Contrib. Plasma Phys. **46**, No. 7-9, 7 – 19 (2006) / **DOI** 10.1002/ctpp.200610035

General Gyrokinetic Equations for Edge Plasmas

H. Qin[*1], **R. H. Cohen**[2], **W. M. Nevins**[2], and **X. Q. Xu**[2]

[1] Princeton Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08543, USA
[2] Lawrence Livermore National Laboratory, Livermore, CA 94550, USA

Received 23 May 2005, accepted 14 September 2005
Published online 22 August 2006

Key words Gyrokinetic equation, edge plasma.
PACS 52.25Fa, 52.25Dg, 52.35Py

During the pedestal cycle of H-mode edge plasmas in tokamak experiments, large-amplitude pedestal build-up and destruction coexist with small-amplitude drift wave turbulence. The pedestal dynamics simultaneously includes fast time-scale electromagnetic instabilities, long time-scale turbulence-induced transport processes, and more interestingly the interaction between them. To numerically simulate the pedestal dynamics from first principles, it is desirable to develop an effective algorithm based on the gyrokinetic theory. However, existing gyrokinetic theories cannot treat fully nonlinear electromagnetic perturbations with multi-scale-length structures in spacetime, and therefore do not apply to edge plasmas. A set of generalized gyrokinetic equations valid for the edge plasmas has been derived. This formalism allows large-amplitude, time-dependent background electromagnetic fields to be developed fully nonlinearly in addition to small-amplitude, short-wavelength electromagnetic perturbations. It turns out that the most general gyrokinetic theory can be geometrically formulated. The Poincaré-Cartan-Einstein 1-form on the 7D phase space determines particles' worldlines in the phase space, and realizes the momentum integrals in kinetic theory as fiber integrals. The infinitesimal generator of the gyro-symmetry is then asymptotically constructed as the base for the gyrophase coordinate of the gyrocenter coordinate system. This is accomplished by applying the Lie coordinate perturbation method to the Poincaré-Cartan-Einstein 1-form. General gyrokinetic Vlasov-Maxwell equations are then developed as the Vlasov-Maxwell equations in the gyrocenter coordinate system, rather than a set of new equations. Because the general gyrokinetic system developed is geometrically the same as the Vlasov-Maxwell equations, all the coordinate-independent properties of the Vlasov-Maxwell equations, such as energy conservation, momentum conservation, and phase space volume conservation, are automatically carried over to the general gyrokinetic system. The pullback transformation associated with the coordinate transformation is shown to be an indispensable part of the general gyrokinetic Vlasov-Maxwell equations. As an example, the pullback transformation in the gyrokinetic Poisson equation is explicitly expressed in terms of moments of the gyrocenter distribution function, with the important gyro-orbit squeezing effect due to the large electric field shearing in the edge and the full finite Larmour radius effect for short wavelength fluctuations. The familiar "polarization drift density" in the gyrocenter Poisson equation is replaced by a more general expression.

1 Introduction

To a large extent, the dynamics of edge plasmas in tokamaks determines the overall confinement properties of the devices. It is necessary to develop a kinetic simulation method that enables large-scale simulations of the edge dynamics based on first principles. The kinetic equation system that is most analytically and algorithmically suitable for this purpose is the gyrokinetic equation system [1–25]. The origin of gyrokinetic theory can be traced back to the early work of extending the Chew-Goldberger-Low theory [26] to higher orders by Frieman, Davidson, and Langdon [1, 2]. The introduction of guiding-center coordinates by Catto [6] and Littlejohn's theory of guiding center using the non-canonical coordinate perturbation method [5, 9, 11] played important role in the development of gyrokinetic theory. Lee [27] first realized that the gyrokinetic Poisson equation is nontrivially different from the regular Poisson equation. The most important difference is the "polarization drift density". Soon, Dubin *et al* [12] applied Hamiltonian non-canonical perturbation method to the derivation

[*] Corresponding author: e-mail: hongqin@princeton.edu

of the gyrokinetic equation, followed by Hahm using the Lagrangian non-canonical perturbation method [13]. Subsequently, many aspects [14–22, 24, 25] of the modern gyrokinetic theory, such as the concept of gyro-center gauge [22], high frequency gyrokinetics [19, 22], and gyro-center pull-back transformation [21, 24] have been worked out. However, it is difficult to apply previously derived gyrokinetic system to the edge plasmas due to the unique features of their dynamics. In the pedestal cycle for H-modes, there exists a long-term dynamics for the pedestal build-up when the plasma is heated by neutral beam injections. The exact dynamics of the pedestal build-up is determined by the short time-scale, nonlinearly saturated microturbulence. The continuous build-up of pedestal eventually will drive edge localized mode (ELM) unstable [28, 29], which is also short time-scale. The nonlinearly evolved ELM reduce the height of the pedestal by a large portion and the pedestal starts to grow again, which marks the beginning of another pedestal cycle. In the present study, we develop a general gyrokinetic system, where the long-term pedestal dynamics is described by a time-dependent background, and the microturbulence and ELMs are described by nonlinear perturbations on the dynamic background. Such a split between dynamic background and perturbations is also convenient when studying the physics associated with the electric field in the radial direction $\mathbf{E}_r$ in the edge. Because the pedestal width L_p is much smaller than the minor radius, the $\mathbf{E}_r$ developed is much bigger than that in the core region. Since the pedestal is time-dependent, so is $\mathbf{E}_r$. It is therefore necessary to allow a large background electric field $\mathbf{E}_0(t)$ to nonlinearly evolve in the gyrokinetic equation system. The background magnetic field $\mathbf{B}_0(t)$ is allowed to be time-dependent as well, which will conveniently include the change of magnetic equilibrium during the pedestal cycle or the ramp-up phase of the toroidal current. In previous gyrokinetic systems, the nonlinear dynamics of the background electromagnetic field was not treated.

Another important new feature of the present study is that a geometric method is adopted. In its most general form, gyrokinetic theory is about a symmetry, called gyro-symmetry, for magnetized plasmas. Our objective is to decouple the gyro-phase dynamics from the rest of particle dynamics by finding the gyro-symmetry. Obviously, this is fundamentally different from the conventional gyrokinetic concept of "averaging out" the "fast gyro-motion". This objective is accomplished by asymptotically constructing a good coordinate system, which is of course a nontrivial task. Indeed, it is almost impossible without using the Lie coordinate perturbation method [11, 30–32] enabled by the geometric nature of the phase space dynamics. We will develop the gyrokinetic Vlasov-Maxwell equations as the Vlasov-Maxwell equations in the gyrocenter coordinates, rather than a new set of equations. Compared with other methods of deriving gyrokinetic equations, the advantage of the geometric approach is that it automatically guarantees the physics described by the gyrokinetic system is the exactly the same as those contained in the Vlasov-Maxwell equations in the laboratory coordinates when the gyrokinetic system is valid, i.e., when the gyro-symmetry exists. Physics is geometry; it does not depend on which coordinate system is used. Therefore all the coordinate-independent properties of the Vlasov-Maxwell equations, such as energy conservation, momentum conservation and phase space volume conservation, are automatically satisfied by the gyrokinetic system. The essential component of the geometric gyrokinetic theory that guarantees the invariance of physics content is the pullback transformation of the distribution function associated with the coordinate transformation. The importance of the pullback transformation can't be over-emphasized. Without this vital element, many important physics will be lost in the gyrokinetic theory. As an example, the pullback transformation in the gyrokinetic Poisson equation is explicitly expressed in terms of moments of the gyrocenter distribution function, with the important gyro-orbit squeezing effect due to the large electric field shearing in the edge and the full finite Larmour radius effect for short wavelength fluctuations. Even though all the coordinate systems are equivalent in describing the physics, the computational complexity for different coordinate systems are different. In this sense, many physics theories and algorithms of computational physics are quests of good coordinates. For the gyrokinetic theory and numerical simulation, the good coordinate system is the gyrocenter coordinate system that explicitly displays the gyro-symmetry as its gyro-phase coordinate.

2 Gyro-symmetry and Lie coordinate perturbation method

The natural geometric object that determines a charged particle's dynamics in an electromagnetic field is given by the Poincaré-Cartan-Einstein 1-form

$$\gamma = A + p = (\mathbf{A} + \mathbf{v}) \cdot d\mathbf{x} - \left[\frac{v^2}{2} + \phi\right] dt \;, \tag{1}$$

constructed by taking the only two geometric objects related to the dynamics of charged particles, the momentum 1-form $p \equiv (-v^2/2, \mathbf{p})$ and the potential 1-form $A \equiv (-\phi, \mathbf{A})$, and then performing the simplest nontrivial operation, i.e., addition with the right units, to let particles interact with fields. Here, the bold mathematical symbols $\mathbf{A}$ and $\mathbf{p}$ represent the $i = 1, 2, 3$ components of the 1-forms A and p, $d\mathbf{x}$ represents dx^i $(i = 1, 2, 3)$, and $(\mathbf{A} + \mathbf{v}) \cdot d\mathbf{x}$ is just a shorthand notation for $\sum_{i=1,2,3}(A_i + v_i)dx^i$. We have normalized γ by m, A by mc/e, and ϕ by m/e. These normalizations will be used thereafter, unless it is explicitly stated otherwise. Particles' dynamics is determined by Hamilton's equation

$$i_\tau d\gamma = 0 \,, \tag{2}$$

where τ is a vector field, whose integrals are particles' worldlines on the 7D phase space P (including time). Here $d\gamma$ is the exterior derivative of γ and $i_\tau d\gamma$ is the inner product between $d\gamma$ and τ. Very elegantly, the Poincaré-Cartan-Einstein 1-form γ also gives the necessary "volume form" needed for the fundamental "velocity integrals" in kinetic theory. However, this topic is beyond the mathematical sophistication of the present paper. A complete geometric setting for the gyrokinetic theory can be found in Ref. [25].

A symmetry vector field η (infinitesimal generator) of γ is defined to be a vector field that satisfies

$$L_\eta \gamma = ds \tag{3}$$

for some function s on the phase space, where L_η is the Lie derivative along η. Vector field η generates a 1-parameter symmetry group for γ. The symmetry for γ that we are interested is an approximate one. It is an exact symmetry when the electromagnetic fields are constant in spacetime. To demonstrate the basic concept, we first consider the case of constant magnetic field without electrical field. Because of its simplicity, there are several symmetries admitted by γ. The gyro-symmetry is the symmetry given by

$$\eta = v_x \left(\frac{1}{B} \frac{\partial}{\partial x} + \frac{\partial}{\partial v_y} \right) + v_y \left(\frac{1}{B} \frac{\partial}{\partial y} - \frac{\partial}{\partial v_x} \right) \,. \tag{4}$$

To find out the corresponding invariant, we need Noether's theorem which links symmetries and invariants. Here, we cast the theorem in the form of forms. For a symmetry vector field η, using Cartan's formula $L_\eta \gamma = d(i_\eta \gamma) + i_\eta d\gamma$, we have

$$d(i_\eta \gamma) + i_\eta d\gamma = ds \,. \tag{5}$$

For the vector field τ of a worldline,

$$d(\gamma \cdot \eta) \cdot \tau = ds \cdot \tau \,, \tag{6}$$

which implies that $\gamma \cdot \eta - s$ is an invariant. Applying Noether's theorem, we can verify that the corresponding invariant is the magnetic moment

$$\mu = \frac{v_x^2 + v_y^2}{2B} \,, \tag{7}$$

as expected. The gyro-symmetry η has a rather complicated expression in the Cartesian coordinates (x, y, v_x, v_y). It is desirable to construct a new coordinate such that η is a coordinate base

$$\eta = \frac{\partial}{\partial \theta} \,, \tag{8}$$

where θ is the gyrophase coordinate. Eq. (4) indicates that the gyro-symmetry η is neither a rotation in the momentum space, nor a rotation in the configuration space. Therefore, θ is not a momentum coordinate or a configuration coordinate. It is a phase-space coordinate that depends on particles' momentum as well as their spacetime positions.

When the fields are not constant in spacetime, the gyro-symmetry η in Eq. (4) is broken. We therefore seek an asymptotic symmetry when the spacetime inhomogeneity is weak. First, we construct a non-canonical phase space coordinate system $\bar{Z} = (\bar{\mathbf{X}}, \bar{u}, \bar{w}, \bar{\theta})$ such that γ can be expanded into an asymptotic series

$$\gamma = \bar{\gamma}_0 + \bar{\gamma}_1 + \bar{\gamma}_2 + \ \dots \,, \tag{9}$$

where $\bar{\gamma}_1 \sim \varepsilon\bar{\gamma}_0$, $\bar{\gamma}_2 \sim \varepsilon\bar{\gamma}_1$, and $\varepsilon \ll 1$. By construction, $\bar{\gamma}_0$ admits the gyro-symmetry $\eta = \partial/\partial\bar{\theta}$, but $\bar{\gamma}_1$ does not necessarily. $\bar{Z}$ is the called the zeroth order gyrocenter coordinate. Then, a coordinate perturbation transformation $g : \bar{Z} \rightarrow Z = g(\bar{Z})$ is introduced such that in the new coordinates $Z = (\mathbf{X}, u, w, \theta)$, γ_1 and/or γ_2 admit the gyro-symmetry $\eta = \partial/\partial\theta$. In fact, we will seek a stronger symmetry condition

$$\partial\gamma/\partial\theta = 0 \ ,$$

which is sufficient for $\eta = \partial/\partial\theta$ to satisfy Eq. (3). Z is the called the first and/or second-order gyrocenter coordinate. The small parameter ε measures the weakness of spacetime inhomogeneity of the fields. The coordinate perturbation transformation procedure indicates that the most relaxed conditions for the existence of an asymptotic gyro-symmetry is

$$\mathbf{E} \equiv \mathbf{E}_0 + \mathbf{E}_1, \ \mathbf{B} \equiv \mathbf{B}_0 + \mathbf{B}_1, \tag{10}$$

$$\mathbf{E}_0 \sim \frac{\mathbf{v} \times \mathbf{B}_0}{c} \ , \ \mathbf{E}_1 \sim \varepsilon_1 \frac{\mathbf{v} \times \mathbf{B}_0}{c} \ , \ \mathbf{B}_1 \sim \varepsilon_1 \mathbf{B}_0, \tag{11}$$

$$\left(|\rho| \frac{\nabla E_0}{E_0}, \frac{1}{\Omega E_0}\frac{\partial E_0}{\partial t} \right) \sim \left(|\rho| \frac{\nabla B_0}{B_0}, \frac{1}{\Omega B_0}\frac{\partial B_0}{\partial t} \right) \sim \varepsilon_0 \ , \tag{12}$$

$$\left(|\rho| \frac{\nabla E_1}{E_1}, \frac{1}{\Omega E_1}\frac{\partial E_1}{\partial t} \right) \sim \left(|\rho| \frac{\nabla B_1}{B_1}, \frac{1}{\Omega B_1}\frac{\partial B_1}{\partial t} \right) \sim 1 \ , \tag{13}$$

where the fields were split into two parts. $(\mathbf{E}_0, \mathbf{B}_0)$ are the time-dependent background fields with long spacetime scale length compared with the spacetime gyroradius $\rho = (\rho, 1/\Omega)$. The weak spacetime inhomogeneities of the background fields are measured by the small parameter ε_0. For edge plasmas, the background electric field is large. The order of $\mathbf{E}_0$ implies that the potential drop of background field can be comparable to the thermal energy of the particles, *i.e.*, $e\mathbf{E}_0 \cdot \rho \sim 1$. $(\mathbf{E}_1, \mathbf{B}_1)$ are the perturbation parts with spacetime scale length comparable to the spacetime gyroradius, and the perturbation amplitude is measured by the small parameter ε_1. Both ε_0 and ε_1 measure the weak spacetime inhomogeneities of the overall fields. In general, we assume $\varepsilon \sim \varepsilon_0 \sim \varepsilon_1$.

The coordinate perturbation method we adopt belongs to the class of perturbation techniques generally referred as the Lie perturbation method [11, 30–32]. A coordinate transformation for the 7D phase space P can be locally represented by a map between two subsets of the R^7 space, $g : z \longmapsto Z = g(z)$. In the Lie coordinate perturbation method, g is a continuous group generated by a vector field G with $g : z \longmapsto Z = g(z, \varepsilon)$ and $G = dg/d\varepsilon|_{\varepsilon=0}$. Under the coordinate transformation g, γ transforms as a function, i.e., it is pulled-back.

$$\begin{aligned} \Gamma(Z) &= g^{-1*}\gamma(z) = \gamma\left[g^{-1}(Z)\right] = \gamma(Z) - L_{G(Z)}\gamma(Z) + O(\varepsilon^2) \\ &= \gamma(Z) - i_{G(Z)}d\gamma(Z) - d\left[\gamma \cdot G(Z)\right] + O(\varepsilon^2) \ , \end{aligned} \tag{14}$$

where use has been made of $-G = dg^{-1}/d\varepsilon|_{\varepsilon=0}$. If γ is an asymptotic series as in Eq. (9), let $Z = g_1(z, \varepsilon)$ and we have

$$\Gamma(Z) = \Gamma_0(Z) + \Gamma_1(Z) + O(\varepsilon^2) \ , \tag{15}$$

$$\Gamma_0(Z) = \gamma_0(Z) \ , \tag{16}$$

$$\Gamma_1(Z) = \gamma_1(Z) - i_{G_1(Z)}d\gamma_0(Z) - d\left[\gamma_0 \cdot G_1(Z)\right] \ . \tag{17}$$

A similar procedure can be straightforwardly carried out to the second order. Let $Z = g_2(g_1(z, \varepsilon), \delta)$ and $\delta \sim \varepsilon^2$,

$$\Gamma_2(Z) = \gamma_2(Z) - L_{G_1(Z)}\gamma_1(Z) + \left(\frac{1}{2}L^2_{G_1(Z)} - L_{G_2(Z)}\right)\gamma_0(Z) \ , \tag{18}$$

where $G_2 = dg_2/d\delta|_{\delta=0}$.

3 Gyrocenter Coordinates

To construct the zeroth order gyrocenter coordinate $\bar{Z} = (\bar{\mathbf{X}}, \bar{u}, \bar{w}, \bar{\theta})$, we first define two vector fields

$$\mathbf{D}(y) \equiv \frac{\mathbf{E}_0(y) \times \mathbf{B}_0(y)}{\left[B_0(y)\right]^2} \ , \ \mathbf{b}(y) \equiv \frac{\mathbf{B}_0(y)}{B_0(y)} \ , \tag{19}$$

where y is a point in the spacetime M. In addition, we define the following vector fields which also depend on $\mathbf{v}_x$, the velocity at another spacetime position $x \in M$,

$$u(y, \mathbf{v}_x)\mathbf{b}(y) \equiv [\mathbf{v}_x - \mathbf{D}(y)] \cdot \ \mathbf{b}(y)\, \mathbf{b}(y) \ , \tag{20}$$

$$w(y, \mathbf{v}_x)\mathbf{c}(y, \mathbf{v}_x) \equiv [\mathbf{v}_x - \mathbf{D}(y)] \times \mathbf{b}(y) \times \mathbf{b}(y) \ , \tag{21}$$

$$\mathbf{c}(y, \mathbf{v}_x) \cdot \mathbf{c}(y, \mathbf{v}_x) = 1 \, , \tag{22}$$

$$\mathbf{a}(y, \mathbf{v}_x) \equiv \mathbf{b}(y) \times \mathbf{c}(y, \mathbf{v}_x) \ , \tag{23}$$

$$\rho(y, \mathbf{v}_x) \equiv \frac{\mathbf{b}(y) \times [\mathbf{v}_x(y) - \mathbf{D}(y)]}{B_0(y)} \, . \tag{24}$$

Velocity $\mathbf{v}_x(y)$ has the following partition

$$\mathbf{v}_x(y) \equiv \mathbf{D}(y) + u(y, \mathbf{v}_x)\mathbf{b}(y) + w(y, \mathbf{v}_x)\mathbf{c}(y, \mathbf{v}_x) \ . \tag{25}$$

The zeroth order gyrocenter coordinate transformation

$$g_0 : z = (\mathbf{x}, \mathbf{v}, t) \mapsto \bar{Z} = (\bar{\mathbf{X}}, \bar{u}, \bar{w}, \bar{\theta}, t) \tag{26}$$

is defined by

$$\mathbf{x} \equiv \bar{\mathbf{X}} + \rho(\bar{\mathbf{X}}, \mathbf{v}) \ , \ \bar{u} \equiv u(\bar{\mathbf{X}}, \mathbf{v}) \ , \ \bar{w} \equiv w(\bar{\mathbf{X}}, \mathbf{v}) \ , \ \sin\bar{\theta} \equiv -\mathbf{c}(\bar{\mathbf{X}}) \cdot \mathbf{e}_1(\bar{\mathbf{X}}) \ , \ t \equiv t \ , \tag{27}$$

where $\mathbf{e}_1(\bar{\mathbf{X}})$ is an arbitrary unit vector field in the perpendicular direction, and it can depend on t as well. Consequently,

$$\mathbf{v} = \mathbf{D}(\bar{\mathbf{X}}) + \bar{u}\mathbf{b}(\bar{\mathbf{X}}) + \bar{w}\mathbf{c}(\bar{\mathbf{X}}) \ . \tag{28}$$

Substituting Eqs. (27) and (28) into Eq. (1), and expanding terms using the ordering Eqs. (10)-(13), we have

$$\gamma = \bar{\gamma}_0 + \bar{\gamma}_1 + \ O(\varepsilon^2) \ , \tag{29}$$

$$\bar{\gamma}_0 = (A_0 + \bar{u}\mathbf{b} + \mathbf{D}) \cdot d\bar{\mathbf{X}} + \frac{\bar{w}^2}{2B_0} d\bar{\theta} - \left(\frac{\bar{u}^2 + \bar{w}^2 + D^2}{2} + \phi_0 \right) dt \ , \tag{30}$$

$$\begin{aligned}
\bar{\gamma}_1 = & \left[\frac{\bar{w}}{B_0} \nabla\mathbf{a} \cdot \left(\bar{u}\mathbf{b} + \frac{\bar{w}\mathbf{c}}{2} \right) + \frac{1}{2}\rho \cdot \nabla\mathbf{B}_0 \times \rho - \frac{\bar{w}}{B_0} \nabla\mathbf{D} \cdot \mathbf{a} + \mathbf{A}_1(\bar{\mathbf{X}} + \rho) \right] \cdot d\bar{\mathbf{X}} \\
& + \left[-\frac{\bar{w}^3}{2B_0^3} \mathbf{a} \cdot \nabla\mathbf{B}_0 \cdot \mathbf{b} + \frac{\bar{w}}{B_0} \mathbf{A}_1(\bar{\mathbf{X}} + \rho) \cdot \mathbf{c} \right] d\bar{\theta} + \left[\frac{1}{B_0} \mathbf{A}_1(\bar{\mathbf{X}} + \rho) \cdot \mathbf{a} \right] d\bar{w} \\
& - \left[\phi_1(\bar{\mathbf{X}} + \rho) + \rho \cdot \frac{\partial \mathbf{D}}{\partial t} - \frac{1}{2}\rho \cdot \nabla\mathbf{E}_0 \cdot \rho - \left(\bar{u}\mathbf{b} + \frac{\bar{w}\mathbf{c}}{2} \right) \cdot \frac{\bar{w}}{B_0} \frac{\partial \mathbf{a}}{\partial t} \right] dt \ .
\end{aligned} \tag{31}$$

Here, every field is evaluated at $\bar{Z}$ and can depend on t, and exact terms of the form $d\alpha$ for some $\alpha : P \to R$ have been discarded because their insignificance in Hamilton's equation (2). It can be easily verified that $\partial\bar{\gamma}_0/\partial\bar{\theta} = 0$, but $\partial\bar{\gamma}_1/\partial\bar{\theta} \neq 0$. As discussed before, we now introduce a coordinate perturbation to the zeroth order gyrocenter coordinates $\bar{Z}$,

$$Z = g_1(\bar{Z}, \varepsilon), \ \ \frac{dg_1}{d\varepsilon}|_{\varepsilon=0} = G_1(\bar{Z}) \ , \tag{32}$$

such that $\partial\gamma_1/\partial\theta = 0$ in the first order gyrocenter coordinates $Z = (\mathbf{X}, u, w, \theta)$. Considering the fact that an arbitrary exact term of the form $d\alpha$ can be added to γ_1, we have

$$\gamma_1(Z) = \bar{\gamma}_1(Z) - i_{G_1(Z)} d\gamma_0(Z) + dS_1(Z) \ , \tag{33}$$

which, with $G_t = 0$, expands into

$$\begin{aligned}
\gamma_1(Z) = &\left[\mathbf{G}_{1\mathbf{X}} \times \mathbf{B}_0 - G_{1u}\mathbf{b} + \nabla S_1 + \frac{w}{B_0}\nabla\mathbf{a}\cdot\left(u\mathbf{b} + \frac{w\mathbf{c}}{2}\right) + \frac{1}{2}\rho\cdot\nabla\mathbf{B}_0\times\rho\right.\\
&\left.-\frac{w}{B_0}\nabla\mathbf{D}\cdot\mathbf{a} + \mathbf{A}_1(\mathbf{X}+\rho)\right]\cdot d\mathbf{X} + \left[\mathbf{G}_{1\mathbf{X}}\cdot\mathbf{b} + \frac{\partial S_1}{\partial u}\right]du + \left[\frac{w}{B_0}G_{1\theta} + \frac{\partial S_1}{\partial w} +\right.\\
&\left.+\frac{1}{B_0}\mathbf{A}_1(\mathbf{X}+\rho)\cdot\mathbf{a}\right]dw + \left[-\frac{w}{B_0}G_{1w} + \frac{\partial S_1}{\partial\theta} - \frac{w^3}{2B_0^3}\mathbf{a}\cdot\nabla\mathbf{B}_0\cdot\mathbf{b}\right.\\
&\left.+\frac{w}{B_0}\mathbf{A}_1(\mathbf{X}+\rho)\cdot\mathbf{c}\right]d\theta + \left[-\mathbf{E}_0\cdot\mathbf{G}_{1\mathbf{X}} + uG_{1u} + wG_{1w} + \frac{\partial S_1}{\partial t} - \phi_1(\mathbf{X}+\rho)\right.\\
&\left.-\rho\cdot\frac{\partial\mathbf{D}}{\partial t} + \frac{1}{2}\rho\cdot\nabla\mathbf{E}_0\cdot\rho + \left(u\mathbf{b} + \frac{w\mathbf{c}}{2}\right)\cdot\frac{w}{B_0}\frac{\partial\mathbf{a}}{\partial t}\right]dt\,. \qquad (34)
\end{aligned}$$

In Eq. (34), every field is evaluated at Z and can depend on t. Extensive calculations are needed to solve for G_1 and S_1 from the requirement that $\partial\gamma_1/\partial\theta = 0$. We list the results without giving the details of the derivation,

$$\begin{aligned}
\mathbf{G}_{1\mathbf{X}} = &-\frac{\partial S_1}{\partial u}\mathbf{b} + \frac{w^2}{2B_0^3}\mathbf{aa}\cdot\nabla\mathbf{B}_0 + \frac{wu}{B_0^2}(\nabla\mathbf{a}\cdot\mathbf{b})\times\mathbf{b} - \frac{w}{B_0^2}(\nabla\mathbf{D}\cdot\mathbf{a})\times\mathbf{b}\\
&+\frac{\nabla S_1 + \mathbf{A}_1(\mathbf{X}+\rho)}{B_0}\times\mathbf{b} \qquad (35)\\
G_{1u} = &\frac{w^2}{2B_0^2}\mathbf{a}\cdot\nabla\mathbf{B}_0\cdot\mathbf{c} + \frac{wu}{B_0}\mathbf{b}\cdot\nabla\mathbf{a}\cdot\mathbf{b} - \frac{w}{B_0}\mathbf{b}\cdot\nabla\mathbf{D}\cdot\mathbf{a} - \mathbf{b}\cdot[\nabla S_1 + \mathbf{A}_1(\mathbf{X}+\rho)]\,, \qquad (36)\\
G_{1w} = &\frac{B_0}{w}\frac{\partial S_1}{\partial\theta} - \frac{w^2}{2B_0^2}\mathbf{a}\cdot\nabla\mathbf{B}_0\cdot\mathbf{b} + \mathbf{c}\cdot\mathbf{A}_1(\mathbf{X}+\rho)\,, \qquad (37)\\
G_{1\theta} = &-\frac{B_0}{w}\frac{\partial S_1}{\partial w} - \frac{1}{w}\mathbf{a}\cdot\mathbf{A}_1(\mathbf{X}+\rho)\,. \qquad (38)
\end{aligned}$$

The determining equation for S_1 is

$$\begin{aligned}
&\frac{\partial S_1}{\partial t} + \left(\frac{\mathbf{E}_0\times\mathbf{b}}{B_0} + u\mathbf{b}\right)\cdot\nabla S_1 + E_{0\parallel}\frac{\partial S_1}{\partial u} + B_0\frac{\partial S_1}{\partial\theta} = \mathbf{E}_{0\perp}\cdot\left[\frac{w^2}{2B_0^3}\widetilde{\mathbf{aa}}\cdot\nabla B_0\right.\\
&\left.+\frac{wu}{B_0^2}(\nabla\mathbf{a}\cdot\mathbf{b})\times\mathbf{b} - \frac{w}{B_0^2}(\nabla\mathbf{D}\cdot\mathbf{a})\times\mathbf{b}\right] - \frac{w^2u}{2B_0^2}\nabla\mathbf{B}_0{:}\widetilde{\mathbf{ca}} - \frac{wu^2}{B_0}\mathbf{b}\cdot\nabla\mathbf{a}\cdot\mathbf{b}\\
&+\frac{wu}{B_0}\mathbf{b}\cdot\nabla\mathbf{D}\cdot\mathbf{a} + \frac{w^3}{2B_0^2}\mathbf{a}\cdot\nabla\mathbf{B}_0\cdot\mathbf{b} + \frac{w}{B_0}\mathbf{a}\cdot\frac{\partial\mathbf{D}}{\partial t} + \widetilde{\psi_1} - \frac{w^2}{2B_0^2}\nabla\mathbf{E}_0{:}\widetilde{\mathbf{aa}} + \frac{uw}{B_0}\mathbf{a}\cdot\frac{\partial\mathbf{b}}{\partial t}\,. \qquad (39)
\end{aligned}$$

The G_1 and S_1 in Eqs. (35)-(39) remove the θ-dependence in γ_1, i.e.,

$$\gamma(Z) = \gamma_0(Z) + \gamma_1(Z)\,, \tag{40}$$

$$\gamma_0 = (A_0 + u\mathbf{b} + \mathbf{D}) \cdot d\mathbf{X} + \frac{w^2}{2B_0} d\theta - \left(\frac{u^2 + w^2 + D^2}{2} + \phi_0\right) dt\,, \tag{41}$$

$$\gamma_1(Z) = -\frac{w^2}{2B_0}\mathbf{R} \cdot d\mathbf{X} - H_1 dt\,, \tag{42}$$

$$H_1 = \mathbf{E}_0 \cdot \frac{w^2}{2B_0^3}\nabla\mathbf{B}_0 + \frac{w^2 u}{4B_0}\mathbf{b} \cdot \nabla \times \mathbf{b} + \langle\psi_1\rangle - \frac{w^2}{4B_0^2}(\nabla \cdot \mathbf{E}_0 - \mathbf{bb} : \nabla\mathbf{E}_0) - \frac{w^2}{2B_0}R_0\,, \tag{43}$$

$$\mathbf{R} \equiv \nabla\mathbf{c} \cdot \mathbf{a}\,,\; R_0 \equiv -\frac{\partial\mathbf{c}}{\partial t} \cdot \mathbf{a}\,, \tag{44}$$

$$\psi_1 \equiv \phi_1(\mathbf{X} + \rho) - \frac{\mathbf{E}_{0\perp} \times \mathbf{b}}{B_0} \cdot \mathbf{A}_1(\mathbf{X} + \rho) - w\mathbf{c} \cdot \mathbf{A}_1(\mathbf{X} + \rho)\,, \tag{45}$$

$$\langle\alpha\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} \alpha d\theta\,,\; \widetilde{\alpha} \equiv \alpha - \langle\alpha\rangle\,. \tag{46}$$

The perturbation procedure can be carried out to the second order by introducing another coordinate transformation $g_2 : g_1(\bar{Z}) \to Z = g_2 \circ g_1(\bar{Z})$. For simplicity, we only display the results up to $O(\varepsilon_1^2)$.

$$\gamma_2 = \langle\psi_2\rangle\, dt\,, \tag{47}$$

$$\psi_2 \equiv \frac{1}{2}\mathbf{E}_{0\perp} \cdot \left[\left(\mathbf{G}_1^\dagger \times \mathbf{B}_1\right) \times \mathbf{b}\right] - \frac{1}{2}(u\mathbf{b} + w\mathbf{c}) \cdot \left(\mathbf{G}_1^\dagger \times \mathbf{B}_1\right) + \mathbf{E}_1^\dagger \cdot \mathbf{G}_1^\dagger \tag{48}$$

$$\mathbf{G}_1^\dagger \equiv \mathbf{G}_{1\mathbf{x}} + \frac{\mathbf{a}}{B_0}G_{1w} + \frac{\mathbf{c}w}{B_0}G_{1\theta}\,, \tag{49}$$

$$\mathbf{E}_1^\dagger \equiv -\nabla\phi_1 - \frac{\partial\mathbf{A}_1}{\partial t} - \nabla\langle\psi_1\rangle\,. \tag{50}$$

The corresponding vector field G_2 for g_2 is

$$\mathbf{G}_{2\mathbf{X}} = -\frac{\partial S_2}{\partial u}\mathbf{b} + \frac{1}{B_0}\nabla S_2 \times \mathbf{b} + \frac{1}{2B_0}\left(\mathbf{G}_1^\dagger \times \mathbf{B}_1\right) \times \mathbf{b}\,, \tag{51}$$

$$G_{2u} = \mathbf{b} \cdot \nabla S_2 + \frac{1}{2}\mathbf{b} \cdot \left(\mathbf{G}_1^\dagger \times \mathbf{B}_1\right)\,, \tag{52}$$

$$G_{2w} = \frac{B_0}{w}\frac{\partial S_2}{\partial\theta} + \frac{1}{2}\mathbf{c} \cdot \left(\mathbf{G}_1^\dagger \times \mathbf{B}_1\right)\,, \tag{53}$$

$$G_{2\theta} = -\frac{B_0}{w}\frac{\partial S_2}{\partial w} + \frac{1}{2}\mathbf{a} \cdot \left(\mathbf{G}_1^\dagger \times \mathbf{B}_1\right)\,, \tag{54}$$

and the gauge function S_2 satisfies

$$\frac{\partial S_2}{\partial t} + \left(\frac{\mathbf{E}_0 \times \mathbf{b}}{B_0} + u\mathbf{b}\right) \cdot \nabla S_1 + E_{0\parallel}\frac{\partial S_2}{\partial u} + B_0\frac{\partial S_2}{\partial\theta} = \widetilde{\psi_2}\,. \tag{55}$$

A particle's trajectory (worldline) is given by a vector field τ on the phase space P which satisfies

$$i_\tau d\gamma = 0\,. \tag{56}$$

The gyrocenter motion equation in its conventional form can be obtained through

$$\frac{d\mathbf{X}}{dt} = \frac{\tau_{\mathbf{X}}}{\tau_t}\,,\; \frac{du}{dt} = \frac{\tau_u}{\tau_t}\,,\; \frac{dw}{dt} = \frac{\tau_w}{\tau_t}\,,\; \frac{d\theta}{dt} = \frac{\tau_\theta}{\tau_t}\,. \tag{57}$$

After some calculation, we obtain the following explicit expressions up to order $O(\varepsilon_0)$ and $O(\varepsilon_1^2)$ for gyrocenter dynamics,

$$\frac{d\mathbf{X}}{dt} = \frac{\mathbf{B}^\dagger}{\mathbf{b}\cdot\mathbf{B}^\dagger}(u + \frac{\mu}{2}\mathbf{b}\cdot\nabla\times\mathbf{b}) - \frac{\mathbf{b}\times\mathbf{E}^\dagger}{\mathbf{b}\cdot\mathbf{B}^\dagger} \ , \tag{58}$$

$$\frac{du}{dt} = \frac{\mathbf{B}^\dagger\cdot\mathbf{E}^\dagger}{\mathbf{B}^\dagger\cdot\mathbf{b}} \ , \tag{59}$$

$$\begin{aligned}\frac{d\theta}{dt} &= B_0 + \mathbf{R}\cdot\frac{d\mathbf{X}}{dt} - R_0 + \frac{\mathbf{E}_0\cdot\nabla B_0}{B_0^2} + \frac{u}{2}\mathbf{b}\cdot\nabla\times\mathbf{b} \\ &+ \frac{\partial}{\partial\mu}\left\langle\psi_1+\psi_2\right\rangle - \frac{1}{2B_0}\left[\nabla\cdot\mathbf{E}_0 - \mathbf{bb}:\nabla\mathbf{E}_0\right]\end{aligned} \tag{60}$$

$$\frac{d\mu}{dt} = 0 \ , \ \mu \equiv \frac{w^2}{2B_0} \ , \tag{61}$$

$$\mathbf{B}^\dagger \equiv \nabla\times(\mathbf{A}_0 + u\mathbf{b} + \mathbf{D}) \ , \tag{62}$$

$$\mathbf{E}^\dagger \equiv \mathbf{E}_0 - \nabla\left[\mu B_0 + \frac{D^2}{2} + \left\langle\psi_1+\psi_2\right\rangle\right] - u\frac{\partial\mathbf{b}}{\partial t} - \frac{\partial\mathbf{D}}{\partial t} \ . \tag{63}$$

The modified fields $\mathbf{B}^\dagger$ and $\mathbf{E}^\dagger$ can be viewed as those generated by a modified potential $A^\dagger = (\phi^\dagger, \mathbf{A}^\dagger)$,

$$\phi^\dagger \equiv \phi_0 + \mu B_0 + \frac{D^2}{2} + \left\langle\psi_1+\psi_2\right\rangle \ , \tag{64}$$

$$\mathbf{A}^\dagger \equiv \mathbf{A}_0 + u\mathbf{b} + \mathbf{D} \ , \tag{65}$$

$$\mathbf{B}^\dagger = \nabla\times\mathbf{A}^\dagger, \ \mathbf{E}^\dagger = -\nabla\phi^\dagger - \frac{\partial\mathbf{A}^\dagger}{\partial t} \ . \tag{66}$$

In the right hand sides of Eqs. (58)-(66), every field is evaluated at the gyrocenter coordinate Z and can depend on t. Note that in Eq. (58) the curvature drift is hidden in the first term on the right hand side. The second term is the Banos drift [33]. The last term is the generalized $\mathbf{E}\times\mathbf{B}$ drift that contains the gradient $\mathbf{B}$ drift along with several other terms, such as the space-time inhomogeneities of $\mathbf{E}_0$, which also induces cross-$\mathbf{B}$ drift. The requirement $\partial\gamma/\partial\theta = 0$ does not uniquely determine the coordinate perturbation G and the gauge function S, and therefore the gyrocenter coordinates. There are freedoms in defining the zeroth order gyrocenter coordinates as well. For example, in Ref. [34], a different definition of the zeroth order gyrocenter coordinates are used, which results in more terms in the expression for $\bar{\gamma}_1$. We will call the freedoms in selecting the gyrocenter coordinates gyrocenter gauges. In Eq. (44), $\mathbf{R}$ and R_0 are θ-independent, even though $\mathbf{a}$ and $\mathbf{c}$ are θ-dependent. Let $R = (R_0, \mathbf{R})$, $X = (t, \mathbf{X})$, and $\nabla = (-\partial/\partial t, \nabla)$. The γ in Eq. (40) is invariant under the following group of transformation

$$R \longrightarrow R' + \nabla\delta(X) \ , \ \theta \longrightarrow \theta' + \delta(X) \ . \tag{67}$$

Apparently, this is a gauge group associated how the gyrophase θ is measured. Naturally, an appropriate name for this gauge would be gyro-gauge.

4 Pullback transformation of the distribution function

Even though the γ in Eq. (40) is gyro-gauge invariant, it does not need to be. Different gyro-center gauges can be chosen such that γ is not gyro-gauge invariant. The gyrocenter coordinate system constructed is just a useful coordinate system for physics, but not the physics itself. It can depend on the gauges (freedoms) we choose, as long as it is useful. Gyrocenter coordinate system and the gyrokinetic equation are not the total of physics under investigation. What is gauge invariant is the system of gyrokinetic equation and the gyrokinetic Maxwell equations. The key element which makes this gyrokinetic system gauge invariant is the pullback transformation of the distribution function associated with the gyrocenter coordinate system. Kinetic theory deals with particle distribution function f, which is a function defined on the phase space P, $f : P \to R$. To complete the equation system, the familiar density and momentum velocity integrals are needed for Maxwell's equations at $x \in M$.

$$j(x) = \int f(z) v \, d^3\mathbf{v} \,,$$

where $j(x) = [-n(x), \mathbf{j}(x)]$ is the spacetime flux and $v \equiv [-1, \mathbf{v}]$. In gyrokinetic theory, however, the $\mathbf{X}$ coordinates in the gyrocenter coordinate system are not coordinates for spacetime. The gyrocenter transformation $g : z \longmapsto Z$ does not preserve the coordinates $\mathbf{x}$ for the spacetime M. However, no matter which coordinate system is used, the moment integrals are still defined at each x. For the new coordinate system Z to be useful, it is necessary to know the construction of $j(x)$ in it. To be specific, the current scenario is that the distribution function f is known in the transformed coordinate system Z as $F(Z)$. Given $F(Z)$, we need to pull back the distribution function $F(Z)$ into $f(z)$,

$$f(z) = g^* \left[F(Z) \right] = F\left(g(z) \right) . \tag{68}$$

Considering the asymptotic nature of the construction of the gyrocenter transformation g,

$$g = g_2 \circ g_1 \circ g_0 \,, \; g_0 : z \longmapsto \bar{Z} \,, \; g_2 \circ g_1 : \bar{Z} \longmapsto Z \,, \tag{69}$$

we have the following pull-back transformation

$$\begin{aligned} f(z) &= g^* F(Z) = g_0^* \circ g_1^* \circ g_2^* F(Z) = g_0^* F\left[g_2(g_1(\bar{Z})) \right] \\ &= g_0^* \left[F(\bar{Z}) + G_1 \cdot \nabla F(\bar{Z}) + \frac{1}{2} \left(G_1 \cdot \nabla \right)^2 F(\bar{Z}) + G_2 \cdot \nabla F(\bar{Z}) + O(\varepsilon^3) \right] \\ &= \left[F(\bar{Z}) + G_1 \cdot \nabla F(\bar{Z}) + \frac{1}{2} \left(G_1 \cdot \nabla \right)^2 F(\bar{Z}) + G_2 \cdot \nabla F(\bar{Z}) \right]_{\bar{Z} \rightarrow g_0(z)} + O(\varepsilon^3) \,. \end{aligned} \tag{70}$$

In Eq. (70), the pullbacks associated with g_1 and g_2 are treated perturbatively, consistent with the perturbative nature of g_1 and g_2. However, at this stage there is no asymptotic expansion for the pullback associated with g_0, because g_0 is not a perturbative coordinate transformation. The importance of the pullback transformation can't be over-emphasized. Without this vital element, many important physics will be lost in the gyrokinetic theory. We will discuss the physics of the pullback transformation in the next section.

5 General gyrokinetic Vlasov-Maxwell equations

After constructing the gyrocenter coordinates and the corresponding pullback transformation, we are ready to cast the Vlasov-Maxwell equations in the gyrocenter coordinates to obtain the general gyrokinetic Vlasov-Maxwell equations. The gyrokinetic Vlasov equation is simply the Vlasov equation $df(\tau) = 0$ in the gyrocenter coordinates Z, which is explicitly

$$\frac{dZ_j}{dt} \frac{\partial F}{\partial Z_j} = 0 \,, \; (0 \leq j \leq 6) \;. \tag{71}$$

Because

$$\frac{\partial}{\partial \theta} \left(\frac{dZ}{dt} \right) = 0 \,, \tag{72}$$

the gyrokinetic equation can be easily split into two parts

$$F = \langle F \rangle + \widetilde{F} \,, \tag{73}$$

$$\frac{\partial \langle F \rangle}{\partial t} + \frac{d\mathbf{X}}{dt} \cdot \nabla_{\mathbf{X}} \langle F \rangle + \frac{du}{dt} \frac{\partial \langle F \rangle}{\partial u} = 0 \,, \tag{74}$$

$$\frac{\partial \widetilde{F}}{\partial t} + \frac{d\mathbf{X}}{dt} \cdot \nabla_{\mathbf{X}} \widetilde{F} + \frac{du}{dt} \frac{\partial \widetilde{F}}{\partial u} + \frac{d\theta}{dt} \frac{\partial \widetilde{F}}{\partial \theta} = 0 \,, \tag{75}$$

where $d\mathbf{X}/dt$, du/dt, and $d\theta/dt$ are given by Eqs. (58)-(60). The gyrokinetic Maxwell's equation can be written as

$$\nabla^2 \mathbf{A} = 4\pi \sum_s q_s \int \left[F(\bar{Z}) + G_1 \cdot \nabla F(\bar{Z}) + \frac{1}{2} \left(G_1 \cdot \nabla \right)^2 F(\bar{Z}) + G_2 \cdot \nabla F(\bar{Z}) \right]_{\bar{Z} \to g_0(z)} \mathbf{v}\, d^3\mathbf{v} \,, \quad (76)$$

$$\nabla^2 \phi = -4\pi \sum_s q_s \int \left[F(\bar{Z}) + G_1 \cdot \nabla F(\bar{Z}) + \frac{1}{2} \left(G_1 \cdot \nabla \right)^2 F(\bar{Z}) + G_2 \cdot \nabla F(\bar{Z}) \right]_{\bar{Z} \to g_0(z)} d^3\mathbf{v} \,. \quad (77)$$

We emphasize that Eqs. (76) and (77) are not new equations which contain different physics than the original Maxwell's equations with moment integrals. The more appropriate name for this equation should be "Maxwell's equations with pulled-back distribution from the gyrocenter coordinates".

The spirit of the general gyrokinetic theory is to decouple the gyro-phase dynamics from the rest of particle dynamics by finding the gyro-symmetry, instead of "averaging out" the "fast gyro-motion". This objective is accomplished by asymptotically constructing a good coordinate system using the Lie coordinate perturbation method enabled by the geometric nature of the phase space dynamics. The general gyrokinetic Vlasov-Maxwell equations are not developed as a new set of equations, but rather as the Vlasov-Maxwell equations in the gyrocenter coordinates. Because the general gyrokinetic system developed is geometrically the same as the Vlasov-Maxwell equations, all the coordinate independent properties of the Vlasov-Maxwell equations, such as energy conservation, momentum conservation and phase space volume conservation, are automatically carried over to the general gyrokinetic system.

The gyrophase dependent $\widetilde{F}$ can be decoupled from the system. Letting $\widetilde{F} = 0$, Eqs. (74), (76), and (77) form a close system for $\langle F \rangle$ and $A = (-\phi, \mathbf{A})$. We note that $\widetilde{F} = 0$ does not imply that $\widetilde{f} = 0$. The distribution function f in the laboratory coordinates becomes gyrophase dependent through the pullback transformation (70) and G. Indeed, the pullback transformation contains significant amount of important physics.

The most famous example is the "polarization drift density" in the gyrokinetic Poisson equation [12,27], which has played an important role in the development of gyrokinetic simulation methods using explicit algorithm [35–42]. It is interesting to note that a term almost the same exists in the Poisson equation for the implicit algorithm [43]. However, the interpretation of this term in the context of implicit algorithm is algorithmic. This a example of the consistency between elegant theories and efficient algorithms. From the viewpoint of modern gyrokinetic theory, the "polarization drift density" can be rigorously derived from the first principles in the most general form. Actually, it is just one of the many terms that appear naturally in the pullback transformation. To illustrate the importance and the basic feature of the pullback transformation for edge plasmas, we carry out the pullback transformation in the gyrokinetic Poisson equation up to the first order of the gyrocenter coordinate perturbation for low frequency, electrostatic physics. In addition, we will take a sub-ordering for edge plasmas to keep only the weak inhomogeneities associated with $\mathbf{E}_0$ in the pullback transformation. Under these assumptions, the gauge function S_1 can be solved for as

$$S_1 = \frac{w}{B_0^3} \mathbf{E} \cdot (\nabla \mathbf{D} \cdot \mathbf{c}) \times \mathbf{b} - \frac{wu}{B_0^2} \mathbf{b} \cdot \nabla \mathbf{D} \cdot \mathbf{c} + \int \tilde{\phi}_1 d\theta + \frac{w^2}{4B_0^2} \nabla \mathbf{D} \cdot \mathbf{ac} \,, \quad (78)$$

from which we can calculate the first order pullback transformation $\mathbf{G}_1 \cdot \nabla F\left(Z\right)$. After some detailed calculation, the Poisson equation (in unnormalized units) can be reduced to

$$\nabla^2 \phi(\mathbf{x}) = -4\pi \sum_s q_s \left[N + N_{\phi_0} + N_{\phi_1} \right] \,, \quad (79)$$

$$N(\mathbf{x}) \equiv \int 2\pi w \, dw du \, I_0\left(\rho \nabla_\perp \right) F\left(\mathbf{x}, w, u \right) \,, \quad (80)$$

$$N_{\phi_0}(\mathbf{x}) \equiv \frac{1}{\Omega_0^2} \left(\mathbf{e}_1 \mathbf{e}_1 + \mathbf{e}_2 \mathbf{e}_2 \right) : \nabla \left[n(\mathbf{x}) \left(\mathbf{D} + V_\parallel(\mathbf{x}) \mathbf{b} \right) \cdot \nabla \mathbf{D} \right] \,, \quad (81)$$

$$N_{\phi_1}(\mathbf{x}) \equiv -\frac{q}{m}\phi_1(\mathbf{x})\sum_{i=1}^{\infty}\frac{2i}{(i!)^2}\left(\frac{\nabla_\perp^2}{4\Omega_0^2}\right)^i M_{2i-2}(\mathbf{x}) \tag{82}$$

$$+\frac{q}{m}\sum_{i,j=0}^{\infty}\frac{2(i+j)}{(i!j!)^2}\left(\frac{\nabla_\perp^2}{4\Omega_0^2}\right)^i\left[M_{2(i+j)-2}(\mathbf{x})\left(\frac{\nabla_\perp^2}{4\Omega_0^2}\right)^j\phi_1(\mathbf{x})\right],$$

$$I_0(\rho\nabla_\perp) \equiv \sum_{i=0}^{\infty}\frac{1}{(i!)^2}\left(\frac{\nabla_\perp^2 w^2}{4\Omega_0^2}\right)^i,\ \Omega_0 \equiv \frac{qB_0}{mc}, \tag{83}$$

$$n(\mathbf{x}) \equiv \int 2\pi w dw du\, F(\mathbf{x},w,u)\ , \tag{84}$$

$$V_\parallel(\mathbf{x}) \equiv \frac{1}{n(\mathbf{x})}\int 2\pi w dw du\, F(\mathbf{x},w,u)\ , \tag{85}$$

$$M_i(\mathbf{x}) \equiv \int 2\pi w dw du\, w^i F(\mathbf{x},w,u)\ . \tag{86}$$

Here, $\mathbf{e}_1$ and $\mathbf{e}_2$ are two perpendicular directions; $n(\mathbf{x})$, $V_\parallel(\mathbf{x})$, and $M_n(\mathbf{x})$ are moments calculated from $F(Z)$, which is the total distribution function. All quantities are evaluated at particle coordinates $\mathbf{x}$. Obviously, $N_{\phi_0}(\mathbf{x})$ is the leading order pullback associated with the inhomogeneities of the background electric field, which capture the important physics of gyro-orbit squeezing effect due to the large $\mathbf{E}_r$ shearing in the edge region. $N_{\phi_1}(\mathbf{x})$ is pullback associated with the short wavelength small amplitude fluctuation. When the scale-length of ϕ_1 is bigger than the gyroradius, it is valid to keep the leading order of these terms,

$$N_{\phi_1}(\mathbf{x}) \equiv \frac{q}{m\Omega_0^2}\left(\nabla_\perp n\cdot\nabla_\perp\phi_1 + n\nabla_\perp^2\phi_1\right) + O\left(\rho^4\nabla_\perp^4\right) = \frac{q}{m\Omega_0^2}\nabla_\perp\cdot(n\nabla_\perp\phi_1) + O\left(\rho^4\nabla_\perp^4\right)\ , \tag{87}$$

which is the "polarization drift density". When the scale-length of ϕ_1 is comparable to the gyroradius, which is often the case for edge plasmas, all the terms on the right hand side of Eq. (82) need to be kept for the finite Larmour radius effect. The "polarization drift density" should be replaced by the more general expression in Eq. (82), systematically derived from the pullback transformation. Sosenko *et al* [23] discussed the possibility of including the polarization drift due to ϕ_1 in the gyrocenter dynamics rather than in the Poisson equation.

If we ignore the spatial variation of the moments $M_n(\mathbf{x})$ associated with the total distribution function $F(Z)$, the expression for N_{ϕ_1} can be simplified into

$$N_{\phi_1}(\mathbf{x}) = \frac{q}{m}\sum_{i,j=0}^{\infty}\frac{2(i+j)}{(i!j!)^2}M_{2(i+j)-2}(\mathbf{x})\left(\frac{\nabla_\perp^2}{4\Omega_0^2}\right)^{i+j}\phi_1(\mathbf{x}) \tag{88}$$

$$= \left[-2\pi\int\frac{\partial F}{\partial w}I_0^2(\rho\nabla_\perp)\,dw du - 2\pi\int F[w=0]du\right]\frac{q}{m}\phi_1(\mathbf{x})\ .$$

If we further assume F is Maxwellian in the transverse direction,

$$F = n\left(\frac{1}{\sqrt{2\pi}v_t}\right)^2\exp\left(-\frac{w^2}{2v_t^2}\right)F_\parallel(u) \tag{89}$$

with

$$\int_{-\infty}^{+\infty}F_\parallel(u)du = 1\ , \tag{90}$$

then

$$N_{\phi_1}(\mathbf{x}) = \frac{qn}{mv_t^2}\left[e^{-b}I_0(b) - 1\right]\phi_1(\mathbf{x})\ , \tag{91}$$

$$b \equiv \frac{v_t^2\nabla_\perp^2}{\Omega_0^2}\ . \tag{92}$$

Finally, we need to separate the long wavelength, large amplitude component of $N(\mathbf{x})$ from the short wavelength, small amplitude component,

$$N(\mathbf{x}) = N_0(\mathbf{x}) + N_1(\mathbf{x}) \ , \tag{93}$$

where $N_0(\mathbf{x})$ is the long wavelength, large amplitude component and $N_1(\mathbf{x})$ is the short wavelength, small amplitude component. The gyrokinetic Poisson equation is then split into

$$\nabla^2 \phi_0(\mathbf{x}) = -4\pi \sum_s q_s N_0 \ , \tag{94}$$

$$\nabla^2 \phi_1(\mathbf{x}) = -4\pi \sum_s q_s \left[N_1 + N_{\phi_0} + N_{\phi_1} \right] \ . \tag{95}$$

The pullback transformation in Ampere's law is equally important. Many more physics, which were previously thought to be incompatible with the gyrokinetic theory, have been included into the gyrokinetic theory by applying the pullback transformation. For example, it has been shown that gyrokinetic theory can describe all the plasma waves in magnetized plasmas, including the high frequency cyclotron waves and compressional Alfvén wave [22].

Acknowledgements This research was supported by the U.S. Department of Energy under contract AC02-76CH03073, and by Lawrence Livermore National Laboratory's LDRD Project 04-SI-03, Kinetic Simulation of Boundary Plasma Turbulent Transport. We are grateful to Prof. Ronald C. Davidson and Dr. Janardhan Manickam for their continuous support of this work. We thank Drs. Peter J. Catto, Bruce I. Cohen, Andris Dimits, Alex Friedman, Gregory W. Hammet, W. Wei-li Lee, Lynda L. Lodestro, Thomas D. Rognlien, Philip B. Snyderfor, and William M. Tang for fruitful discussions.

References

[1] E.A. Frieman, R.C. Davidson, and B. Langdon, Physics of Fluids **9**, 1475 (1966).
[2] R.C. Davidson, Physics of Fluids **10**, 669 (1967).
[3] P.H. Rutherford and E.A. Frieman, Physics of Fluids **11**, 569 (1968).
[4] J.B. Taylor and R.J. Hastie, Plasma Physics **10**, 479 (1968).
[5] R.G. Littlejohn, Journal of Mathematical Physics **20**, 2445 (1979).
[6] P.J. Catto, Plasma Physics and Controlled Fusion **20**, 719 (1978).
[7] T.M. Antonsen and B. Lane, Physics of Fluids **23**, 1205 (1980).
[8] P.J. Catto, W.M. Tang, and D.E. Baldwin, Plasma Physics and Controlled Fusion **23**, 639 (1981).
[9] R.G. Littlejohn, Physics of Fluids **24**, 1730 (1981).
[10] E.A. Frieman and L. Chen, Physics of Fluids **25**, 502 (1982).
[11] R.G. Littlejohn, Physica Scripta **T2**, 119 (1982).
[12] D.H.E. Dubin, J.A. Krommes, C. Oberman, and W.W. Lee, Physics of Fluids **26**, 3524 (1983).
[13] T.S. Hahm, Physics of Fluids **31**, 2670 (1988).
[14] A. Brizard, Journal of Plasma Physics **41**, 541 (1989).
[15] A.M. Dimits, L.L. Lodestro, and D.H.E. Dubin, Physics of Fluids B-Plasma Physics **4**, 274 (1992).
[16] T.S. Hahm, Physics of Plasmas **3**, 4658 (1996).
[17] H. Qin, *Gyrokinetic Theory and Computational Methods for Electromagnetic Perturbations in Tokamaks*, PhD thesis, Princeton University, Princeton, NJ 08540, 1998.
[18] H. Qin, W.M. Tang, and G. Rewoldt, Physics of Plasmas **5**, 1035 (1998).
[19] H.Qin, W.M. Tang, W.W. Lee, and G. Rewoldt, Physics of Plasmas **6**, 1575 (1999).
[20] H. Qin, W.M. Tang, and G. Rewoldt, Physics of Plasmas **6**, 2544 (1999).
[21] H. Qin, W.M. Tang, G. Rewoldt, and W.W. Lee, Physics of Plasmas **7**, 991 (2000).
[22] H. Qin, W.M. Tang, and W.W. Lee, Physics of Plasmas **7**, 4433 (2000).
[23] P.P. Sosenko, P. Bertrand, and V.K. Decyk, Physica Scripta **64**, 264 (2001).
[24] H. Qin and W.M. Tang, Physics of Plasmas **11**, 1052 (2004).
[25] H. Qin, Fields Institute Communications **46**, 171 (2005).
[26] G.F. Chew, M.L. Goldberger, and F.E. Low, Proceedings of the Royal Society of London Series A-Mathematical and Physical Sciences **236**, 112 (1956).
[27] W.W. Lee, Physics of Fluids **26**, 556 (1983).
[28] J.W. Connor, R.J. Hastie, H.R. Wilson, and R.L. Miller, Physics of Plasmas **5**, 2687 (1998).
[29] P.B. Snyder et al., Physics of Plasmas **9**, 2037 (2002).

[30] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, pages 240–241, Springer-Verlag, New York, second edition, 1989.
[31] J.R. Cary, Physics Report **79**, 129 (1981).
[32] J.R. Cary and R.G. Littlejohn, Annals of Physics **151**, 1 (1983).
[33] A.J. Banos, J. Plasma Physics **1**, 305 (1967).
[34] R.G. Littlejohn, Journal of Plasma Physics **29**, 111 (1983).
[35] B.I. Cohen, T.J. Williams, A.M. Dimits, and J.A. Byers, Physics of Fluids B-Plasma Physics **5**, 2967 (1993).
[36] S.E. Parker, W.W. Lee, and R.A. Santoro, Physical Review Letters **71**, 2042 (1993).
[37] A.M. Dimits, T.J. Williams, J.A. Byers, and B.I. Cohen, Physical Review Letters **77**, 71 (1996).
[38] R.D. Sydora, V.K. Decyk, and J.M. Dawson, Plasma Physics and Controlled Fusion **38**, A281 (1996).
[39] Z. Lin, T.S. Hahm, W.W. Lee, W.M. Tang, and R.B. White, Science **281**, 1835 (1998).
[40] W. Dorland, F. Jenko, M. Kotschenreuther, and B.N. Rogers, Physical Review Letters **85**, 5579 (2000).
[41] J. Candy and R.E. Waltz, Physical Review Letters **91**, 045001 (2003).
[42] Y. Chen and S.E. Parker, Journal of Computational Physics **189**, 463 (2003).
[43] A. Friedman, A.B. Langdon, and B.I. Cohen, Comments Plasma Phys. Controlled Fusion **6**, 225 (1981).