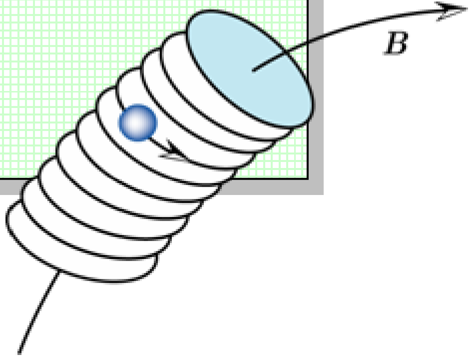
Variational Symplectic Integrator of the Guiding Center Motion



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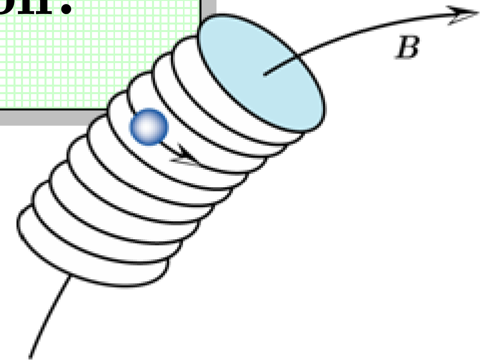
2008 Sherwood Meeting
April 1, 2008

www.pppl.gov/~hongqin/Gyrokinetics.php

How to Calculate Guiding Center Motion?



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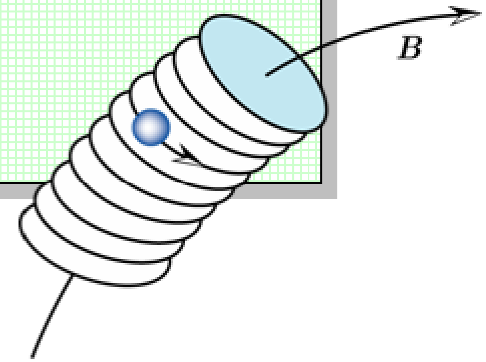
You Don't Know How to Calculate the Guiding Center Motion !



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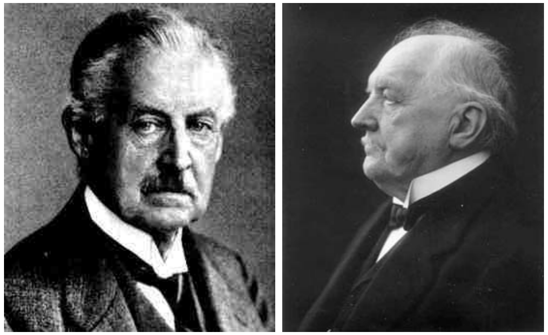
Gyrocenter dynamics and algorithms

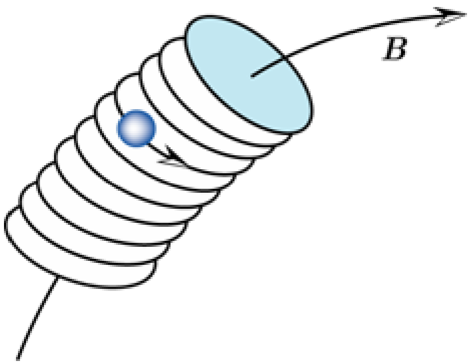
$$\frac{d\boldsymbol{X}}{dt} = u\boldsymbol{b} + \frac{\mu\,\boldsymbol{\nabla} B \times \boldsymbol{b}}{B} + \frac{\boldsymbol{E}\times\boldsymbol{b}}{B} + \frac{u^2\boldsymbol{b}\times(\boldsymbol{b}\cdot\boldsymbol{\nabla} b)}{B}$$

$$\frac{du}{dt} = -\frac{\mu\boldsymbol{b}\cdot\boldsymbol{\nabla} B}{B} + \boldsymbol{b}\cdot\boldsymbol{E}$$



Carl Runge
(1856-1927)

Martin Kutta
(1867-1944)

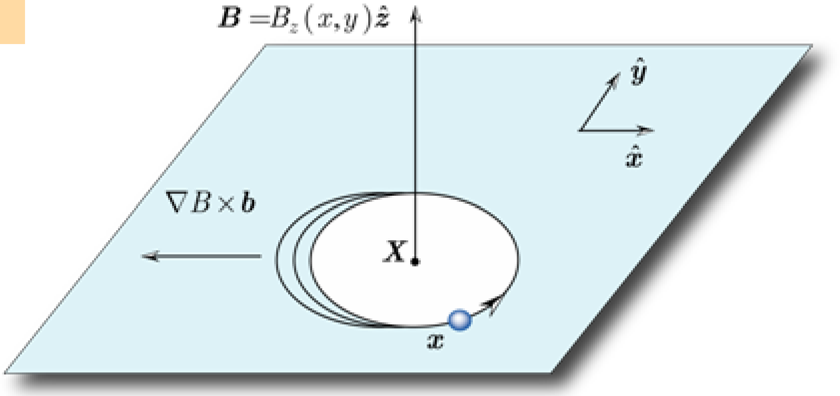


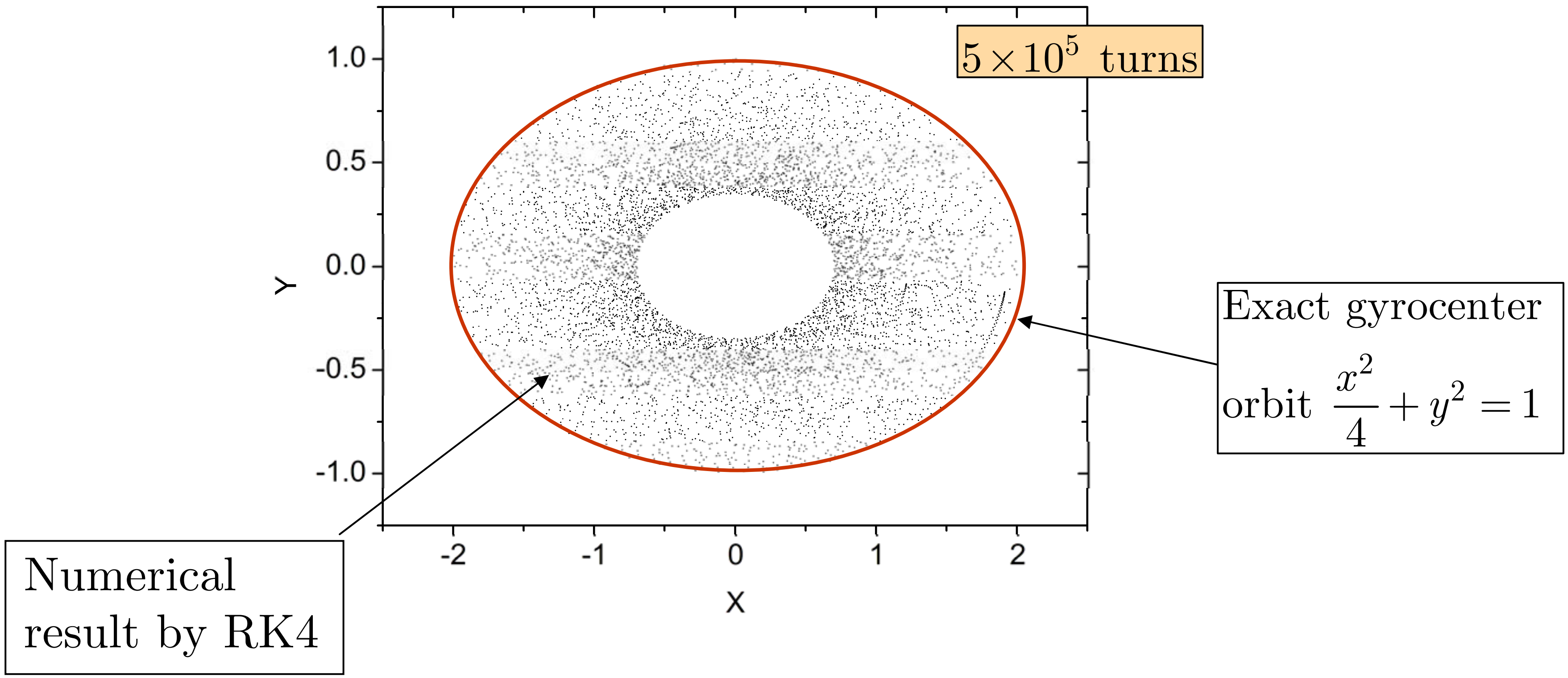
nth (4^{th}) order Runge-Kutta methods

- Long time non-conservation.
- Errors add up coherently.

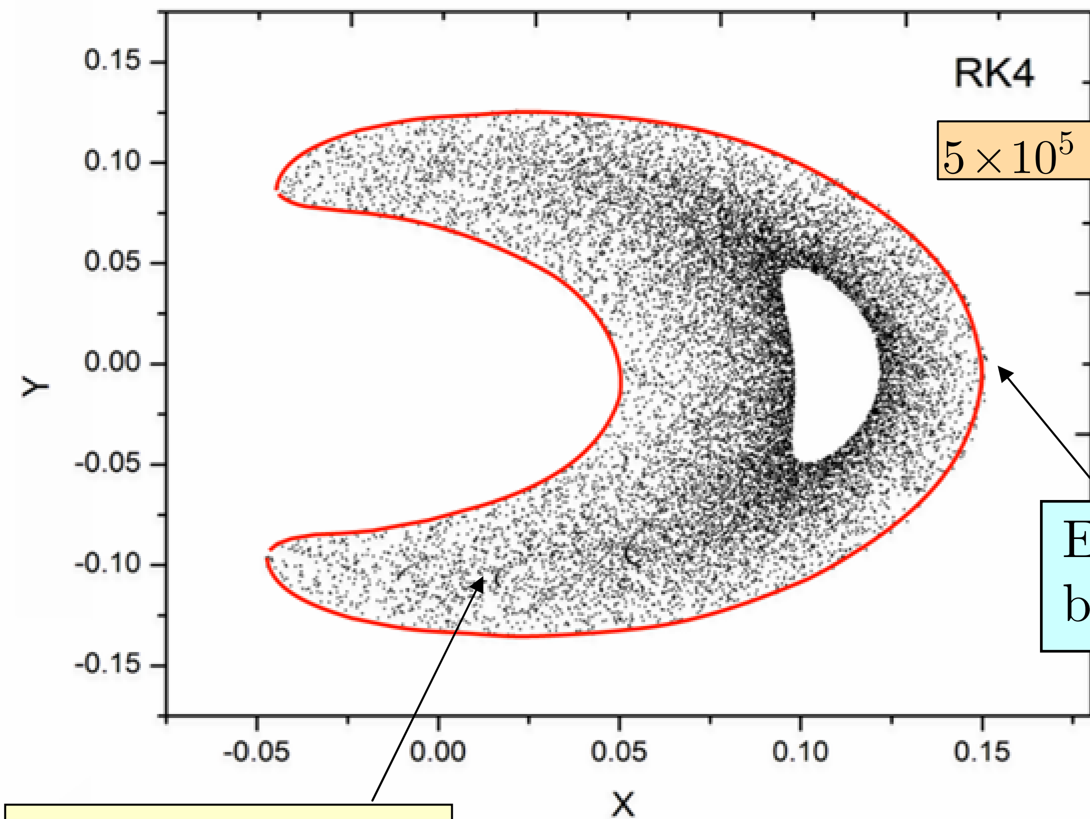
Example 1 – gradient drift

$$B_z(x,y) = 1 + 0.05\left(\frac{x^2}{4} + y^2\right)$$





Example 2 – banana orbit



5×10^5 turns

Exact banana orbit

Numerical result by RK4

ITER: $T_{\text{burn}} \sim 3 \times 10^6 T_{\text{banana}}$

Can we do better than RK4? -- symplectic integrator

Feng (1983)

Ruth (1984)

Conserves symplectic structure;
Bound energy error globally

Symplectic integrator ?

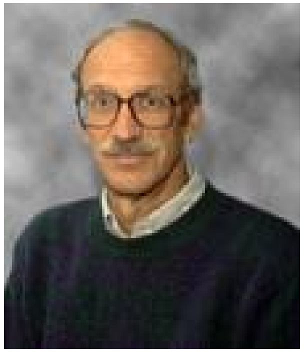
Requires canonical
Hamiltonian structure

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} H_{,q} \\ H_{,p} \end{pmatrix}$$

Application areas:

- Accelerator physics (everybody)
- Planetary dynamics (S. Tremaine)
- Nonlinear dynamics (everybody)
- **Plasma physics (J. Cary 89')**

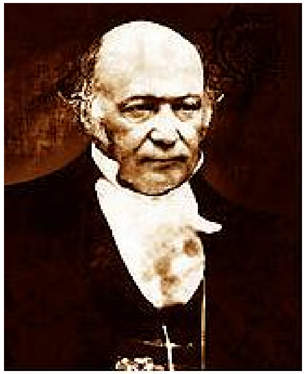
What is the canonical structure for gyrocenter dynamics?



R. White

Gyrocenter dynamics does not have a global canonical structure

Geometry of gyrocenter



Hamilton

Exterior derivative

Inner product

$$i_\tau d\gamma = 0$$

Worldline

$$\gamma = \omega - \left(\tfrac{1}{2}u^2 + \mu B + \phi\right)dt$$
$$\omega = (\,A + u\boldsymbol{b}\,)\cdot d\boldsymbol{X}$$

Only has a non-canonical symplectic structure $\Omega \equiv d\omega$

Gyrocenter dynamics

R. Littlejohn

$$\gamma = (\boldsymbol{A} + u\boldsymbol{b}) \cdot d\boldsymbol{X} + \mu d\theta - \left(\tfrac{1}{2}u^2 + \mu B + \phi\right)dt$$

$$i_\tau d\gamma = 0$$

$$\approx$$

$$L = (\boldsymbol{A} + u\boldsymbol{b}) \cdot \dot{\boldsymbol{X}} + \mu\dot{\theta} - \left(\tfrac{1}{2}u^2 + \mu B + \phi\right)$$

E-L Eq.

$\boldsymbol{E} \times \boldsymbol{B}$, ∇B drift

$$\frac{d\boldsymbol{X}}{dt} = \frac{\boldsymbol{B}^\dagger}{B_\parallel^\dagger}\left(u + \frac{\mu}{2}\boldsymbol{b} \cdot \boldsymbol{\nabla} \times \boldsymbol{b}\right) - \frac{\boldsymbol{b} \times \boldsymbol{E}^\dagger}{B_\parallel^\dagger}$$

$$\frac{du}{dt} = \frac{\boldsymbol{B}^\dagger \cdot \boldsymbol{E}^\dagger}{B_\parallel^\dagger}$$

$$\frac{d\theta}{dt} = B, \quad \frac{d\mu}{dt} = 0$$

curvature drift

Banos drift (1967)

$$\boldsymbol{B}^\dagger \equiv \boldsymbol{\nabla} \times \boldsymbol{A}^\dagger, \quad \boldsymbol{A}^\dagger \equiv \boldsymbol{A} + u\boldsymbol{b}$$

$$B_\parallel^\dagger = \boldsymbol{B}^\dagger \cdot \boldsymbol{b}, \quad \boldsymbol{E}^\dagger \equiv \boldsymbol{E} - \mu\boldsymbol{\nabla}B$$

Darboux Theorem

Jean Gaston Darboux
(1842-1917)

Darboux Theorem (1882):

Every symplectic structure is **locally** canonical.

Gyrocenter dynamics can be canonical **locally**.

No symplectic integrator for gyrocenter?

Variational symplectic integrator

J. Marsden (2001)

$i_\tau d\gamma = 0$ $\approx$ $A = \int_0^{t_1} L dt$

$L = (\boldsymbol{A} + u\boldsymbol{b}) \cdot \dot{\boldsymbol{X}} - \left(\frac{1}{2}u^2 + \mu B + \phi\right)$

discretize on
$t = [0, h, 2h, ..., (N-1)h]$

$$A \approx A_d = \sum_{k=0}^{N-1} h L_d(k, k+1)$$

$$L_d(k, k+1) \equiv L_d(\boldsymbol{x}_k, u_k, \boldsymbol{x}_{k+1}, u_{k+1})$$

minimize w.r.t. $(\mathbf{x}_k, u_k)$

$$\frac{\partial}{\partial x_k^j}[L_d(k-1,k) + L_d(k,k+1)] = 0, (j = 1,2,3)$$

$$\frac{\partial}{\partial u_k}[L_d(k-1,k) + L_d(k,k+1)] = 0$$

discretized Euler-Lagrangian Eq.

$\left[(\mathbf{x}_{k-1}, u_{k-1}), (\mathbf{x}_k, u_k)\right]$
$\rightarrow (\mathbf{x}_{k+1}, u_{k+1})$

Conserved symplectic structure

$$\theta^{+}(k,k+1) \equiv \frac{\partial}{\partial \boldsymbol{x}_{k+1}} L_d(k,k+1)\cdot d\boldsymbol{x}_{k+1} + \frac{\partial}{\partial u_{k+1}} L_d(k,k+1)\,du_{k+1}$$

$$\theta^{-}(k,k+1) \equiv -\frac{\partial}{\partial \boldsymbol{x}_{k}} L_d(k,k+1)\cdot d\boldsymbol{x}_{k} - \frac{\partial}{\partial u_{k}} L_d(k,k+1)\,du_{k}$$

$$dL_d(k,k+1) = \theta^{+}(k,k+1) - \theta^{-}(k,k+1)$$

minimize w.r.t. $(\mathbf{x}_k, u_k)$

$$\Omega_d(k,k+1) \equiv d\theta^{+} = d\theta^{-}$$

$$dA_d = \theta^{+}(0,1) - \theta^{-}(N-1,N)$$

$$\Omega_d(0,1) = \Omega_d(N-1,N)$$

1^{st} order variational symplectic integrator

$$L_d(k,k+1) \equiv \frac{\left[\boldsymbol{A}^{\dagger}(k+1)+\boldsymbol{A}^{\dagger}(k)\right]}{2} \cdot \frac{\left[\boldsymbol{x}_{k+1}-\boldsymbol{x}_k\right]}{h} - \frac{u_k u_{k+1}}{2} - \mu B(k) - \varphi(k)$$

$$\frac{1}{2h} A^{\dagger i}_{,j}(k)\left(x^i_{k+1}-x^i_{k-1}\right) - \frac{1}{2h}\left[A^{\dagger j}(k+1)-A^{\dagger j}(k-1)\right] = \mu B_{,j}(k) + \varphi_{,j}(k)$$

$$\frac{1}{2h} b^i(k)\left(x^i_{k+1}-x^i_{k-1}\right) = \frac{u_{k+1}+u_{k-1}}{2}$$

$$\left[\left(\mathbf{x}_{k-1},u_{k-1}\right),\left(\mathbf{x}_k,u_k\right)\right] \to \left(\mathbf{x}_{k+1},u_{k+1}\right)$$

implicit

Semi-explicit Newton's method

$$\left[A^{\dagger j}(k+1) - A^{\dagger j}(k-1) \right] \approx A^{\dagger j}_{,i}(k) \left(x^i_{k+1} - x^i_{k-1} \right) + b^j(k) \left(u_{k+1} - u_{k-1} \right)$$

$$\frac{1}{2h}\left[A^{\dagger i}_{,j}(k) - A^{\dagger j}_{,i}(k) \right]\left(x^i_{k+1} - x^i_{k-1} \right) - \frac{b^j(k)}{2h}\left[2u_{k-1} - \frac{b^i(k)}{h}\left(x^i_{k+1} - x^i_{k-1} \right) \right]$$

$$= \mu B_{,j}(k) + \varphi_{,j}(k)$$

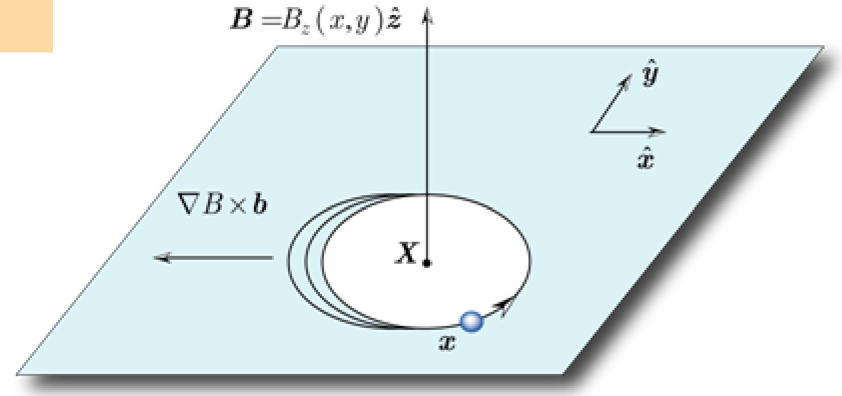
$$\frac{1}{2h} b^i(k) \left(x^i_{k+1} - x^i_{k-1} \right) = \frac{u_{k+1} + u_{k-1}}{2}$$

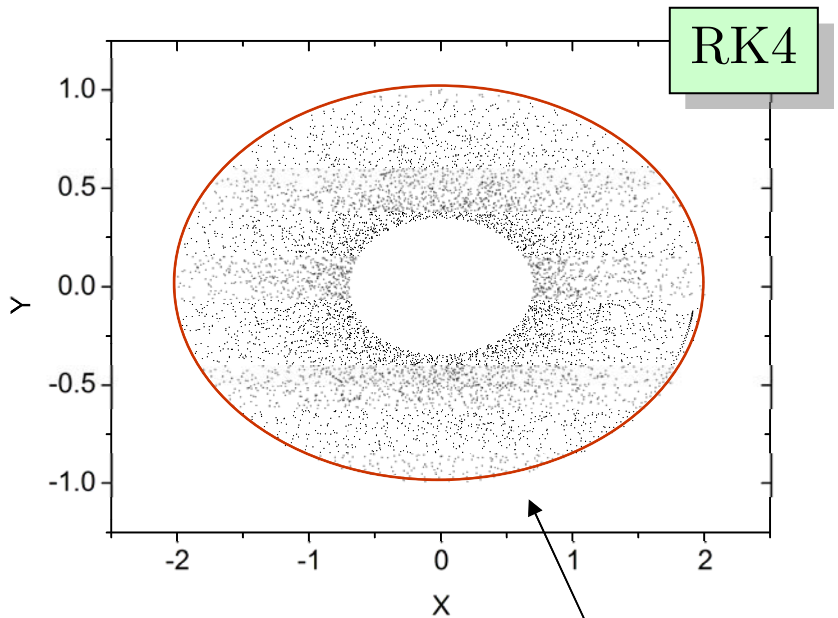
explict, initial guess for the Newton's method

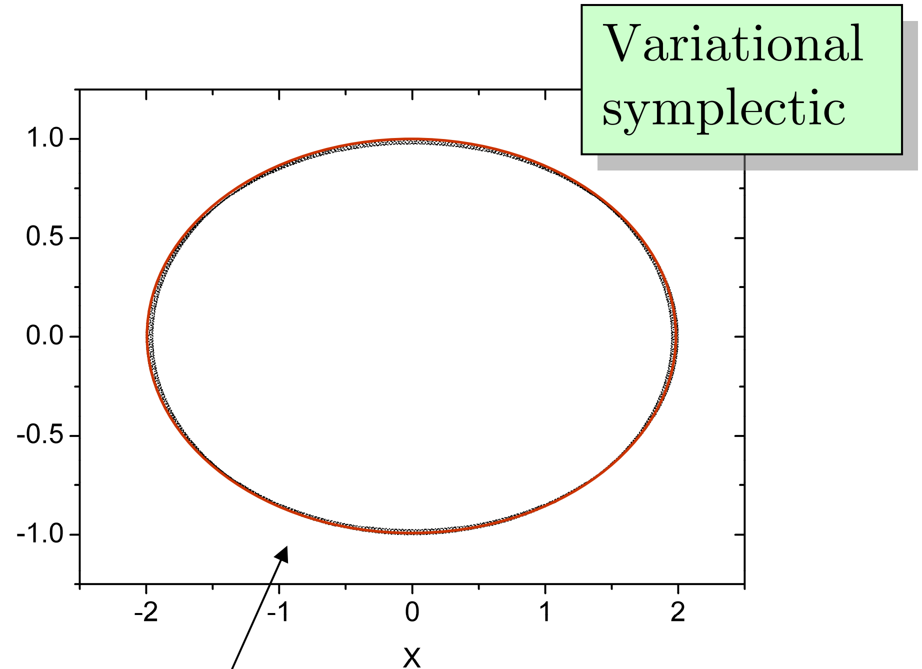
Example – gradient drift

$$\boldsymbol{B} = B(x,y)\hat{\boldsymbol{z}}$$

$$B(x,y) = 1 + 0.05\left(\frac{x^2}{4} + y^2\right)$$





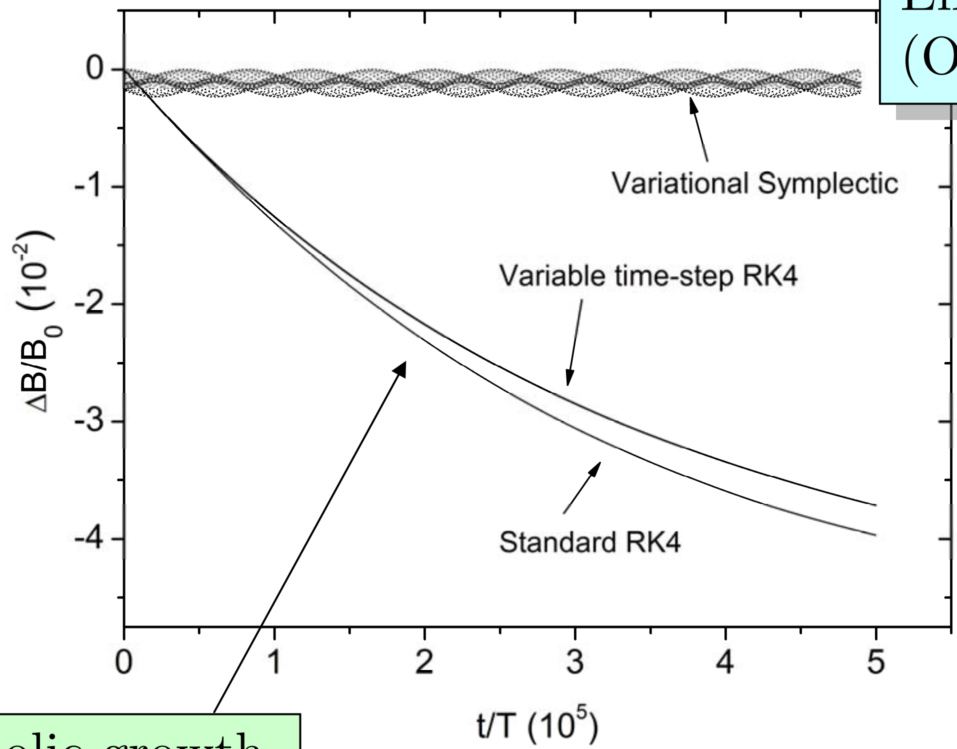


Exact gyrocenter orbit $\dfrac{x^2}{4} + y^2 = 1$

5×10^5 turns

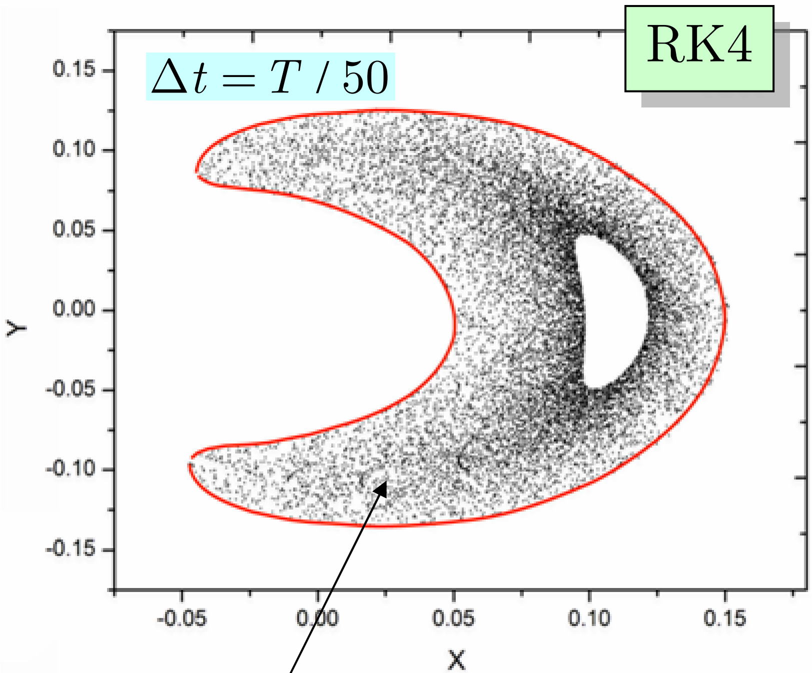
Variational symplectic globally bounds energy error

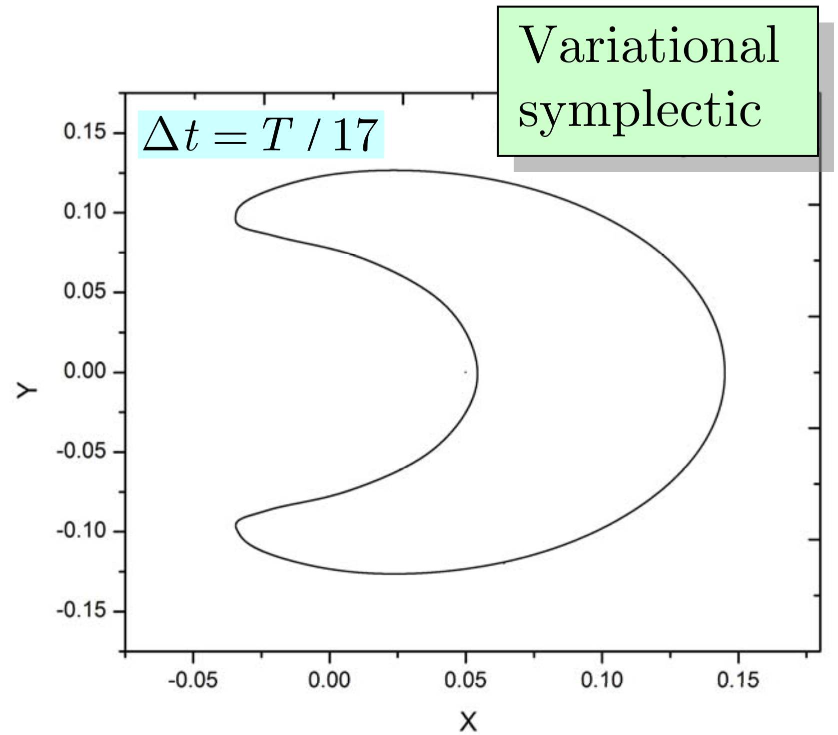
Energy error
(Orbit deviation)



Parabolic growth
of energy error

Example 2 - 4^{th} order variational symplectic integrator, banana orbits



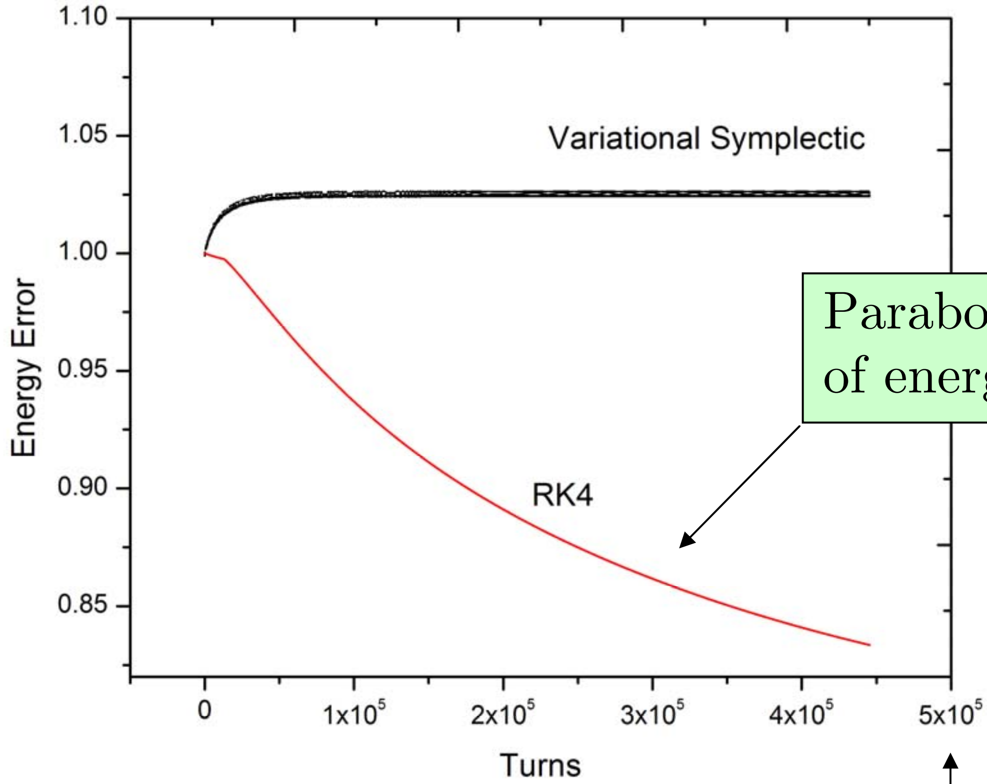


Transport reduction by integration errors

5×10^5 turns

ITER: $T_{\text{burn}} \sim 3 \times 10^6 T_{\text{banana}}$

4^{th} order variational symplectic integrator for banana orbits



Parabolic growth of energy error

$0.2\ T_{\text{burn}}$

Conclusions

How to calculate guiding center motion?

Physics is geometry. So is algorithm.

Symplectic bounds error **globally**, others do not.

“Can Billy come out and compete in the global economy?”

How about collisions?

Collisions do not remove the energy errors of RK4.

Global energy errors of RK4 are numerical noises to collisional physics.

How about implicitness?

Implicit root search does destroy the symplectic structure.

Approximate solution is better than no solution.