On the gyrokinetic equilibrium

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Recent developments in gyrokinetic-magnetohydrodynamics (MHD) theory and in electromagnetic gyrokinetic particle simulations raise the question of consistency between the gyrokinetic model and the fluid model. Due to the special characteristics of the guiding center coordinates, it is a nontrivial exercise to show this consistency. In this paper it is shown, in a very general setting, that the gyrokinetic theory and the fluid equations do give an equivalent description of plasma equilibrium ($\partial/\partial t=0$). The fluid continuity equation and momentum equation for equilibrium plasmas are recovered entirely from the gyrokinetic theory. However, it was Spitzer who first realized the importance of consistency between guiding-center motion and fluid equations. In particular, he studied the ''apparent paradoxical result'' regarding the difference between perpendicular particle flow and guiding-center flow, which will be referred to as the Spitzer paradox in this paper. By recovering the fluid equations from the gyrokinetic theory, we automatically resolve the Spitzer paradox, whose essence is how the perpendicular current and flow are microscopically generated from particles' guiding-center motion. The mathematical construction in the gyrokinetic theory which relates observable quantities in the laboratory frame to the distribution function in the guiding-center coordinates is consistent with Spitzer's original physical picture, while today's gyrokinetic-MHD theory covers a much wider range of problems in a much more general and quantitative way. © *2000 American Institute of Physics.* [S1070-664X(00)05202-2]

I. INTRODUCTION

Since Littlejohn's work on the non-canonical Hamiltonian structure for the guiding center motion,[1–3] modern gyrokinetic theory has gradually been established.[4–14] Particle simulations based on gyrokinetic models have successfully been used to study electrostatic microturbulence and transport in tokamak plasmas.[15–21] Recently, gyrokinetic theory has been developed to study fluid types of modes in general geometry. These modes, such as Toroidal Alfvén Eigenmodes (TAEs) and Compressional Alfvén Waves (CAWs) in tokamak geometry, were studied previously only by fluid equations. The advance of gyrokinetic theory in this direction blurs the conventional boundary between the gyrokinetic theory and the fluid models. The ideal behind the so-called gyrokinetic-magnetohydrodynamic (MHD) theory is to investigate all the macroscopic fluid phenomena entirely from the gyrokinetic side.[11–14] (In this paper, gyrokinetic-MHD theory is defined to be the analysis of electromagnetic fluid motions from the gyrokinetic point of view without utilizing the fluid equations. Here, MHD refers to plasma fluid dynamics in general.) Obviously, there should be no discrepancy between the gyrokinetic and the fluid pictures if the gyrokinetic-MHD theory is meant to be correct in describing the fluid motions. The importance of gyrokinetic-MHD theory is manifested as modern fusion devices approach ignition conditions, where significant numbers of energetic particles are generated by fusion reactions, and the restrictions of fluid models, including the lack of kinetic resonances and inaccuracy in parallel dynamics, become prominent. Fully kinetic models are needed. Gyrokinetic-MHD theory can provide a rigorous, self-consistent, and nonperturbative formalism for those modes having both strong kinetic and strong fluid characteristics, such as fishbone modes and hot particle driven TAEs. For this purpose, it is crucial that gyrokinetic-MHD theory can recover (but is not limited to) fluid models.

What gyrokinetic theory offers is a simplified version of the Vlasov–Maxwell system by utilizing the fact that, in strongly magnetized plasmas, the particle's gyroradius is much smaller than the scale length of the magnetic field: $\epsilon_B \equiv |\rho/L_B| \ll 1$, where $L_B \equiv |B/\nabla \mathbf{B}|$. More fundamentally, gyrokinetic theory is about the construction of a gyrocenter coordinate system in which the particle's gyromotion is decoupled from the rest of the particle dynamics, and deriving the Vlasov–Maxwell equation system in this special coordinate system. (According to the convention in Refs. 9–14, guiding-center coordinates refer to this special coordinates in the magnetostatic case, while gyrocenter coordinates refer to their counterparts when there are electromagnetic perturbations in the system. In the static case, the gyrocenter coordinate system is the guiding-center coordinate system. Since we only consider static cases in this paper, we will use these two terms interchangeably.) Even though all coordinate systems are geometrically equivalent, the algebra involved is different depending on the specific problems being studied. For applications in magnetized plasmas, the advantage of the gyrocenter coordinate system lies at the fact that, in this coordinate system, the fast time scale gyromotion is decoupled from the particle's gyrocenter orbit dynamics. For low frequency electrostatic modes and shear Alfvén modes, the gyromotion is not important and is naturally decoupled from the system, as if it completely ''averaged out.'' The advantage of the gyrokinetic approach does not come without a

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price. Because of the special characteristics of the guiding center coordinates, it is a nontrivial exercise to show the consistency between the gyrokinetic system and the fluid equations. Here, the fluid equations mean the moment equations of the Vlasov equation in the particle coordinates. This undertaking constitutes a major portion of the recent analytical work in the gyrokinetic area. Quite often, gyrokinetic systems are only shown to be able to recover reduced fluid equations.

The purpose of the present analysis is to show, in a very general setting, that gyrokinetic theory and the fluid equations give an equivalent description of plasma equilibrium ($\partial/\partial t=0$). We will recover the fluid continuity equation and the momentum equation (the zeroth and first moment equations of the Vlasov equation in the particle coordinates) for equilibrium plasmas from the gyrokinetic theory. The gyrokinetic equilibrium is of fundamental importance for the widely adopted perturbative gyrokinetic particle simulation (δf method),[17–21] where the equilibrium distribution function and the electromagnetic field are assumed to be known. Gyrokinetic equilibria consistent with the well-studied fluid ones are obviously necessary for the perturbative gyrokinetic particle simulations to be reliable. In particular, recent numerical studies of equilibria with zonal flows[21] raise again the question of how to describe the equilibrium flow from the gyrokinetic point of view. In addition, kinetic-MHD modes are normally driven by both energetic particles and equilibrium inhomogeneities. For example, to investigate internal kink modes by the gyrokinetic models, we have to give a correct equilibrium current distribution in the gyrokinetic equilibrium. A linear gyrokinetic theory for kinetic MHD eigenmodes can be found in Refs. 11–14. The recovery of the complete fluid equations with nonlinear dynamics from gyrokinetic theory will be reported in future publications.

The essence of the problem studied here is how to relate the measurable quantities in the laboratory frame to the information in the guiding center coordinates . Given a distribution function $F(\mathbf{X},V_\parallel,\mu)$ in the guiding-center coordinates $\mathbf{Z}=(\mathbf{X},V_\parallel,\mu,\xi)$, how do we calculate the fluid density, flow, and current? For example, if we want to know the fluid perpendicular current in a plasma without parallel flow and electrical field, we may want to take the velocity moment of $F(\mathbf{X},V_\parallel,\mu)$ to obtain the perpendicular current (see Sec. III for details and more general cases):

$$\sum_s e\int \dot{\mathbf{X}}_\perp 2\pi B_\parallel^*/mF(\mathbf{Z})dV_\parallel d\mu$$
$$=\sum_s e\int \mathbf{V}_d 2\pi B_\parallel^*/mF(\mathbf{Z})dV_\parallel d\mu$$
$$=\frac{c}{B}\mathbf{b}\times\left(p_\perp\frac{\nabla B}{B}+p_\parallel\mathbf{b}\cdot\nabla\mathbf{b}\right)=j_b, \tag{1}$$

where Σ_s is the summation over species. This is only part of the perpendicular current we have obtained from the Vlasov equation in the particle coordinates $\mathbf{z}=(\mathbf{x},\mathbf{v})$ by taking the velocity moments,

$$\mathbf{j}_\perp=\frac{c}{B}\nabla_\perp p_\perp\times\mathbf{b}-\frac{c}{B}(p_\parallel-p_\perp)(\nabla\times\mathbf{b}). \tag{2}$$

By comparison, we find the missing term is

$$\mathbf{j}_{M\perp}=-\left[\nabla\times\left(c\mathbf{b}\frac{p_\perp}{B}\right)\right]_\perp. \tag{3}$$

One may want to try to recover this missing term by manipulating the gyrokinetic equation, but it is not apparent at all how it can be found this way. Why is this term missing?

Obviously, the physics of the missing term is the perpendicular component of the well-known diamagnetic current. If we *imagine* gyrating particles in strong magnetic field to act like small magnets with magnetic moment $\mu=mv_\perp^2/2B$, then $\mathbf{j}_M=c\nabla\times\mathbf{M}$ with $\mathbf{M}=-\mathbf{b}\Sigma_s n\mu_s$. We note that understanding this physical picture cannot be used to replace a rigorous derivation from first principles—the Vlasov–Maxwell system in either the particle coordinates or the guiding center coordinates. If the gyrokinetic description of plasmas is self-consistent and complete, we should be able to systematically recover this term, just as when we start from the Vlasov equation in the particle coordinates, we automatically recover this term.

Even though we pose this problem here using the language of modern gyrokinetic theory, it is Spitzer who first realized this problem and its significance almost a half century ago.[22,23] Spitzer's qualitative solution of this apparently paradoxical result is fundamental and has been widely adopted. At the center of Spitzer's physical picture is the idea that particle flow is different from the guiding-center flow. We will refer to this problem as the ''Spitzer paradox'' to reflect its interesting history. In Sec. II, we will discuss the Spitzer paradox and how it is closely related to the modern gyrokinetic-MHD theory. The importance of revisiting the Spitzer paradox lies at the fact that the gyrokinetic approach, the description of plasmas based on gyrocenter motion, has been developed into a precise quantitative theory. Any ''seeming conflict'' between the gyrokinetic theory and other well-established models should be resolved properly. Different theoretical models should predict the same results for experimental observables, such as the fluid density and current. In this sense, Spitzer's solution to this problem is the first attempt at today's gyrokinetic-MHD theory.

Of course, gyrokinetic-MHD theory has much more content than just the perpendicular current. In Sec. III, we will start from the basic gyrokinetic system and systematically recover, for the static cases, the zeroth and first moment equations of the Vlasov equation in the particle coordinates. By quantitatively recovering the fluid equations from the gyrokinetic theory, we automatically resolve the Spitzer paradox, whose essence is how the perpendicular current and flow are microscopically generated from particles' guiding-center motion. As Spitzer showed for the perpendicular flow, gyrokinetic theory indicates that all observables in the laboratory frame (particle coordinates) are different from their counterparts in the guiding center coordinates. Furthermore, gyrokinetic theory gives exact relations between observables in the laboratory frame and quantities defined in the guiding center coordinates.

The mathematical construction, which relates the distribution function $F(\mathbf{X},V_\parallel,\mu)$ in the guiding-center coordinates $\mathbf{Z}=(\mathbf{X},V_\parallel,\mu,\xi)$ to the observable fluid quantities in the laboratory frame is the pull-back transformation $G^{-1}*$ of the inverse guiding-center transformation G^{-1}. We will show that the physics encapsulated in the pull-back transformation is consistent with Spitzer's physical picture on how the diamagnetic current is generated by guiding-center motion, even though the pull-back transformation is introduced into the gyrokinetic system independently and naturally as a mathematical convenience.

The paper is organized as follows. In Sec. II, the Spitzer paradox and its modern implication are discussed. In Sec. III, we start from the basic gyrokinetic theory, systematically derive the equilibrium $(\partial/\partial t=0)$ fluid continuity equation and momentum equation, and finish with a quantitative analysis of the Spitzer paradox. In the last section, we summarize and discuss future work.

II. THE SPITZER PARADOX AND ITS MODERN IMPLICATIONS

Spitzer first noticed the obvious differences between the currents described by the fluid equations and the guiding-center motion.[22,23] There are two aspects of these obvious differences in an equilibrium plasma without parallel flow and electric field. First, the perpendicular current given by the fluid model is the diamagnetic current $c\mathbf{b}\times\nabla p/B$, which is not in the guiding-center drift motion. (It will be clear later that $c\mathbf{b}\times\nabla p/B$ is only part of the diamagnetic current.) On the other hand, the curvature drift and the gradient drift for the guiding-center motion are not found in the fluid results. This puzzle, first posed and discussed by Spitzer, is what we call the Spitzer paradox. To resolve it, we must explain, qualitatively as well as quantitatively, how the diamagnetic current is microscopically generated, and what happens to the macroscopic counterparts of the curvature drift and the gradient drift.

As to the first part of the puzzle—how the diamagnetic current is generated microscopically, Spitzer gave the well-known physical picture, which is illustrated in Fig. 1. The basic setup is an equilibrium plasma with a constant magnetic field and a pressure (density) gradient in the perpendicular direction. From the fluid equation $\mathbf{j}\times B/c=\nabla p$, we know that the perpendicular current is $c\mathbf{b}\times\nabla p/B$. However, if we look at the microscopic picture, for each guiding center, the drift motion does not produce any current or flow. Spitzer pointed out that there are more particles on the left than on the right; thus macroscopically gyromotion generates current and flow at each spatial location.

Widely adopted[24–26] as it is, Spitzer's picture has constantly been misunderstood. It is easy to observe that each particle spends the same amount of time moving downward on the right as it does moving upward on the left. The guiding centers for the particles do not move vertically. Based on this fact, Krall and Trivelpiece[25] argued that ''the existence

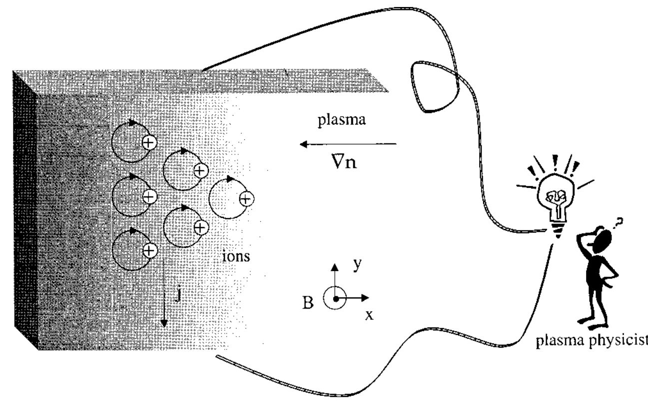

FIG. 1. Spitzer paradox.

of a current **J** does not necessarily mean a transport of particle across the magnetic field. Electric field and magnetic gradients produce particle drifts, but a gradient in the *pressure* does not.'' This viewpoint is not accurate simply because current is physically carried by particles. If there is no particle flow inside the plasma, there is no current. The problem with Krall and Trivelpiece's argument is the underlying assumption that if the guiding centers do not move vertically, then the macroscopic vertical velocity is zero.

Actually, even if the guiding centers do not move vertically, we can still observe vertical macroscopic flow. This is obvious from Fig. 2. Suppose that there are 3, 2, and 1 ions on the gyro-orbits centered at $O3$, $O2$, and $O1$, respectively. We observe gyrophase-averaged downward flows at points B, C, and D with current intensity 1, and an upward flow at point A with current intensity 3. It is clear now that zero guiding center velocity does not imply zero *local* macroscopic current or flow, because it only implies zero *averaged* macroscopic current or flow. More importantly, this average is the average over gyrophase and over different spatial locations. Macroscopic and microscopic pictures are consistent if the averaged current or flow over gyrophase and over the configuration space are the same, which is the case for Fig. 2.

The average over the configuration space inevitably involves the boundary conditions, which was realized by Spitzer to be necessary to reconcile the macroscopic and the microscopic picture. The boundary conditions can be quite subtle. Let's consider the setup in Fig. 1. Whether the light will be on depends on the boundary conditions. If the left and right boundaries are perfect reflecting walls [see Fig. 3(a)], which is case considered by Van Leevwen, Bohr, and

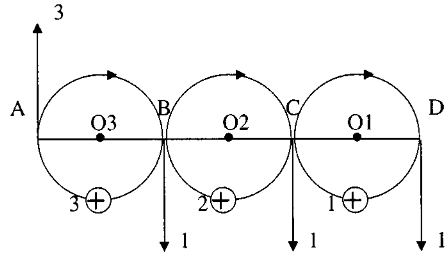

FIG. 2. Guiding-center motion and diamagnetic current and flow.

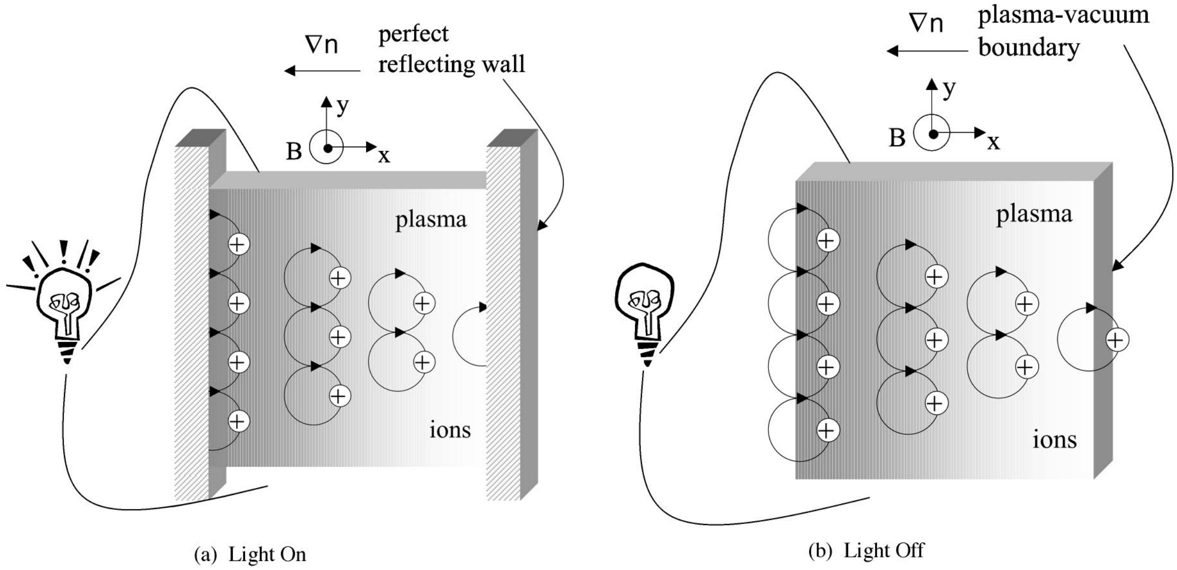

FIG. 3. Boundary conditions.

Spitzer,[23] the light is on, because there is a net current and flow downward. Microscopically, gyromotions are not completed on the boundary, therefore, the spatially averaged guiding center flow is downward. If the boundaries are sharp boundaries between vacuum and the plasma [see Fig. 3(b)], the light is off, because there is no net current. Macroscopically, the density gradient on the left plasma-vacuum boundary is reversed and approaches infinity. There is a surface flow upward on the boundary, such that the spatially averaged macroscopic flow is zero. In both situations, the microscopic and macroscopic pictures agree, because the spatially averaged current and flow are the same. We do not discuss how these boundary conditions can be achieved, but assume theoretically that they can be achieved. The discussion of boundary conditions in the gyrokinetic formalism is actually of practical interests. In the process of developing the gyrokinetic particle simulation methods, boundary conditions were identified as important issues at the very beginning. When studying the drift waves using the gyrokinetic particle simulation method, Lee and Okuda[27] adopted a boundary condition which is equivalent to the scenario illustrated in Fig. 3(b).

For the second part of the puzzle—what is the macroscopic counterpart of the magnetic drifts, Spitzer[23] argued that ''in a region where the density and pressure are uniform, no macroscopic velocities or currents can appear, regardless of what magnetic field, **B**, may be present, provided that $\partial \mathbf{B}/\partial t$ vanishes.'' Chen[26] concluded that ''the curvature drift exists in the fluid picture, ... The gradient-**B** drift, however, does not exist for fluids.'' Krall and Trivelpiece's conclusion[25] argued that ''Electric field and magnetic gradients produce particle drifts, but a gradient in the *pressure* does not.''

Even though the basic idea that the guiding center flow is different from the particle flow is fundamental, the quantitative microscopic picture for the diamagnetic current given by Spitzer[22] was complicated and difficult to apply to general geometries. Chen[26] concluded that ''it can be quite tricky to work with the single-particle picture. The fluid theory usually gives the right results when applied straightforwardly, even though it contains ''fictitious'' drifts like the diamagnetic drift.''

The Spitzer paradox is important, because it highlights the seeming conflict between the theory of gyromotion and the fluid equations, two most fundamental concepts in plasma physics. After Spitzer posed this paradox, gyrokinetic theory, the theory of gyromotion, has been developed into a powerful tool for the study of plasma physics. Recent development in gyrokinetic-MHD theory and gyrokinetic particle simulation raises again the necessity to show the consistency between the gyrokinetic model and the fluid model, but in a wider range and in a more general geometry. These are the modern implications of the Spitzer paradox. Therefore, Spitzer's effort in this area should be regarded as the beginning of today's gyrokinetic-MHD theory. In the next section, we will recover the fluid continuity equation and the momentum equation from the gyrokinetic system. Part of the analysis is to derive the perpendicular flow (current) from the gyrokinetic theory, which is exactly the essence of the Spitzer paradox. In another words, the analysis presented here includes a quantitative solution to the Spitzer paradox in a very general setting.

III. GYROKINETIC EQUILIBRIUM

A. Posing the problem

Gyrokinetic theory assumes and takes advantage of the fact that the plasma is strongly magnetized, that is, the particles' gyro-radii are much smaller than the scale-length of the magnetic field

$$\epsilon \equiv \left| \frac{\rho \nabla \mathbf{B}}{\mathbf{B}} \right| \ll 1. \tag{4}$$

The gyrokinetic theory for equilibrium plasmas is given by phase space Poisson bracket

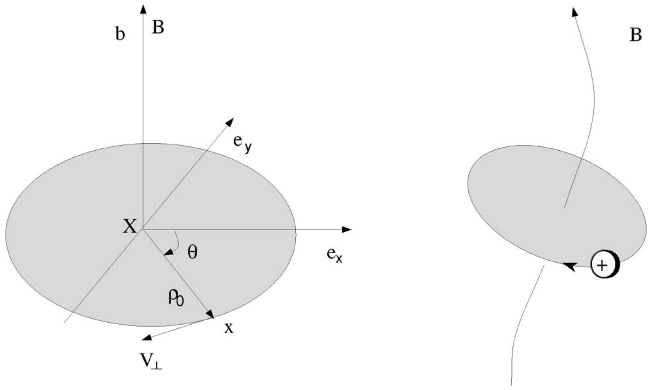

FIG. 4. Guiding-center coordinate system.

$$\{\cdot,\cdot\}:\ \mathcal{F}(\mathbf{Z})\times\mathcal{F}(\mathbf{Z})\rightarrow\mathcal{F}(\mathbf{Z}),\tag{5}$$

Hamiltonian $H(\mathbf{Z})\in\mathcal{F}(\mathbf{Z})$, and phase space distribution function $F(\mathbf{Z})\in\mathcal{F}(\mathbf{Z})$, where $\mathbf{Z}=(\mathbf{X},V_\parallel,\mu,\xi)$ is the phase space coordinates which are normally referred to as the guiding-center coordinates and $\mathcal{F}(\mathbf{Z})$ is the set of all smooth phase space functions.

The Hamiltonian $H(\mathbf{Z})$ and the Poisson bracket $\{\cdot,\cdot\}$[1–3,9,10] determine the particle dynamics in $\mathbf{Z}$, and are given by

$$H=\frac{mV_\parallel^2}{2}+\mu B+e\phi,\tag{6}$$

and

$$\begin{aligned}\{F,G\}=&\frac{e}{mc}\left(\frac{\partial F}{\partial\xi}\frac{\partial G}{\partial\mu}-\frac{\partial F}{\partial\mu}\frac{\partial G}{\partial\xi}\right)-\frac{c\mathbf{b}}{eB_\parallel^*}\\&\cdot\left[\left(\frac{\partial}{\partial\mathbf{X}}F+\mathbf{W}\frac{\partial F}{\partial\xi}\right)\times\left(\frac{\partial}{\partial\mathbf{X}}G+\mathbf{W}\frac{\partial G}{\partial\xi}\right)\right]+\frac{\mathbf{B}^*}{mB_\parallel^*}\\&\cdot\left[\left(\frac{\partial}{\partial\mathbf{X}}F+\mathbf{W}\frac{\partial F}{\partial\xi}\right)\frac{\partial G}{\partial V_\parallel}-\left(\frac{\partial}{\partial\mathbf{X}}G+\mathbf{W}\frac{\partial G}{\partial\xi}\right)\frac{\partial F}{\partial V_\parallel}\right],\end{aligned}\tag{7}$$

where

$$\mathbf{B}^*=\mathbf{B}+\frac{cmV_\parallel}{e}\nabla\times\mathbf{b},\quad B_\parallel^*=\mathbf{b}\cdot\mathbf{B}^*.\tag{8}$$

The guiding-center velocity is then given by

$$\begin{aligned}\dot{\mathbf{X}}=\{\mathbf{X},H\}&=\frac{1}{m}\frac{\mathbf{B}^*}{B_\parallel^*}\frac{\partial H}{\partial V_\parallel}+\frac{c}{e}\frac{\mathbf{b}}{B_\parallel^*}\times\frac{\partial H}{\partial\mathbf{X}}\\&=V_\parallel\mathbf{b}+\mathbf{V}_{\mathbf{E}\times\mathbf{B}}+\mathbf{V}_d+O(\epsilon^2),\\\mathbf{V}_{\mathbf{E}\times\mathbf{B}}&=\frac{c\mathbf{b}}{B}\times\nabla\phi,\\\mathbf{V}_d&=\frac{c}{eB}\mathbf{b}\times(\mu\nabla B+mV_\parallel^2\mathbf{b}\cdot\nabla\mathbf{b}).\end{aligned}\tag{9}$$

In this paper, we use the gyroradius $\boldsymbol{\rho}_0$ and the thermal velocity v_{th} as the basic scale parameters for length and velocity. $\mathbf{E}=-\nabla\phi$ is thus treated as $O(\epsilon)$ for simplicity. The case in which $\mathbf{E}$ is $O(\epsilon^0)$ will be considered elsewhere. The basic ordering can be summarized as

$$u_\parallel\sim v_{\text{th}}\sim O(\epsilon^0)\gg u_\perp\sim\frac{c\nabla p\times\mathbf{b}}{enB}\sim\frac{c\mathbf{E}\times\mathbf{B}}{B^2}\sim O(\epsilon).\tag{10}$$

For the current purpose, we only need to know the leading order expression for the guiding-center transformation $G:\mathbf{z}\mapsto\mathbf{Z}$, which transforms the particle coordinates $\mathbf{z}=(\mathbf{x},\mathbf{v})$ into the guiding-center coordinates $\mathbf{Z}=(\mathbf{X},V_\parallel,\mu,\xi)$

$$\begin{aligned}&\mathbf{X}=\mathbf{x}-\boldsymbol{\rho}_0+O(\epsilon),\quad V_\parallel=v_\parallel+O(\epsilon),\\&\mu=\mu_0+O(\epsilon),\quad \xi=\theta+O(\epsilon),\end{aligned}\tag{11}$$

where $\mu_0=mv_\perp^2/2B$ and $(\mathbf{x},v_\parallel,v_\perp,\theta)$ is the usual local particle coordinates. $\boldsymbol{\rho}_0$, defined in particle coordinates, is the usual gyroradius. θ is chosen such that $\hat{\mathbf{v}}_\perp=-e/|e|(\mathbf{e}_x\sin\theta+\mathbf{e}_y\cos\theta)$. $\mathbf{e}_x$ and $\mathbf{e}_y$ are two perpendicular directions in the configuration space, and $(\mathbf{e}_x,\mathbf{e}_y,\mathbf{b})$ is a right-handed orthogonal frame. The guiding center coordinate system in a static magnetic field is illustrated in Fig. 4.

The gyrokinetic equation for $F(\mathbf{Z})$ is

$$\{F,H\}=\dot{\mathbf{X}}\cdot\frac{\partial F}{\partial\mathbf{X}}+\dot{V}_\parallel\frac{\partial F}{\partial V_\parallel}=0,\tag{12}$$

where we have made use of the fact that for low frequency phenomena, $F(\mathbf{Z})$ is gyrophase independent.[11–14] It is easy to verify that phase space volume is conserved by the Poisson bracket (Liouville theorem)

$$\frac{\partial}{\partial\mathbf{X}}\cdot(B_\parallel^*\dot{\mathbf{X}})+\frac{\partial}{\partial V_\parallel}(B_\parallel^*\dot{V}_\parallel)=0.\tag{13}$$

Therefore, the gyrokinetic equation can also be expressed as

$$\frac{\partial}{\partial\mathbf{X}}\cdot(B_\parallel^*\dot{\mathbf{X}}F)+\frac{\partial}{\partial V_\parallel}(B_\parallel^*\dot{V}_\parallel F)=0.\tag{14}$$

Our goal here is to recover from the gyrokinetic theory summarized above the fluid continuity equation and motion equation derived directly from the Vlasov equation for each species

$$\nabla\cdot n\mathbf{u}=0\tag{15}$$

$$mn\mathbf{u}\cdot\nabla\mathbf{u}+\nabla\cdot\mathbf{p}-\frac{1}{c}en\mathbf{u}\times\mathbf{B}-en\mathbf{E}=0,\tag{16}$$

where

$$n\equiv\int f(\mathbf{x},\mathbf{v})\,d^3\mathbf{v},\tag{17}$$

$$\mathbf{u}\equiv\frac{1}{n}\int\mathbf{v}f(\mathbf{x},\mathbf{v})d^3\mathbf{v},\tag{18}$$

$$\mathbf{p}\equiv\int m(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})f(\mathbf{x},\mathbf{v})d^3\mathbf{v}=p_\perp\mathbf{I}+(p_\parallel-p_\perp)\mathbf{bb},\tag{19}$$

$$p_\perp=\int\frac{1}{2}m(\mathbf{v}_\perp-\mathbf{u}_\perp)\cdot(\mathbf{v}_\perp-\mathbf{u}_\perp)f(\mathbf{x},\mathbf{v})d^3\mathbf{v},\tag{20}$$

$$p_{\|}=\int m(\mathbf{v}_{\|}-\mathbf{u}_{\|})\cdot(\mathbf{v}_{\|}-\mathbf{u}_{\|})f(\mathbf{x},\mathbf{v})d^3\mathbf{v}. \tag{21}$$

$f(\mathbf{x},\mathbf{v})$ is the distribution function in particle coordinates, and $\mathbf{I}$ is the unit tensor.

As pointed out before, gyrokinetic theory assumes particles' gyro-radii are much smaller than the scale length of the magnetic field ($\epsilon \ll 1$). We do not expect gyrokinetic theory to be valid when ϵ is not small. As a result, the recovery of the fluid equations from the gyrokinetic model is for strongly magnetized plasmas only. Under this assumption, we have taken the fluid pressure tensor $\mathbf{p}$ to be isotropic in the perpendicular direction. Since the gyrokinetic theory adopted in this paper is correct to order $O(\epsilon)$, our recovery of the fluid equations from the gyrokinetic side is carried out to order $O(\epsilon)$ as well.

The fluid momentum equation [Eq. (16)] is a vector equation. We will recover the parallel and perpendicular components of it separately. Making use of the identity

$$\begin{aligned}\nabla\cdot(\mathbf{bb})&=(\nabla\cdot\mathbf{b})\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{b}\\ &=(\nabla_\perp-\nabla_{\|})\ln B-\frac{1}{B}\mathbf{b}\times(\nabla\times\mathbf{B}),\end{aligned} \tag{22}$$

we have

$$\begin{aligned}\nabla\cdot\mathbf{p}&=\nabla_\perp p_\perp+\nabla_{\|}p_{\|}\\ &\quad+(p_{\|}-p_\perp)\left[(\nabla_\perp-\nabla_{\|})\ln B-\frac{1}{B}\mathbf{b}\times(\nabla\times\mathbf{B})\right].\end{aligned} \tag{23}$$

The parallel and perpendicular components of the fluid momentum equation [Eq. (16)] are

$$mn(\mathbf{u}\cdot\nabla\mathbf{u})_{\|}+\nabla_{\|}p_{\|}-(p_{\|}-p_\perp)\nabla_{\|}\ln B-en\mathbf{E}_{\|}=0 \tag{24}$$

and

$$\begin{aligned}n\mathbf{u}_\perp=&-\frac{c}{eB}[mn(\mathbf{u}\cdot\nabla\mathbf{u})\times\mathbf{b}+\nabla_\perp p_\perp\times\mathbf{b}\\ &-(p_{\|}-p_\perp)(\nabla\times\mathbf{b})_\perp-en\mathbf{E}\times\mathbf{b}],\end{aligned} \tag{25}$$

where we have utilized the following identity:

$$(\nabla\times\mathbf{b})_\perp=\frac{1}{B}(\nabla\times\mathbf{B})_\perp-\nabla_\perp\ln B\times\mathbf{b}. \tag{26}$$

B. Pull-back formula for fluid density and velocity in particle coordinates

As Spitzer first noticed, the fluid velocity is different from the averaged drift velocity. But how can we relate quantitatively the fluid velocity $\mathbf{u}(\mathbf{r})$ in the particle coordinates to the information about the gyrokinetic distribution function $F(\mathbf{Z})$ in the guiding-center coordinates? The general formula for this purpose is the so-called ''pull-back transformation.'' Generally, for a macroscopic quantity $q(\mathbf{r})$ in the particle coordinates, we have

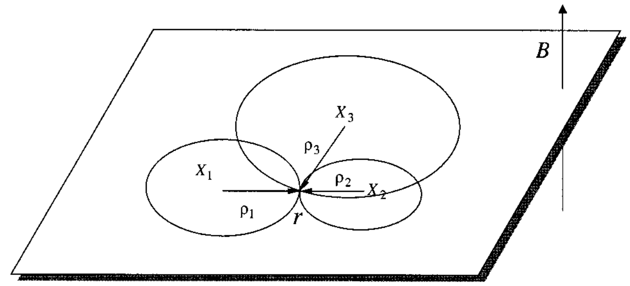

FIG. 5. The physics of the pull-back formula.

$$\begin{aligned}q(\mathbf{r})&=\int q(\mathbf{r},\mathbf{v})f(\mathbf{r},\mathbf{v})d^3\mathbf{v}\\ &=\int q(\mathbf{z})f(\mathbf{z})\delta(\mathbf{x}-\mathbf{r})d^6\mathbf{z}\\ &=\int G^{-1*}[q(\mathbf{z})\delta(\mathbf{x}-\mathbf{r})]F(\mathbf{Z})d^6\mathbf{Z}\\ &=\int Q(\mathbf{Z})\delta[\mathbf{X}(\mathbf{Z})+\boldsymbol{\rho}-\mathbf{r}]F(\mathbf{Z})d^6\mathbf{Z},\end{aligned} \tag{27}$$

where we have assumed that the guiding-center transformation G is a diffeomorphism (one-one onto and smooth), and

$$d^6\mathbf{Z}\equiv B_{\|}^*/m\,d^3\mathbf{X}dV_{\|}d\mu d\xi, \tag{28}$$

$$Q(\mathbf{Z})=G^{-1*}q(\mathbf{z}), \tag{29}$$

$$\boldsymbol{\rho}=G^{-1*}\boldsymbol{\rho}_0. \tag{30}$$

Here G^{-1*} is the pull-back of G^{-1}, which maps any function on $\mathbf{z}=(\mathbf{x},\mathbf{v})$ into a function on $\mathbf{Z}=(\mathbf{X},V_{\|},\mu,\xi)$

$$G^{-1*}:f(\mathbf{z})\mapsto F(\mathbf{Z})\equiv f(\mathbf{x}(\mathbf{Z})). \tag{31}$$

The physics encapsulated in the pull-back formula is illustratively shown in Fig. 5. An observable $q(\mathbf{r})$ at certain location $\mathbf{r}$ in the laboratory frame is the average of its microscopic counterpart expressed in the guiding center coordinates $Q(\mathbf{Z})$ over nearby guiding centers with $\mathbf{X}(\mathbf{Z})+\boldsymbol{\rho}(\mathbf{Z})=\mathbf{r}$. In Fig. 5, three examples of such guiding centers are shown. Obviously, this mechanism is consistent with Spitzer's original qualitative picture of how the diamagnetic current is generated by the gyromotion (see Sec. II), but expressed using the gyrokinetic language, and in a quantitative way.

For the number density in particle coordinates, we use $q(\mathbf{z})=1$ and $G^{-1*}1=1$.

$$\begin{aligned}n(\mathbf{r})&=\int\delta(\mathbf{X}+\boldsymbol{\rho}-\mathbf{r})F(\mathbf{Z})\,d^6\mathbf{Z}\\ &=\int\delta(\mathbf{X}-\mathbf{r})F(\mathbf{Z})\,d^6\mathbf{Z}+O(\epsilon^2)\\ &=2\pi\int F(\mathbf{Z})B_{\|}^*/m\,dV_{\|}d\mu\Big|_{\mathbf{X}\mapsto\mathbf{r}}+O(\epsilon^2),\end{aligned} \tag{32}$$

where ''$|_{\mathbf{X}\mapsto\mathbf{r}}$'' means replacing $\mathbf{X}$ by $\mathbf{r}$.

For the fluid velocity in particle coordinates $\mathbf{u}(\mathbf{r})$, we have $q(\mathbf{z})=\mathbf{v}=\dot{\mathbf{x}}$, $G^{-1*}\mathbf{v}=\dot{\mathbf{X}}+\dot{\boldsymbol{\rho}}(\mathbf{X})+O(\epsilon^2)$, and

$$n(\mathbf{r})\mathbf{u}(\mathbf{r})=\int(\dot{\mathbf{X}}+\dot{\boldsymbol{\rho}})\,\delta(\mathbf{X}+\boldsymbol{\rho}-\mathbf{r})F(\mathbf{Z})d^6\mathbf{Z}+O(\epsilon^2)$$
$$=\int[V_\parallel\mathbf{b}+\mathbf{V}_{\mathbf{E}\times\mathbf{B}}+\mathbf{V}_d]\,\delta(\mathbf{X}-\mathbf{r})F(\mathbf{Z})d^6\mathbf{Z}$$
$$+\int\dot{\boldsymbol{\rho}}\delta(\mathbf{X}+\boldsymbol{\rho}-\mathbf{r})F(\mathbf{Z})d^6\mathbf{Z}+O(\epsilon^2). \quad (33)$$

We now look at the expression for $n(\mathbf{r})$ term by term. The first term is part of the parallel flow.

$$\int V_\parallel\mathbf{b}\delta(\mathbf{X}-\mathbf{r})F(\mathbf{Z})d^6\mathbf{Z}=nU_\parallel\mathbf{b}\Big|_{\mathbf{X}\mapsto\mathbf{r}}+O(\epsilon^2), \quad (34)$$

$$U_\parallel\equiv\frac{2\pi}{n}\int V_\parallel B_\parallel^*/mF(\mathbf{Z})dV_\parallel d\mu. \quad (35)$$

For the $\mathbf{E}\times\mathbf{B}$ term, we have

$$\int\mathbf{V}_{\mathbf{E}\times\mathbf{B}}\delta(\mathbf{X}-\mathbf{r})F(\mathbf{Z})d^6\mathbf{Z}=\frac{c}{B}n\mathbf{E}\times\mathbf{b}\Big|_{\mathbf{X}\mapsto\mathbf{r}}. \quad (36)$$

The magnetic drift term is

$$\int\mathbf{V}_d\delta(\mathbf{X}-\mathbf{r})F(\mathbf{Z})d^6\mathbf{Z}$$
$$=\frac{c}{eB}\mathbf{b}\times\left(W_\perp\frac{\nabla B}{B}+W_\parallel\mathbf{b}\cdot\nabla\mathbf{b}\right)\Big|_{\mathbf{X}\mapsto\mathbf{r}}, \quad (37)$$

where

$$W_\perp\equiv2\pi\int B\mu F(\mathbf{Z})B_\parallel^*/mdV_\parallel d\mu, \quad (38)$$

$$W_\parallel\equiv2\pi\int mV_\parallel^2F(\mathbf{Z})B_\parallel^*/mdV_\parallel d\mu. \quad (39)$$

The last term is the diamagnetic flow, which can be simplified in terms of $W_\perp$,

$$\int\dot{\boldsymbol{\rho}}\delta(\mathbf{X}+\boldsymbol{\rho}-\mathbf{r})F(\mathbf{Z})d^6\mathbf{Z}$$
$$=\int\dot{\boldsymbol{\rho}}\boldsymbol{\rho}\cdot\nabla\,\delta(\mathbf{X}-\mathbf{r})F(\mathbf{Z})d^6\mathbf{Z}+O(\epsilon^2)$$
$$=-\int\nabla\cdot[\boldsymbol{\rho}\dot{\boldsymbol{\rho}}B_\parallel^*/mF(\mathbf{Z})]\,\delta(\mathbf{X}-\mathbf{r})dV_\parallel d\mu d\xi+O(\epsilon^2)$$
$$=-\frac{c}{e}\nabla\times\left(\mathbf{b}\frac{W_\perp}{B}\right)\Big|_{\mathbf{X}\mapsto\mathbf{r}}+O(\epsilon^2). \quad (40)$$

In deriving above equation, the following equations are used:

$$\dot{\boldsymbol{\rho}}=\{\boldsymbol{\rho},H\}=\sqrt{\frac{2\mu B}{m}}\mathbf{e}_\xi+O(\epsilon), \quad (41)$$

$$\left(\int\dot{\boldsymbol{\rho}}\boldsymbol{\rho}d\xi\right)_{ij}=\frac{2\pi\mu c}{e}\epsilon_{ij\mathbf{b}}+O(\epsilon). \quad (42)$$

Here, $\epsilon_{ij\mathbf{b}}$ is the Kronecker symbol, and the subscript $\mathbf{b}$ represents the dimension parallel to $\mathbf{B}$.

It is important to observe that this diamagnetic flow appears naturally in our gyrokinetic system. Understanding diamagnetic flow from the gyrokinetic point of view is the key issue of the Spitzer paradox. We will come back to this issue later on in Sec. III D.

Overall, the fluid density and flow in the particle coordinates can be expressed in terms of the gyrokinetic distribution function $F(\mathbf{Z})$ in the guiding-center coordinates $\mathbf{Z}=(\mathbf{X},V_\parallel,\mu,\xi)$ as

$$n(\mathbf{r})=2\pi\int F(\mathbf{Z})B_\parallel^*/mdV_\parallel d\mu\Big|_{\mathbf{X}\mapsto\mathbf{r}}+O(\epsilon^2), \quad (43)$$

$$n(\mathbf{r})\mathbf{u}(\mathbf{r})=\left[nU_\parallel\mathbf{b}+\frac{c\mathbf{b}}{eB}\times\left(W_\perp\frac{\nabla B}{B}+W_\parallel\mathbf{b}\cdot\nabla\mathbf{b}\right)\right.$$
$$\left.+\frac{cn}{B}\mathbf{E}\times\mathbf{b}-\frac{c}{e}\nabla\times\left(\frac{W_\perp}{B}\mathbf{b}\right)\right]\Big|_{\mathbf{X}\mapsto\mathbf{r}}+O(\epsilon^2). \quad (44)$$

The replacement of $\mathbf{X}$ by $\mathbf{r}$ after the velocity space integral in the above equations brings quantities from the guiding-center coordinate $\mathbf{Z}=(\mathbf{X},V_\parallel,\mu,\xi)$ back into the particle coordinates $\mathbf{z}=(\mathbf{x},\mathbf{v})$, as if we are working in the ''mixed coordinates'' $(\mathbf{r},V_\parallel,\mu,\xi)$.

C. Recovery of fluid equations

We now recover the fluid continuity equation [Eq. (15)] from the gyrokinetic model. From our result for $n(\mathbf{r})\mathbf{u}(\mathbf{r})$,

$$\nabla\cdot(n(\mathbf{r})\mathbf{u}(\mathbf{r}))=\left[\nabla\cdot2\pi\int\dot{\mathbf{X}}F(\mathbf{Z})B_\parallel^*/mdV_\parallel d\mu\right.$$
$$\left.-\frac{c}{e}\nabla\cdot\nabla\times\left(\frac{W_\perp\mathbf{b}}{B}\right)\right]\Big|_{\mathbf{X}\mapsto\mathbf{r}}+O(\epsilon^3)$$
$$=\left[\nabla\cdot2\pi\int\dot{\mathbf{X}}F(\mathbf{Z})B_\parallel^*/mdV_\parallel d\mu\right]\Big|_{\mathbf{X}\mapsto\mathbf{r}}$$
$$+O(\epsilon^3). \quad (45)$$

Applying $\int dV_\parallel d\mu d\xi$ to the gyrokinetic equation [Eq. (14)] gives

$$\nabla\cdot\int2\pi\dot{\mathbf{X}}F(\mathbf{Z})B_\parallel^*\,dV_\parallel d\mu=0. \quad (46)$$

Therefore,

$$\nabla\cdot(n(\mathbf{r})\mathbf{u}(\mathbf{r}))=O(\epsilon^3). \quad (47)$$

To recover the fluid momentum equation [Eq. (16)], we first invoke the basic ordering

$$u_\parallel\sim v_{\mathrm{th}}\sim O(\epsilon^0)\gg u_\perp\sim\frac{c\nabla p\times\mathbf{b}}{enB}\sim\frac{c\mathbf{E}\times\mathbf{B}}{B^2}\sim O(\epsilon). \quad (48)$$

It is then clear that

$$\mathbf{u}\cdot\nabla\mathbf{u}=u_\parallel\mathbf{b}\cdot\nabla u_\parallel\mathbf{b}+O(\epsilon^2)$$
$$=-u_\parallel^2(\mathbf{b}\times\nabla\times\mathbf{b})+u_\parallel\mathbf{b}\mathbf{b}\cdot\nabla u_\parallel+O(\epsilon^2), \quad (49)$$

$$
\begin{aligned}
0=\nabla\cdot(n\mathbf{u})&=\nabla\cdot(n\mathbf{b}u_\parallel)+O(\epsilon^2)\\
&=n\mathbf{b}\cdot\nabla u_\parallel+u_\parallel\nabla\cdot(n\mathbf{b})+O(\epsilon^2)\\
&=n\mathbf{b}\cdot\nabla u_\parallel+u_\parallel\mathbf{B}\cdot\nabla\frac{n}{B}+O(\epsilon^2). \qquad (50)
\end{aligned}
$$

We also can verify that

$$
\begin{aligned}
W_\perp(\mathbf{r})&=2\pi\int B\mu F(\mathbf{Z})B_\parallel^*/m\ dV_\parallel d\mu\Big|_{\mathbf{X}\mapsto\mathbf{r}}\\
&=\int\frac{1}{2}mv_\perp^2 fd^3\mathbf{v}+O(\epsilon)=p_\perp(\mathbf{r})+O(\epsilon), \qquad (51)
\end{aligned}
$$

$$
\begin{aligned}
W_\parallel(\mathbf{r})&=2\pi\int mV_\parallel^2F(\mathbf{Z})B_\parallel^*/mdV_\parallel d\mu\Big|_{\mathbf{X}\mapsto\mathbf{r}}\\
&=\Big[mnU_\parallel^2+2\pi\int m(V_\parallel-U_\parallel)^2\\
&\quad\times F(\mathbf{Z})B_\parallel^*/mdV_\parallel d\mu\Big]\Big|_{\mathbf{X}\mapsto\mathbf{r}}\\
&=mnu_\parallel^2(\mathbf{r})+p_\parallel(\mathbf{r})+O(\epsilon). \qquad (52)
\end{aligned}
$$

To derive the parallel component of the fluid momentum equation [Eq. (24)], we apply $2\pi\int V_\parallel B_\parallel^*/m\ dV_\parallel d\mu$ to the gyrokinetic equation

$$
V_\parallel\mathbf{b}\cdot\frac{\partial F}{\partial\mathbf{X}}-(\mu\mathbf{b}\cdot\nabla B-eE_\parallel)\frac{\partial F}{\partial V_\parallel}=O(\epsilon^2), \qquad (53)
$$

and get

$$
B_\parallel^*\mathbf{b}\cdot\frac{\partial}{\partial\mathbf{X}}\frac{W_\parallel}{B_\parallel^*}+\frac{W_\perp}{B}\mathbf{b}\cdot\nabla B-eE_\parallel n=O(\epsilon^2). \qquad (54)
$$

Replacing $\mathbf{X}$ by $\mathbf{r}$ and using Eqs. (51) and (52), we have

$$
\mathbf{B}\cdot\nabla\frac{mnu_\parallel^2}{B}+\mathbf{B}\cdot\nabla\frac{p_\parallel}{B}+\frac{p_\perp}{B}\mathbf{b}\cdot\nabla B-enE_\parallel n=O(\epsilon^2). \qquad (55)
$$

The first term of this equation can be simplified by Eqs. (49) and (50).

$$
\begin{aligned}
\mathbf{B}\cdot\nabla\frac{mnu_\parallel^2}{B}&=2u_\parallel mn\mathbf{b}\cdot\nabla u_\parallel+mu_\parallel^2\mathbf{B}\cdot\nabla\frac{n}{B}\\
&=mnu_\parallel\mathbf{b}\cdot\nabla u_\parallel\\
&=mn(u_\parallel\mathbf{b}\cdot\nabla u_\parallel\mathbf{b})_\parallel\\
&=mn(\mathbf{u}\cdot\nabla\mathbf{u})_\parallel+O(\epsilon^2). \qquad (56)
\end{aligned}
$$

In addition

$$
\mathbf{B}\cdot\nabla\frac{p_\parallel}{B}=\nabla_\parallel p_\parallel-p_\parallel\nabla_\parallel\ \ln B. \qquad (57)
$$

We therefore obtain from the gyrokinetic theory

$$
mn(\mathbf{u}\cdot\nabla\mathbf{u})_\parallel+\nabla_\parallel p_\parallel-(p_\parallel-p_\perp)\nabla_\parallel\ \ln B-enE_\parallel=O(\epsilon^2), \qquad (58)
$$

which is the parallel component of the fluid momentum equation [Eq. (24)], correct to order $O(\epsilon^2)$.

For the perpendicular component of the fluid equation, we have, from Eq. (44),

$$
\begin{aligned}
n\mathbf{u}_\perp&=\left\{\frac{c\mathbf{b}}{eB}\times\left(\frac{W_\perp}{B}\nabla B+W_\parallel\mathbf{b}\cdot\nabla\mathbf{b}\right)+\frac{cn}{B}\mathbf{E}\times\mathbf{b}\right.\\
&\quad\left.-\frac{c}{e}\left[\nabla\times\left(\frac{W_\perp}{B}\mathbf{b}\right)\right]_\perp\right\}\Bigg|_{\mathbf{X}\mapsto\mathbf{r}}+O(\epsilon^2)\\
&=\frac{c}{e}\left\{-\frac{\nabla p_\perp\times\mathbf{b}}{B}+p_\perp\left[\frac{\mathbf{b}\times\nabla B}{B^2}-\left(\nabla\times\frac{\mathbf{b}}{B}\right)_\perp\right]\right.\\
&\quad\left.+p_\parallel\frac{(\nabla\times\mathbf{b})_\perp}{B}+mnu_\parallel^2\frac{(\nabla\times\mathbf{b})_\perp}{B}\right\}+n\frac{\mathbf{E}\times\mathbf{b}}{B}c+O(\epsilon^2).
\end{aligned}
\qquad (59)
$$

Using

$$
(\mathbf{u}\cdot\nabla\mathbf{u})\times\mathbf{b}=-u_\parallel^2(\nabla\times\mathbf{b})_\perp+O(\epsilon^2), \qquad (60)
$$

$$
\frac{\mathbf{b}\times\nabla B}{B^2}-\left(\nabla\times\frac{\mathbf{b}}{B}\right)_\perp=-\frac{(\nabla\times\mathbf{b})_\perp}{B}, \qquad (61)
$$

we finally have

$$
\begin{aligned}
n\mathbf{u}_\perp&=-\frac{c}{eB}[mn\mathbf{u}\cdot\nabla\mathbf{u}\times\mathbf{b}+\nabla_\perp p_\perp\times\mathbf{b}\\
&\quad-(p_\parallel-p_\perp)(\nabla\times\mathbf{b})_\perp-en\mathbf{E}\times\mathbf{b}]+O(\epsilon^2), \qquad (62)
\end{aligned}
$$

which is the perpendicular fluid momentum equation [Eq. (25)], correct to order $O(\epsilon^2)$.

The above equations are derived for a single species, from which one-fluid equations can be derived trivially by the usual procedure. Since fluid equations for a single species have been recovered from the gyrokinetic side, so have the one-fluid equations.

D. Quantitative analysis of the Spitzer paradox

By quantitatively recovering the fluid equations from the gyrokinetic theory, we automatically resolve the Spitzer paradox, whose essence is how the perpendicular current and flow are microscopically generated from particles' guiding-center motion. For the discussion in this section, we assume there is no macroscopic parallel flow. Using the results in previous sections, we can express the perpendicular current from the gyrokinetic viewpoint as

$$
\begin{aligned}
\mathbf{j}_\perp&=\sum_s ne\mathbf{u}_\perp\\
&=\sum_s\left[\int e(\dot{\mathbf{X}}+\dot{\boldsymbol{\rho}})\delta(\mathbf{X}+\boldsymbol{\rho}-\mathbf{r})F(\mathbf{Z})d^6\mathbf{Z}\right]_\perp\\
&=\sum_s\mathbf{j}_{M\perp}+\mathbf{j}_d\\
&=\frac{c}{B}\left[\mathbf{b}\times\nabla\sum_s p_\perp+\left(\sum_s p_\parallel-\sum_s p_\perp\right)(\nabla\times\mathbf{b})_\perp\right], \qquad (63)
\end{aligned}
$$

where Σ_s is the summation over species. As we have pointed out before, the physics of this equation is consistent with Spitzer's original picture, but is expressed from the modern viewpoint of gyrokinetic theory, and in a quantitative way. The macroscopic flow at some spatial location in the particle coordinates, $\mathbf{u}(\mathbf{r})$, is the averaged instantaneous particle velocity of the guiding centers nearby, $1/n\int(\dot{\mathbf{X}}+\dot{\boldsymbol{\rho}})\delta(\mathbf{X}+\boldsymbol{\rho}-\mathbf{r})F(\mathbf{Z})d^6\mathbf{Z}$. The mathematics here is simple and accurate. Physically, the perpendicular current consists of two parts, the perpendicular component of the diamagnetic current and the drift current. The $\mathbf{E}\times\mathbf{B}$ currents due to different species cancel out in neutral plasmas.

In the above derivation from the gyrokinetic theory, $\mathbf{j}_M$ and $\mathbf{j}_d$ are naturally separated. They have different physical meaning as they first appear in the equation. However, if we derive $\mathbf{j}_\perp$ from the velocity moment of the Vlasov equation in the particle coordinates, we can only directly obtain

$$\mathbf{j}_\perp=\frac{c}{B}\nabla_\perp\sum_s p_\perp\times\mathbf{b}-\frac{c}{B}\left(\sum_s p_\parallel-\sum_s p_\perp\right)(\nabla\times\mathbf{b}). \tag{64}$$

Then, if we define $\mathbf{j}_M$ and $\mathbf{j}_d$ as they are, we find algebraically $\mathbf{j}_\perp=\mathbf{j}_{M\perp}+\mathbf{j}_d$. In this approach, $\mathbf{j}_M$ and $\mathbf{j}_d$ are merely definitions. The theory itself does not indicate such a split. As a matter of fact, we believe this split was first discovered when the problem was studied from the gyrokinetic side.

For each species, we have derived the diamagnetic current as

$$\begin{aligned}\mathbf{j}_M(\mathbf{r})&=e\int\dot{\boldsymbol{\rho}}\delta(\mathbf{X}+\boldsymbol{\rho}-\mathbf{r})F(\mathbf{Z})d^6\mathbf{Z}\\&=-\nabla\times\left(c\mathbf{b}\frac{p_\perp}{B}\right)+O(\epsilon^2)\\&=-\frac{c\nabla p_\perp\times\mathbf{b}}{B}-cp_\perp\nabla\times\frac{\mathbf{b}}{B}+O(\epsilon^2).\end{aligned} \tag{65}$$

Diamagnetic current in an electromagnetic medium of the form $c\nabla\times\mathbf{M}$ is well-known. $\mathbf{M}$ is the magnetic moment of the medium. If we define the plasma magnetic moment to be $-\mathbf{b}W_\perp/B$, then $c\nabla\times(-\mathbf{b}p_\perp/B)$ is the diamagnetic current of the plasma. However, as first realized by Northrop,[28] this physical picture should quantitatively be proved from first principles—gyrokinetic theory. In other words, the fact that in plasmas there is a current of the form $c\nabla\times(-\mathbf{b}p_\perp/B)$ should not be taken granted, but as a result of correct microscopic theories. Northrop proved this fact from the viewpoint of guiding-center motion. But his method is quite complicated and is only outlined in Ref. 28. Obviously, our derivation here is simpler and valid for general 3D (three-dimensional) geometry. We note that the diamagnetic current is not $c\mathbf{b}\times\nabla p_\perp/B$, but rather $c\nabla\times(-p_\perp\mathbf{b}/B)$. In other words, the $c\mathbf{b}\times\nabla p_\perp/B$ term in the fluid model is only part of the diamagnetic current. To get the whole story, we need to consider the second half of the Spitzer paradox, which is about the macroscopic counterparts of the curvature drift and the gradient drift.

From gyrokinetic theory, the current due to the magnetic drift for each species is

$$\mathbf{j}_d(\mathbf{r})=\frac{c\mathbf{b}}{B}\times\left[p_\perp\frac{\nabla B}{B}-p_\parallel\mathbf{b}\times\nabla\times\mathbf{b}\right]+O(\epsilon^2). \tag{66}$$

Therefore, contrary to those viewpoints cited in Sec. II, our conclusion is that both the curvature drift and the gradient drift have their macroscopic counterparts.

Putting together $\mathbf{j}_M$ and $\mathbf{j}_d$ calculated from the gyrokinetic theory, we have

$$\begin{aligned}\mathbf{j}_\perp&=\sum_s \mathbf{j}_{M\perp}+\mathbf{j}_d\\&=\frac{c}{B}\left[\mathbf{b}\times\nabla\sum_s p_\perp+\left(\sum_s p_\parallel-\sum_s p_\perp\right)(\nabla\times\mathbf{b})_\perp\right],\end{aligned} \tag{67}$$

where part of the diamagnetic current $c\Sigma_s p_\perp\nabla\times(\mathbf{b}/B)$ cancels part of the current due to the gradient drift. As discussed before, this result agrees with the fluid equations, correct to order $O(\epsilon^2)$. When the distribution function F_0 is isotropic, $\Sigma_s p_\parallel=\Sigma_s p_\perp=\Sigma_s p$, we recover the familiar fluid result $\mathbf{j}_\perp=c/B\mathbf{b}\times\nabla\Sigma_s p$.

To summarize, using the gyrokinetic theory, we found that Spitzer's original picture as to how the diamagnetic current is generated is correct. However, accurate analysis shows that the fluid perpendicular current in isotropic plasmas, $c\mathbf{b}\times\nabla\Sigma_s p_\perp/B$, contains both the diamagnetic current and part of the current caused by the gradient drift. The other part of the current caused by the gradient drift cancels the current generated by the curvature drift for isotropic distribution functions. This explains why it seems that, for isotropic plasmas, there are no counterparts in the fluid model for the curvature drift and the gradient drift, if $c\mathbf{b}\times\nabla\Sigma_s p_\perp/B$ is mistakenly believed to be exactly the diamagnetic drift.

IV. SUMMARY AND FUTURE WORK

The physical picture of guiding-center motion and its modern quantitative formulation, gyrokinetic theory, is an effective description of magnetized plasmas. On the other hand, fluid equations can be derived exactly as the moment equations of the Vlasov equation without referring to the notion of guiding-center motion. These fluid equations are well-known and valid for both magnetized plasmas and non-magnetized plasmas. It is interesting to compare the gyrokinetic description with the fluid description in magnetized plasmas where both descriptions are correct. The Spitzer paradox, a seeming conflict between the macroscopic fluid picture and the microscopic guiding-center picture regarding the perpendicular current, was noticed almost a half century ago. In general, it is necessary to show the consistency between the gyrokinetic model and the fluid model in all aspects. We have demonstrated this consistency, for the case of equilibrium plasmas ($\partial/\partial t=0$), by systematically deriving the basic fluid equations from the basic gyrokinetic theory. Besides its theoretical importance, recovering fluid equations from the gyrokinetic side has its practical value as well. Gyrokinetic-MHD theory will provide a rigorous theoretical formalism for studying the gyrokinetic-MHD phenomena in ignited tokamaks by both particle simulations and numerical solutions.

Only nonlinear equilibrium is considered in this paper. Recent advances in linear gyrokinetic theory are reported in Refs. 11–14. It is shown there that linear gyrokinetic theory can successfully recover shear Alfvén types of modes in general geometry. In tokamak plasmas, internal kink modes and TAEs, conventionally only studied by fluid models, have been recovered from the gyrokinetic models. The newly developed gyrokinetic perpendicular dynamics enables us to include the compressional Alfvén wave in the gyrokinetic theory. The recovery of general nonlinear fluid dynamics equations from the gyrokinetic theory will be presented in future publications.

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