup>) order Runge-Kutta methods

- Long time non-conservation.
- Errors add up coherently.

## Example – gradient drift

$$B_z(x, y) = 1 + 0.05 \left( \frac{x^2}{4} + y^2 \right)$$



# Can we do better than RK4? -- symplectic integrator



Feng (1983)



Ruth (1984)

Conserves symplectic structure;  
Bound energy error globally

Symplectic integrator ?

Requires canonical  
Hamiltonian structure

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} H_{,q} \\ H_{,p} \end{pmatrix}$$

Application areas:

- Accelerator physics (everybody)
- Planetary dynamics (S. Tremaine)
- Nonlinear dynamics (everybody)

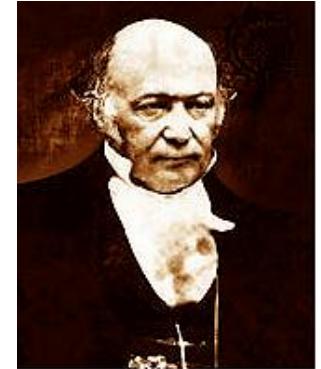
## What is the canonical structure for gyrocenter dynamics?



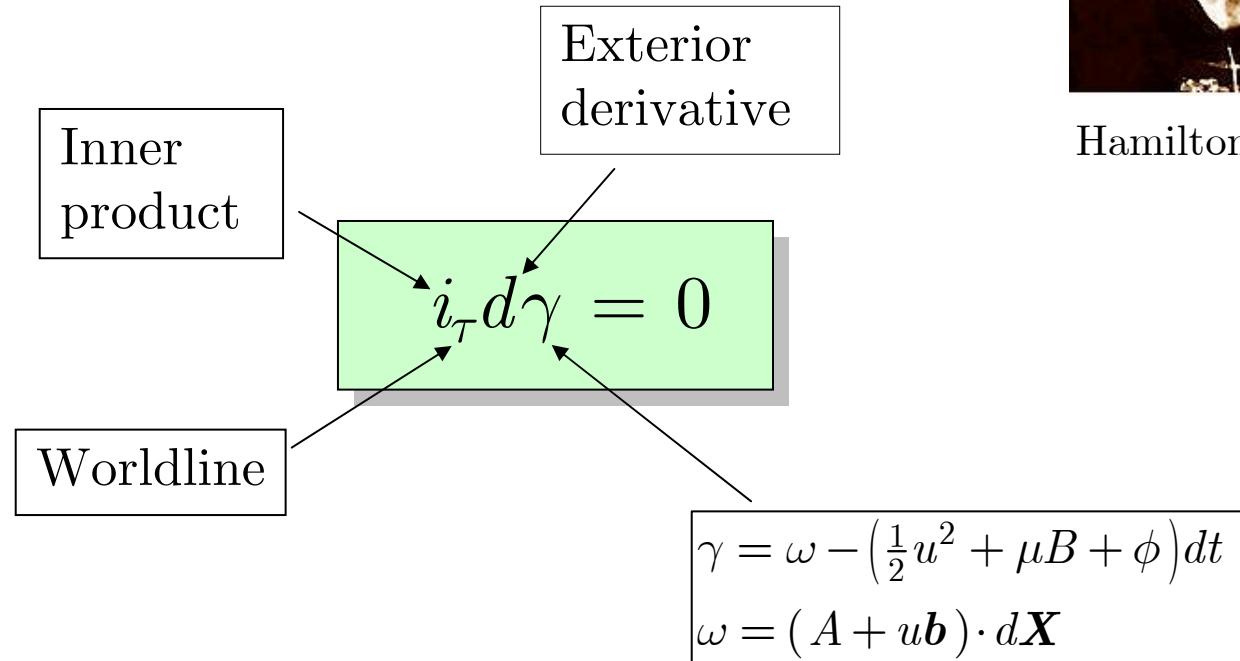
R. White

# Gyrocenter dynamics does not have a global canonical structure

## Geometry of gyrocenter

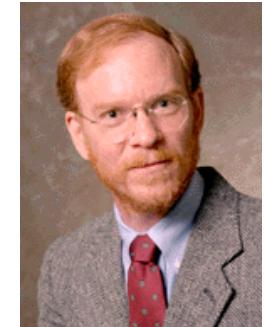


Hamilton



Only has a non-canonical  
symplectic structure  $\Omega \equiv d\omega$

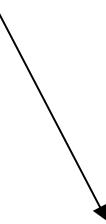
# Gyrocenter dynamics



$$i_\tau d\gamma = 0$$

$\approx$

$$\begin{aligned} A &= \int_0^{t_1} L dt \\ L &= (\mathbf{A} + u\mathbf{b}) \cdot \dot{\mathbf{X}} - \left( \frac{1}{2} u^2 + \mu B + \phi \right) \end{aligned}$$



$$\frac{d\mathbf{X}}{dt} = \frac{\mathbf{B}^\dagger}{B_\parallel^\dagger} \left( u + \frac{\mu}{2} \mathbf{b} \cdot \nabla \times \mathbf{b} \right) - \frac{\mathbf{b} \times \mathbf{E}^\dagger}{B_\parallel^\dagger}$$

$$\frac{du}{dt} = \frac{\mathbf{B}^\dagger \cdot \mathbf{E}^\dagger}{B_\parallel^\dagger}$$

$$\mathbf{B}^\dagger \equiv \nabla \times \mathbf{A}^\dagger, \quad \mathbf{A}^\dagger \equiv \mathbf{A} + u\mathbf{b}$$

$$B_\parallel^\dagger = \mathbf{B}^\dagger \cdot \mathbf{b}, \quad \mathbf{E}^\dagger \equiv \mathbf{E} - \mu \nabla B$$

# Darboux Theorem



Jean Gaston Darboux  
(1842-1917)

Darboux Theorem (1882):

Every symplectic structure is **locally** canonical.



Gyrocenter dynamics can be canonical **locally**.



No symplectic integrator for gyrocenter?

# Variational symplectic integrator



$$i_{\tau} d\gamma = 0$$

$\approx$

$$\begin{aligned} A &= \int_0^{t_1} L dt \\ L &= (\mathbf{A} + u\mathbf{b}) \cdot \dot{\mathbf{X}} - \left( \frac{1}{2} u^2 + \mu B + \phi \right) \end{aligned}$$

discretize on

$$t = [0, h, 2h, \dots, (N-1)h]$$



$$\begin{aligned} A &\approx A_d = \sum_{k=0}^{N-1} h L_d(k, k+1) \\ L_d(k, k+1) &\equiv L_d(\mathbf{x}_k, u_k, \mathbf{x}_{k+1}, u_{k+1}) \end{aligned}$$



minimize w.r.t.  $(\mathbf{x}_k, u_k)$

$$\begin{aligned} &[(\mathbf{x}_{k-1}, u_{k-1}), (\mathbf{x}_k, u_k)] \\ \rightarrow &(\mathbf{x}_{k+1}, u_{k+1}) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x_k^j} [L_d(k-1, k) + L_d(k, k+1)] &= 0, (j = 1, 2, 3) \\ \frac{\partial}{\partial u_k} [L_d(k-1, k) + L_d(k, k+1)] &= 0 \end{aligned}$$

discretized Euler-Lagrangian Eq.

J. Marsden  
(2001)

# Conserved symplectic structure

$$\theta^+(k, k+1) \equiv \frac{\partial}{\partial \mathbf{x}_{k+1}} L_d(k, k+1) \cdot d\mathbf{x}_{k+1} + \frac{\partial}{\partial u_{k+1}} L_d(k, k+1) du_{k+1}$$

$$\theta^-(k, k+1) \equiv -\frac{\partial}{\partial \mathbf{x}_k} L_d(k, k+1) \cdot d\mathbf{x}_k - \frac{\partial}{\partial u_k} L_d(k, k+1) du_k$$



$$dL_d(k, k+1) = \theta^+(k, k+1) - \theta^-(k, k+1)$$

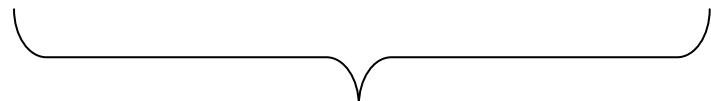
minimize w.r.t.  $(\mathbf{x}_k, u_k)$



$$\Omega_d(k, k+1) \equiv d\theta^+ = d\theta^-$$



$$dA_d = \theta^+(0, 1) - \theta^-(N-1, N)$$



$$\Omega_d(0, 1) = \Omega_d(N-1, N)$$

# 1<sup>st</sup> order variational symplectic integrator

$$L_d(k, k+1) \equiv \frac{[\mathbf{A}^\dagger(k+1) + \mathbf{A}^\dagger(k)]}{2} \cdot \frac{[\mathbf{x}_{k+1} - \mathbf{x}_k]}{h} - \frac{u_k u_{k+1}}{2} - \mu B(k) - \varphi(k)$$



$$\frac{1}{2h} A_{,j}^{\dagger i}(k) (x_{k+1}^i - x_{k-1}^i) - \frac{1}{2h} [A^{\dagger j}(k+1) - A^{\dagger j}(k-1)] = \mu B_{,j}(k) + \varphi_{,j}(k)$$

$$\frac{1}{2h} b^i(k) (x_{k+1}^i - x_{k-1}^i) = \frac{u_{k+1} + u_{k-1}}{2}$$

$[(\mathbf{x}_{k-1}, u_{k-1}), (\mathbf{x}_k, u_k)]$

$\rightarrow (\mathbf{x}_{k+1}, u_{k+1})$

implicit

## Semi-explicit Newton's method

$$[A^{\dagger j}(k+1) - A^{\dagger j}(k-1)] \approx A_{,i}^{\dagger j}(k)(x_{k+1}^i - x_{k-1}^i) + b^j(k)(u_{k+1} - u_{k-1})$$



$$\begin{aligned} & \frac{1}{2h} [A_{,j}^{\dagger i}(k) - A_{,i}^{\dagger j}(k)] (x_{k+1}^i - x_{k-1}^i) - \frac{b^j(k)}{2h} \left[ 2u_{k-1} - \frac{b^i(k)}{h} (x_{k+1}^i - x_{k-1}^i) \right] \\ &= \mu B_{,j}(k) + \varphi_{,j}(k) \end{aligned}$$

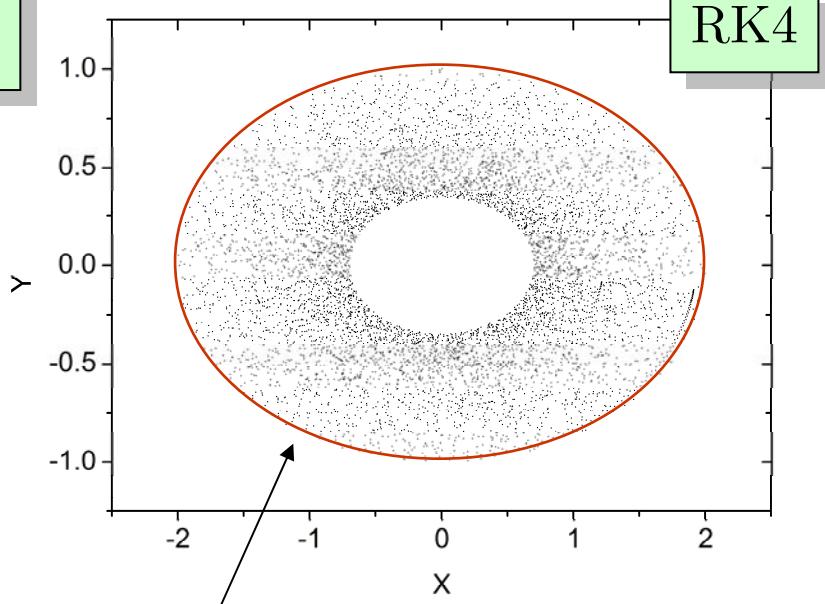
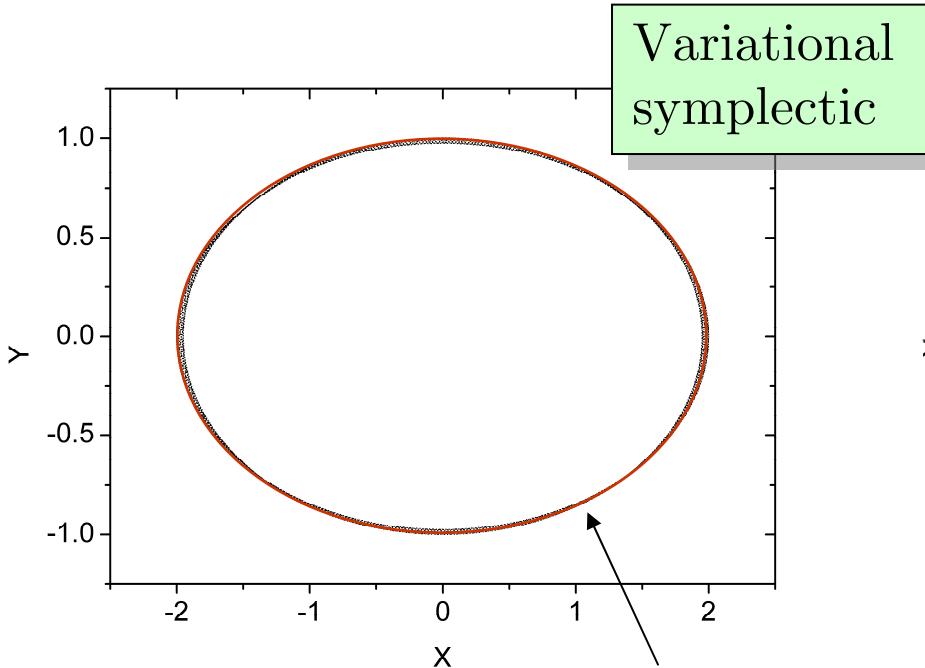
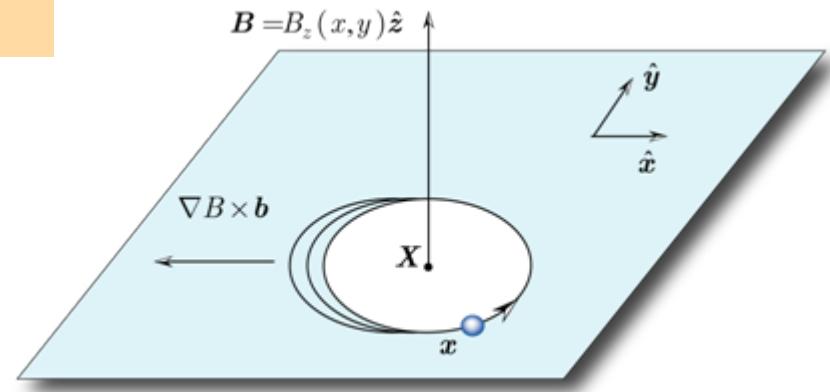
$$\frac{1}{2h} b^i(k) (x_{k+1}^i - x_{k-1}^i) = \frac{u_{k+1} + u_{k-1}}{2}$$

explict, initial guess for  
the Newton's method

## Example – gradient drift

$$\mathbf{B} = B(x, y) \hat{z}$$

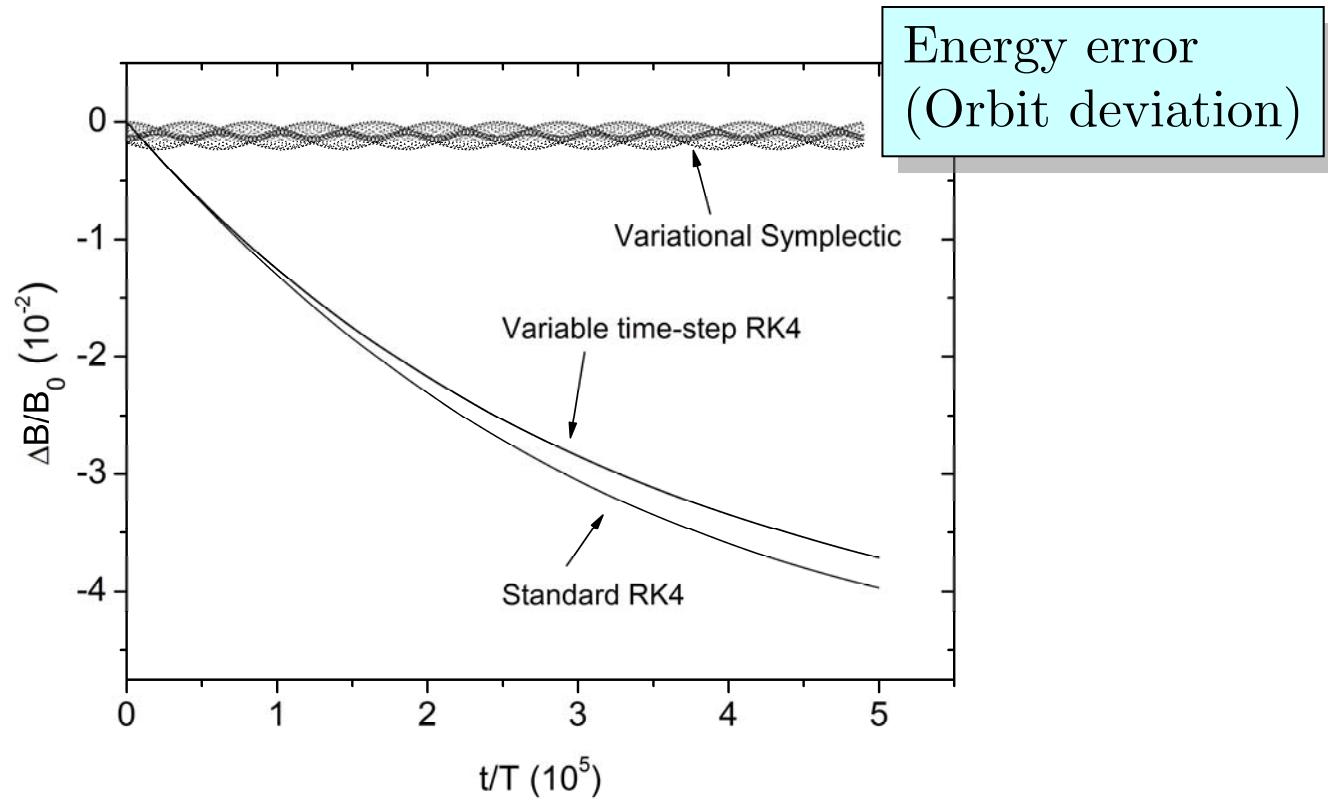
$$B(x, y) = 1 + 0.05 \left( \frac{x^2}{4} + y^2 \right)$$



Exact gyrocenter orbit  $\frac{x^2}{4} + y^2 = 1$

$5 \times 10^5$  turns

# Variational symplectic globally bounds energy error



## Conclusions

Physics is geometry. So is algorithm.

Symplectic bounds error **globally**, others do not.

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"Can Billy come out and compete in  
the global economy?"