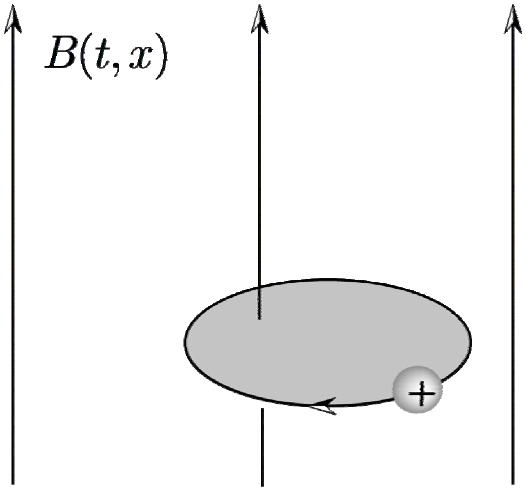
A Footnote on Adiabatic Invariants*

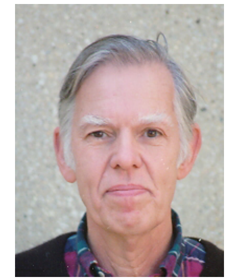
Hong Qin and Ronald C. Davidson

Theory Seminar
Princeton Plasma Physics Laboratory
Princeton, NJ 08543
May 26, 2005

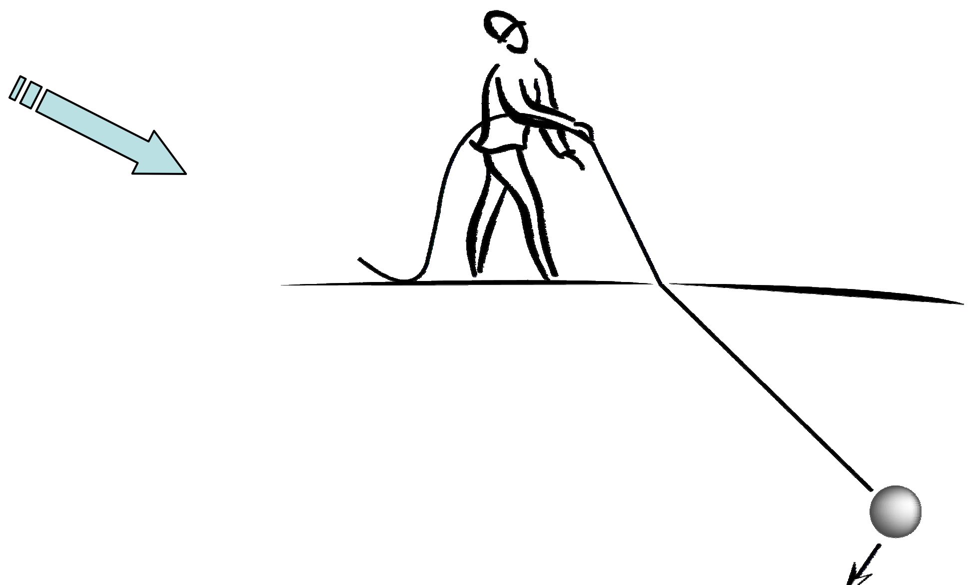
---

* Research supported by U.S. Dept. of Energy. We thank Drs. K. Takayama, T.-S Wang, R. Kulsrud, and E. Startsev for productive discussions.

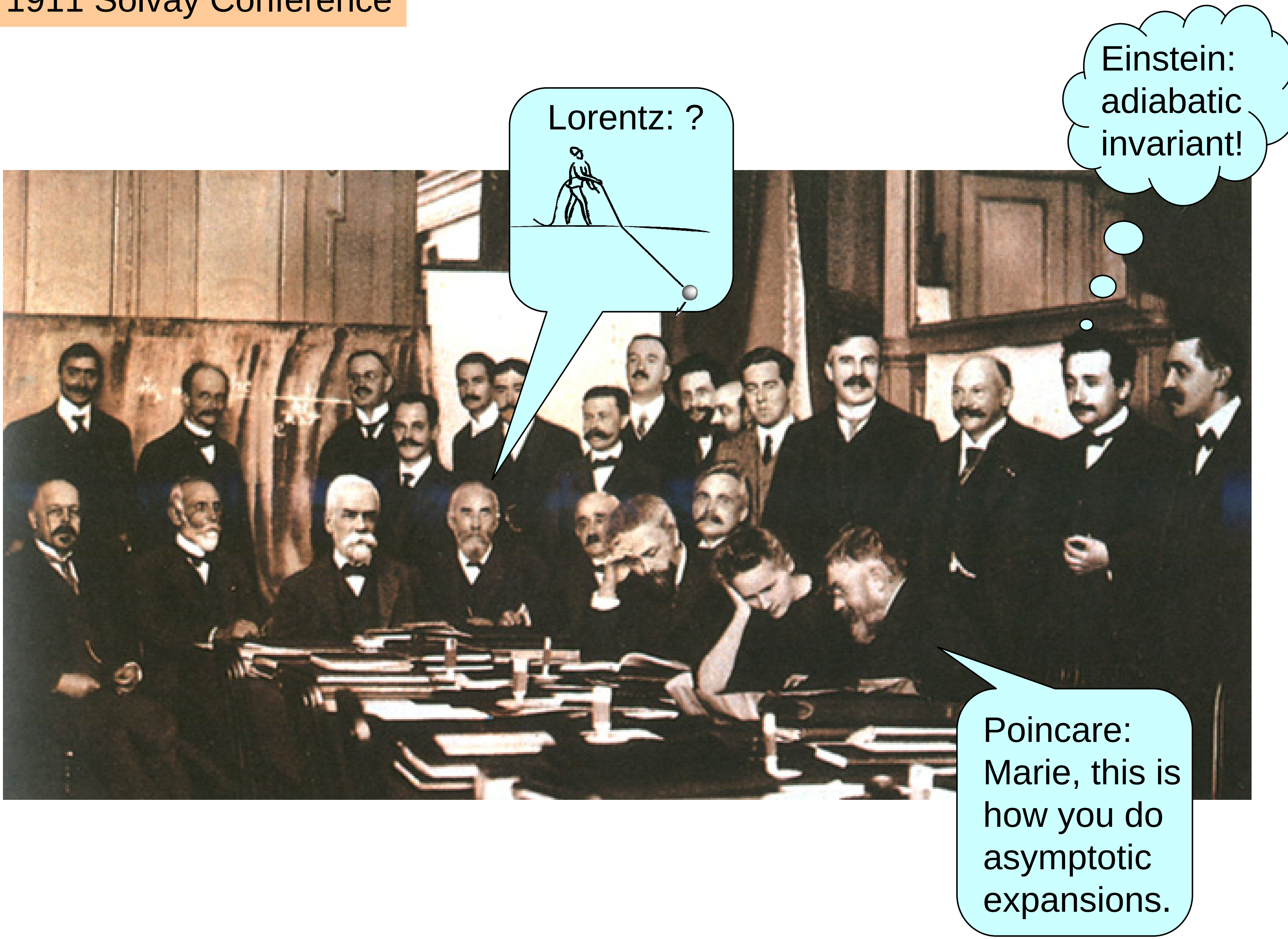




L. Spitzer suggested R. Kulsrud and M. Kruskal to look at a simpler problem first (1950s).



1911 Solvay Conference



Kulsrud's result on adiabatic invariant (1957)

For $u + \kappa(\varepsilon t)u = 0,$

if $\kappa(-\infty) = \kappa_-, \ \kappa(+\infty) = \kappa_+,$

then $\Delta I = I(+\infty) - I(-\infty) \sim o(\varepsilon^n), \ n \in N,$

$$I \equiv \frac{H(t)}{\sqrt{\kappa(t)}}, \ H(t) \equiv \frac{1}{2}\left(u_t^2 + ku^2 \right).$$

But, how about $-\infty < t < +\infty$?

- Kulsrud (1957)
- Dyhne (1960)
- Kruskal (1961)
- Sludskin (1963)
- Howard (1970)

Arnold's result on adiabatic invariant (1977)

For $$u + \kappa(\varepsilon t)u = 0,$$

$$\Delta I = I(t) - I(0) \sim o(1), \text{ for } 0 \le t \le \frac{1}{\varepsilon}.$$

Adiabatic invariant of arbitrary order

For $$u + \kappa(\varepsilon t)u = 0,$$

$$\Delta I = I(t) - I(0) \sim o(1), \text{ for } 0 \le t \le \frac{1}{\varepsilon^n}, \; n \in N.$$

Common mistake

$$\frac{dI}{dt} \sim O(\varepsilon^n),\ n \in N.$$

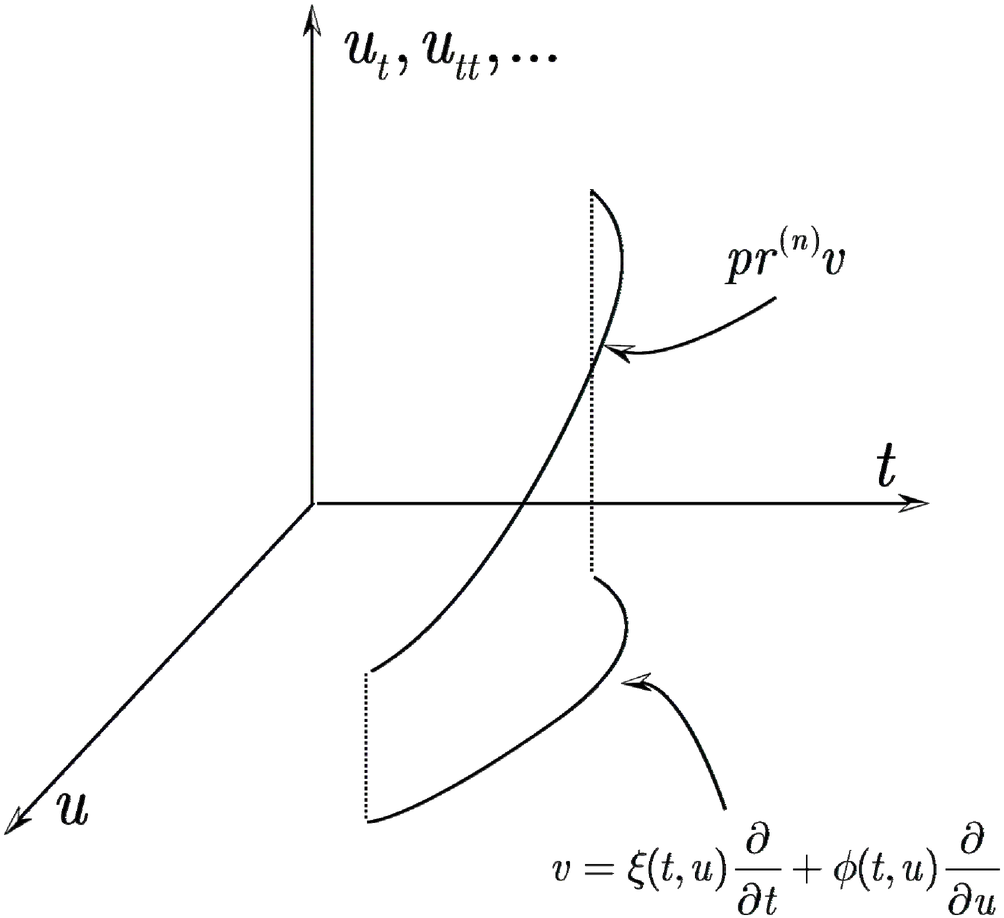
Asymptotic invariant.
Not adiabatic invariant.

- Asymptotic invariant is about “dot”.
- Adiabatic invariant is about “delta”.

- Energy is an asymptotic invariant, but not an adiabatic invariant.

We can do better by looking at the symmetries and (exact) invariants.

What is symmetry?
S. Lie (1890s)



What is symmetry?

$$v = \xi(t,u)\frac{\partial}{\partial t} + \phi(t,u)\frac{\partial}{\partial u}$$

$$pr^{(2)}v = \xi\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} + \phi^{u}\frac{\partial}{\partial u} + \phi^{uu}\frac{\partial}{\partial u},$$

$$\phi^{u} \equiv \phi_t + (\phi_u - \xi_t)u - \xi_u u^2,$$

$$\begin{aligned}\phi^{uu} &\equiv -3\xi_u uu + (\phi_u - 2\xi_t)u - \xi_{uu}u^3 \\ &+(\phi_{uu} - 2\xi_{tu})u^2 + (2\phi_{ut} - \xi_{tt})u + \phi_{tt}\ .\end{aligned}$$

$$pr^{(2)}v\left[u + \kappa(t)u\right] = \phi^{uu} + \kappa\phi + \xi\kappa u = 0.$$

Symmetry

$$\xi = a(t)u + b(t),$$

$$\phi = a(t)u^2 + c(t)u + d(t),$$

$$a + \kappa a = 0,$$

$$d + \kappa d = 0,$$

$$b + 4\kappa b + 2\kappa b = 0,$$

$$c - \frac{b}{2} = 0.$$

Lie algebra and sub-algebras of the symmetry group

(Qin & Davidson 2005)

?! (2D)

Wronskian (2D)

$$g = g_a \oplus g_b \oplus g_c \oplus g_d$$

Courant-Snyder symmetry (3D)

Scaling (1D)

Noether's Theorem (1918) links invariants and symmetries

Infinitesimal divergence symmetry:

$$pr^{(2)}v(L) + L\frac{d\xi}{dt} = \frac{dB(t,u)}{dt}$$

$B(t,u)$ --- arbitrary function.

L --- Lagrangian.

Then $pr^{(2)}v(L) = \frac{dA}{dt} + \xi\frac{dL}{dt}$ for some function $A(t,u)$, and $I = B - A - L\xi$ is an invariant.

Courant-Snyder invariant

$$b = 2w^2,$$

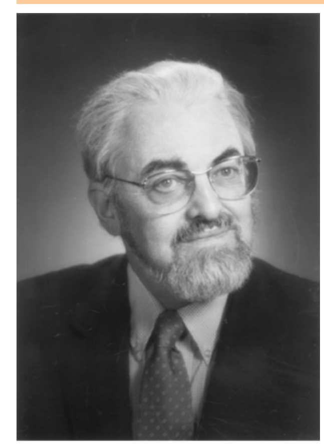
$$g_b : v = 2w^2 \frac{\partial}{\partial t} + wwu \frac{\partial}{\partial u},$$

$$w + \kappa(t) w - \frac{e}{w^3} = 0.$$

$$I_{cs} = \frac{u^2}{w^2} + \left(wu - wu \right)^2 \text{ is an invariant.}$$

Envelope equation.
Birkhoff (1908, concept)

Courant-Snyder invariant (1958)



Ernest D. Courant

Kulsrud (1957) had two equations equivalent to the envelope equation.

Back to the adiabatic invariant

$$\text{Let } \kappa = \kappa(\varepsilon t),$$

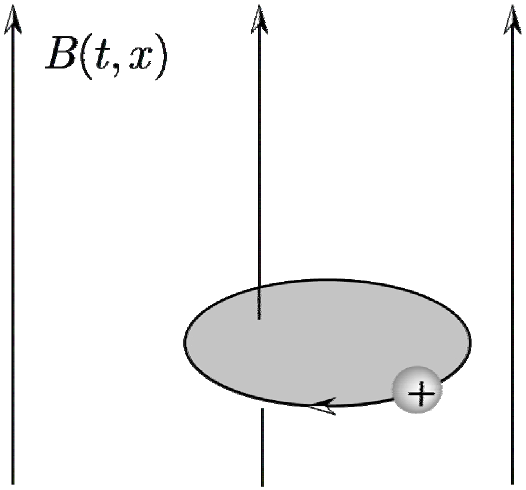
$$w \sim \kappa^{-1/4} - \frac{\left(\kappa^{-1/4}\right)_{tt}}{4\kappa} \text{ for any t.}$$

$$I_{cs} \sim I + O(\varepsilon), \quad I \equiv \frac{H(t)}{\sqrt{\kappa(t)}},$$

$$\Delta I \equiv I(t) - I(0) \sim O(\varepsilon) \text{ for any t.}$$

Courant-Snyder invariant enables strong result.

Back to gyromotion



Conjucture:

Magnetic moment is asymptotic to an exact magnetic invariant.

Symmetry of the envelope equation (Qin & Davidson 2005)

Theorem 1. For an arbitrary function $\kappa(t)$ and w_1, w_2 satisfying

$$w_1 + \kappa w_1 = \frac{\varepsilon_1}{w_1^3},$$

$$w_2 + \kappa w_2 = \frac{\varepsilon_2}{w_2^3},$$

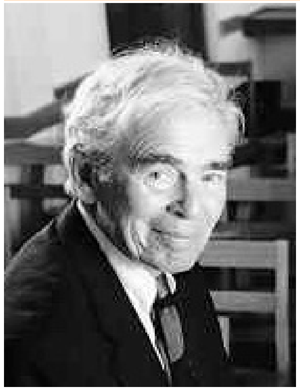
where ε_1 and ε_2 are real constants, the quantity

$$I = \varepsilon_1 \left(\frac{w_2}{w_1} \right)^2 + \varepsilon_2 \left(\frac{w_1}{w_2} \right)^2 + \left(w_2 w_1 - w_2 w_1 \right)^2$$

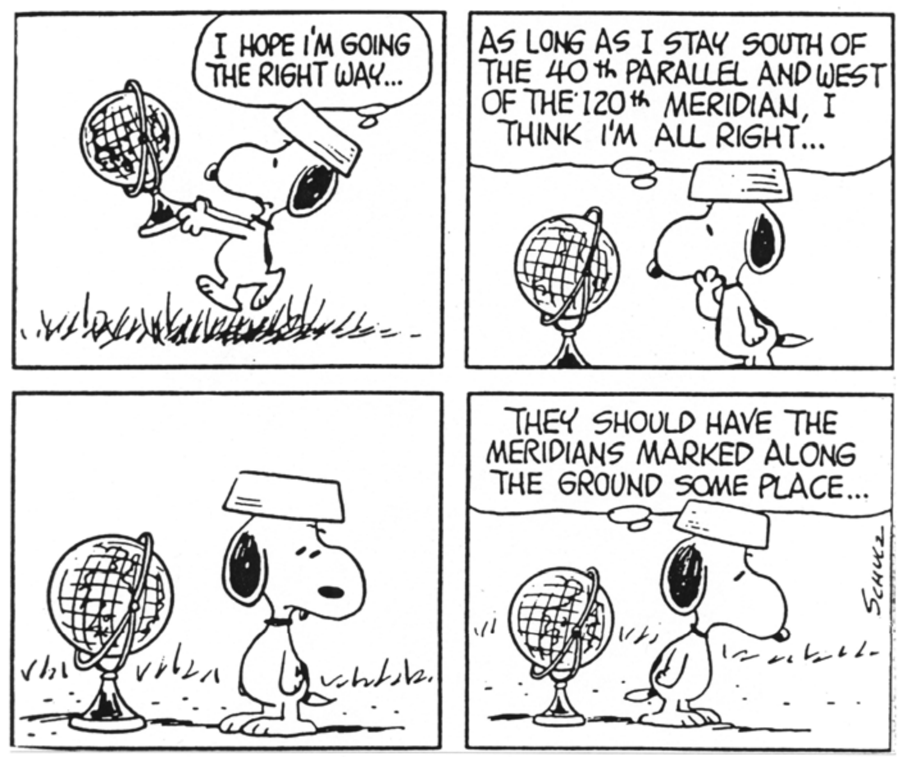
is an invariant.

Application -- Gyrokinetic theory

Frieman, Davidson, Langdon (1966)

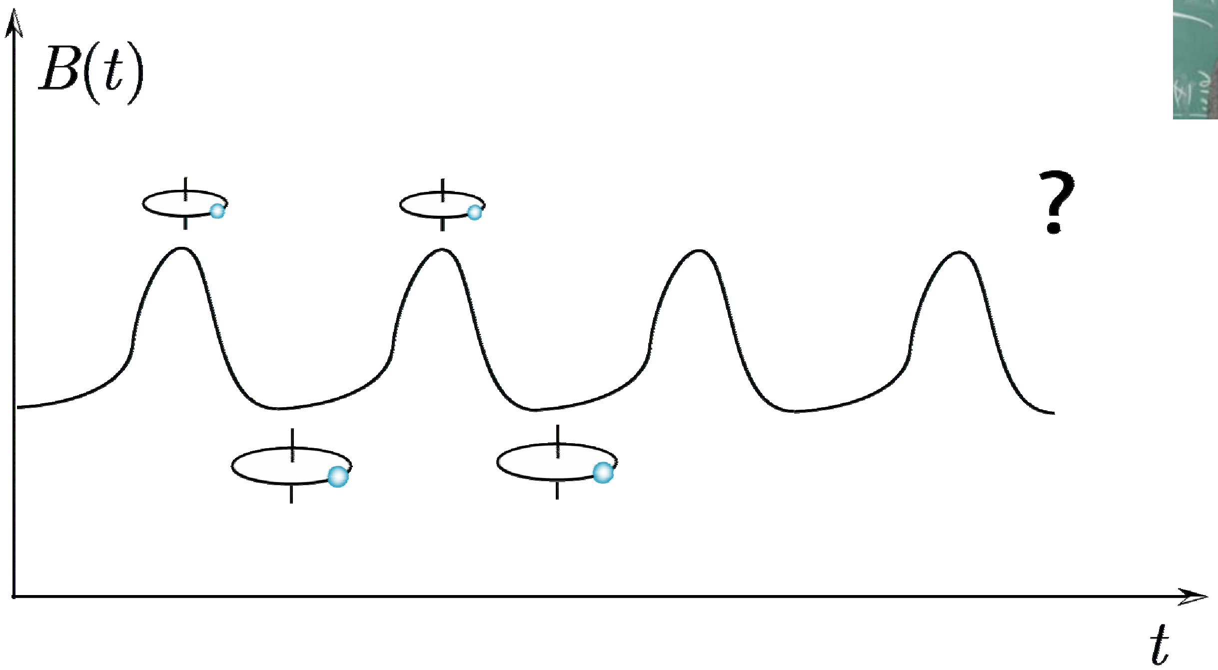


Quest of good coordinates.



Application --- Magnetic pump

GPP I Problem, N. Fisch (1993)



Application – Symplectic integrator

- Symplectic Integrator conserves phase space volume exactly.
- There is an exact invariant to which the energy is asymptotic.

Feng, Ruth, Channell (1983, 1984)

J. Marsden (1991)

Relevant topics

- Adiabatic invariant in strong inhomgeneities (Dodin, Fisch, Weitzner, Chang).

- Anisotropic equilibrium for 3D beams (Qin, Davidson, Startsev, Hudson).