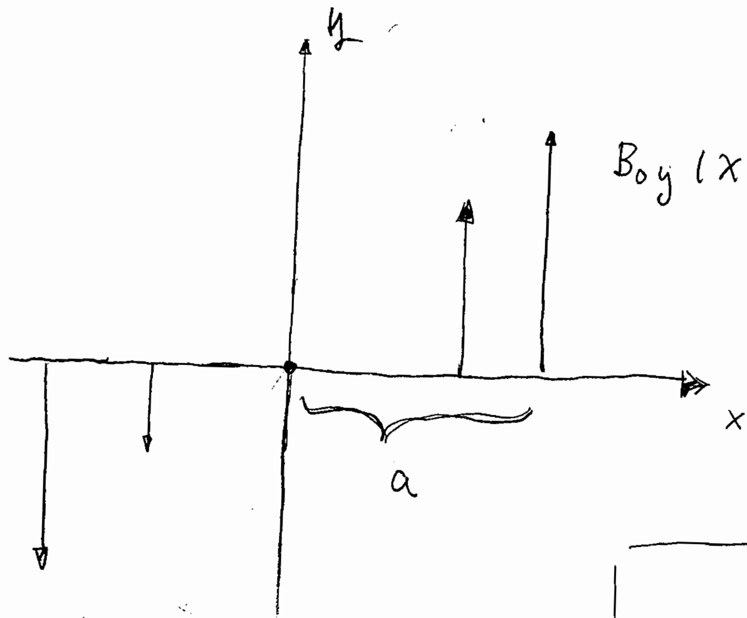
$\boxed{8}$

Resistive mode — tearing mode

Equilibrium



$$\boxed{a \sim \frac{B_{0y}}{B_{0y}'(x)} \; : \text{ scale length of } B_0}$$

$$\vec{B}_0 = B_{0y}(x)\,\hat{y} + B_{0z}\,\hat{z} \quad \text{with } B_{0z} \gg B_{0y}(x)$$

$$B_{0y}(0) = 0$$

$$P_0 = \text{Const}, \quad \rho_0 = \text{Const.}$$

$$j_0 = \frac{c}{4\pi} \nabla \times B_0 = \frac{c}{4\pi} \frac{\partial B_{0y}}{\partial x} \hat{z}$$

Assume incompresshility $\nabla \cdot V = 0$

to replace $\frac{d}{dt}\left(P/\rho^{\gamma}\right) = 0$

$\boxed{\text{Linearized resistive MHD}}$:

$$\begin{cases} \rho_0 \frac{\partial V_1}{\partial t} = -\nabla P_1 + \frac{J_0 \times B_1}{c} + \frac{J_1 \times B_0}{c} & (1) \\ \nabla \cdot V = 0 & (2) \\ \frac{\partial B_1}{\partial t} = \nabla \times (V \times B_0) + \frac{\eta c^2}{4\pi} \nabla^2 B_1 & (3) \end{cases}$$

$\boxed{\text{Looking for mode with}}$

Resistive freq

$$\omega_R \ll \omega \ll \omega_A$$

Alfvén freq

$$\omega_R = \frac{\eta c^2}{4\pi a^2}, \qquad \omega_A = K V_A = K \frac{B_{0y}}{\sqrt{4\pi \rho_0}}$$

and 2D mode with a single k in $\hat{y}$

$$\begin{pmatrix} V \\ B_1 \end{pmatrix} = \begin{pmatrix} V(x) \\ B_1(x) \end{pmatrix} \cdot e^{i(ky - wt)}$$

Eliminate ∇P_1 in (1)

$\nabla \times$ (1) $\Longrightarrow$

$$\rho_0 \frac{\partial}{\partial t} \nabla \times V = - \overbrace{\frac{(j_0 \cdot \nabla) B_1}{c}}^{0} + \frac{(B_1 \cdot \nabla) j_0}{c}$$

$$- \frac{j_1 \cdot \nabla B_0}{c} + \frac{(B_0 \cdot \nabla) j_1}{c} \quad \text{---} \quad (4)$$

$$\nabla \times V = ik V_z \hat{x} - \frac{\partial V_z}{\partial x} \hat{y} + \hat{z} \left[\frac{\partial V_y}{\partial x} - ik V_x \right]$$

$$j_1 = \frac{c}{4\pi} \nabla \times B$$

$$= \frac{c}{4\pi} \left\{ ik B_{1z} \hat{x} - \frac{\partial B_{1z}}{\partial x} \hat{y} + \hat{z} \left[\frac{\partial B_{1y}}{\partial x} - ik B_{1x} \right] \right\}$$

$$(j_0 \cdot \nabla) B_1 = 0$$

$$\frac{(B_1 \cdot \nabla) j_0}{c} = \frac{1}{c} B_{1x} \frac{\partial j_0}{\partial x} \hat{z}$$

$$\frac{j_1 \cdot \nabla B_0}{c} = \frac{1}{c} j_{1x} \cdot \frac{\partial B_{0y}}{\partial x} \hat{y} = \frac{1}{4\pi} i k B_{1z} \frac{\partial B_{0y}}{\partial x} \hat{y}$$

$$\frac{(B_0 \cdot \nabla) j_1}{c} = \frac{1}{4\pi} B_{0y} ik \left[\hat{x}\, ikB_{1z} - \hat{y} \frac{\partial B_{1z}}{\partial x} + \hat{z} \left(\frac{\partial B_{1y}}{\partial x} - ikB_{1x} \right) \right]$$

④ $\underline{\underline{\hat{x}}}$:

$$\omega K V_{1z} \rho_0 = -\frac{1}{4\pi} B_{0y} K^2 B_{1z} \qquad ⑤$$

④ $\hat{y}$:

$$i\omega \rho_0 \frac{\partial V_{1z}}{\partial x} = -\frac{1}{4\pi} ik B_{1z} \frac{\partial B_{0y}}{\partial x} - \frac{1}{4\pi} ikB_{0y} \frac{\partial B_{1z}}{\partial x} \qquad ⑥$$

④ $\hat{z}$:

$$-i\omega \rho_0 \left[\frac{\partial V_y}{\partial x} - ikV_x \right] = \frac{1}{c} B_{1x} \frac{\partial j_0}{\partial x} + \frac{ik}{4\pi} B_{0y} \left(\frac{\partial B_{1y}}{\partial x} - ikB_{1x} \right) \qquad ⑦$$

$$\nabla \cdot V = 0 \Rightarrow \frac{\partial V_x}{\partial x} + ik V_y = 0 \Rightarrow V_y = +\frac{i}{k}\frac{\partial V_x}{\partial x}$$

$$\nabla \cdot B_1 = 0 \Rightarrow \frac{\partial B_{1x}}{\partial x} + ik B_{1y} = 0 \Rightarrow B_{1y} = \frac{i}{k}\frac{\partial B_{1x}}{\partial x}$$

$\rightarrow$ (7) $\Rightarrow$

$$\omega \rho_0 \left[\frac{\partial}{\partial x}\left(\frac{1}{k}\frac{\partial V_x}{\partial x}\right) - k V_x \right] = \frac{1}{4\pi} B_{1x} \frac{\partial^2 B_{0y}}{\partial x^2}$$

$$- \frac{ik}{4\pi} B_{0y} \left[\frac{\partial}{\partial x}\left(\frac{1}{k}\frac{\partial B_{1x}}{\partial x}\right) - k B_{1x} \right]$$

$\Rightarrow$

$$\boxed{\frac{4\pi\omega\rho_0}{k}\frac{\partial^2 V_x}{\partial x^2} - 4\pi\omega\rho_0 k V_x = \underbrace{B_{1x}\frac{\partial^2 B_{0y}}{\partial x^2} - B_{0y}\frac{\partial^2 B_{1x}}{\partial x^2}}_{} + k^2 B_{0y} B_{1x}} \quad \text{---} \ (8)$$

$$\| $$

$$-\frac{\partial}{\partial x}\left[B_{y0}^2 \frac{\partial}{\partial x}\left(\frac{B_{1x}}{B_{y0}}\right) \right]$$

$$③ \Rightarrow \frac{\partial \vec{B}_1}{\partial t} = -(V \cdot \nabla B_0) + (B_0 \cdot \nabla) V + \frac{\eta c^2}{4\pi} \nabla \vec{B}_1$$

$$= -V_x \frac{\partial B_{0y}}{\partial x} \hat{y} + B_{0y} ik \vec{V} + \frac{\eta c^2}{4\pi} \nabla \vec{B}_1$$

$\hat{x}$:

$$\boxed{-i\omega B_{1x} = B_{0y} ik V_x + \frac{\eta c^2}{4\pi} \nabla B_{1x}} \quad \text{--- ⑨}$$

⑧ + ⑨ + B.C. is a well-posed ODE system in terms of V_x, B_{1x},

$\boxed{\text{Outer Region}}$, $\omega_R \ll \omega \ll \omega_A$

$$\cdots (10)$$

(8) $\Rightarrow$ $$\frac{\partial}{\partial x}\left[B_{y0}^2 \frac{\partial}{\partial x}\left(\frac{B_{1x}}{B_{y0}}\right)\right] - K^2 B_{0y} B_{1x} = 0$$

(9) $\Rightarrow$ $$\frac{B_{1x}}{B_{0y}} = -\frac{K}{\omega} V_x \quad \cdots (11)$$

with B.C: $B_{1x} = 0$ at $x = \pm L$

$$\int_{-L}^{0} (10) \times \frac{B_{1x}^*}{B_{0y}}\, dx \Rightarrow$$

$$B_{1x}^* B_{y0} \frac{\partial}{\partial x}\left(\frac{B_{1x}}{B_{0y}}\right)\Bigg|_{-L}^{0} - \int_{-L}^{0} B_{y0}^2 \left(\frac{\partial}{\partial x}\frac{B_{1x}}{B_{0y}}\right)^2 dx$$

$$- \int_{-L}^{0} K^2 B_{1x}^2\, dx = 0 \quad \cdots (12)$$

(11) requires $\dfrac{B_{1x}}{B_{0y}}$ to be well-behaved.

$\Rightarrow$ (12) can not be satisfied unless $B_{1x} = 0$ everywhere

In order for (12) to be true,

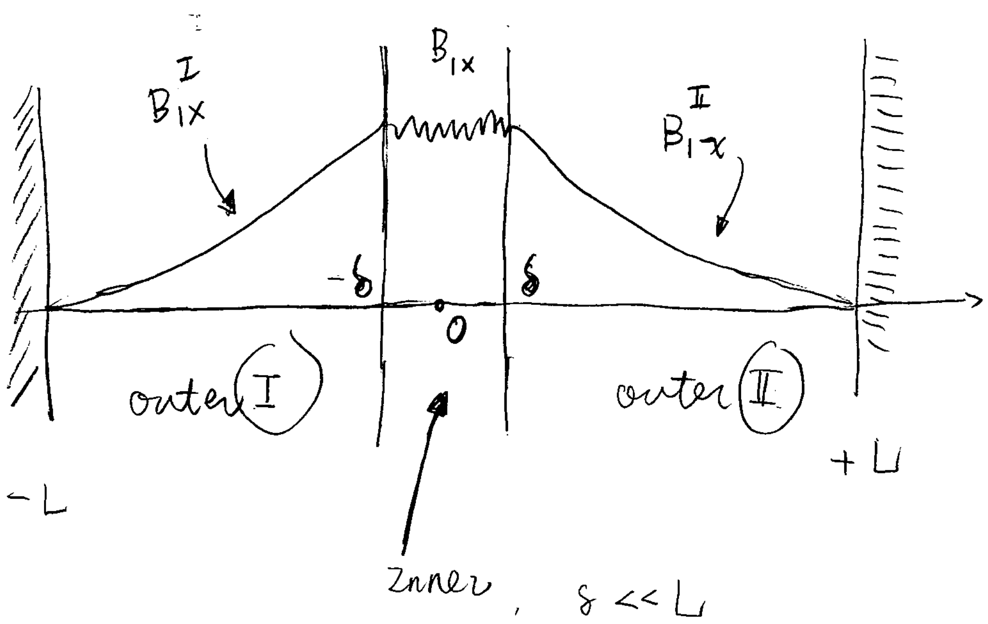
$\dfrac{\partial}{\partial x}\left(\dfrac{B_{1x}}{B_{0y}}\right)$ has to be singular at $x = 0$,

Then, $\dfrac{\eta c^2}{4\pi} \nabla B_{1x}$ in (9) will be important

at $x = 0$, so will be the inertia terms in

(8).

$\Rightarrow$ There exists a boundary layer at $x = 0$.



From outer regions

$$B_{1x}^{I}\Big|_{x=-\delta} = B_{1x}^{II}\Big|_{x=\delta}$$

because $\nabla \cdot B = 0$

But:

$$B_{1y}^{I}\Big|_{x=-\delta} \neq B_{1y}^{II}\Big|_{x=\delta}$$

because the current through the Bondary Layer.

$$\nabla \cdot B = 0 \Rightarrow \frac{\partial B_{1x}}{\partial x} = -ikB_{1y}$$

$$\Rightarrow \quad \Delta' = \frac{1}{B_{1x}^{\mathrm{II}}} \frac{\partial B_{1x}^{\mathrm{II}}}{\partial x}\bigg|_{x=\delta} - \frac{1}{B_{1x}^{\mathrm{I}}} \frac{\partial B_{1x}^{\mathrm{I}}}{\partial x}\bigg|_{x=-\delta} \quad \cdots (13)$$

jump condition determined by outer solutions.

$\boxed{\text{Inner layer}}$: $\frac{\partial^2 V_x}{\partial x^2}$ & $\frac{\partial^2 B_{1x}}{\partial x^2}$ dominate

$$(8) \Rightarrow \quad \frac{4\pi \omega \rho_0}{k} \frac{\partial^2 V_x}{\partial x^2} = -B_{0y} \frac{\partial^2 B_{1x}}{\partial x^2} \quad \cdots (14)$$

$$(9) \Rightarrow \quad -i\omega B_{1x} = B_{0y}\, i k V_x + \frac{\eta c^2}{4\pi} \frac{\partial^2 B_{1x}}{\partial x^2} \quad \cdots (15)$$

$$B_{1x} = \overline{B_{1x}} + \tilde{B}_{1x},$$

Const. across the inner layer, $\overline{B_{1x}} = B_{1x}^{\mathrm{I}}\Big|_{x=-\delta}$

$$\nabla \cdot B = 0 \quad \Rightarrow \quad \frac{\partial B_{1x}}{\partial x} = -ikB_{1y} \ll \frac{B_{1x}}{\delta}$$

$$\Rightarrow \quad \tilde{B}_{1x} \ll \bar{B}_{1x}$$

$$(15) \Rightarrow \quad -\omega \bar{B}_{1x} = B_{oy} k V_x + \frac{\eta c^2}{i 4\pi} \frac{\partial^2}{\partial x^2} \tilde{B}_{1x}$$

$$(14) \Rightarrow \quad \frac{\partial^2 V_x}{\partial x^2} = - \frac{B_{oy} k}{4\pi \omega \rho_o} \frac{\partial^2 \tilde{B}_{1x}}{\partial x^2}$$

$$= - \frac{B_{oy} k}{4\pi \omega \rho_o} \frac{i 4\pi}{\eta c^2} \left[-\omega \bar{B}_{1x} - B_{oy} k V_x \right]$$

$$\Rightarrow \quad \frac{\partial^2 V_x}{\partial x^2} - \frac{B_{oy}^2 k^2 i}{\omega \rho_o \eta c^2} V_x - \frac{i B_{oy} k}{\rho_o \eta c^2} \bar{B}_{1x} = 0$$

because $\delta \ll L$, $\quad B_{oy} \approx B_{oy}'\big|_{x=0} x$

$$\Rightarrow \quad \frac{\partial^2 V_x}{\partial x^2} - \frac{B_{0y}'^2 k^2 i}{\omega \rho_0 \eta c^2} x^2 V_x - \frac{i B_{0y}' k}{\rho_0 \eta c^2} x \bar{B}_{1x}$$

$$= 0$$

Normalized x by $\delta = \left(\frac{\omega \rho_0 \eta c^2}{i B_{0y}'^2 k^2} \right)^{1/4}$

$$= (\gamma \rho_0 \eta)^{1/4} \left(\frac{c}{B_{0y}' k} \right)^{1/2}$$

$\omega = i\gamma$

expecting $\gamma > 0$

$\bar{X} = x/\delta \quad \Rightarrow$

$$\frac{\partial^2 V_x}{\partial \bar{X}^2} - \bar{X}^2 V_x - \underbrace{\frac{i B_{0y}' k \delta^3 \bar{B}_{1x}}{\rho_0 \eta c^2}}_{= \frac{\omega \bar{B}_{1x}}{(B_{0y}' k) \delta}} \bar{X} = 0$$

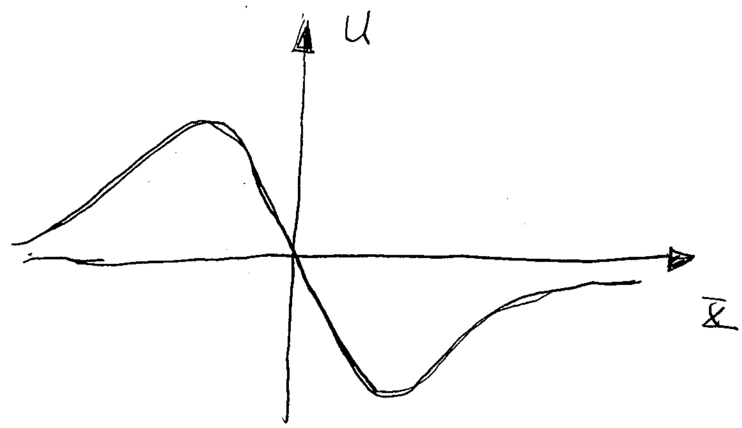
$$u \equiv \frac{v_x \, (B_{0y}' k)\,\delta}{\omega\, \overline{B_{1x}}} \Longrightarrow \frac{\partial^2 u}{\partial \bar{x}^2} = \bar{x}\,(1+\bar{x}u) \quad \cdots (16)$$

B.C. $\bar{x} \to \pm\infty$, $\dfrac{\partial^2 u}{\partial \bar{x}^2}$ finite

$$\Longrightarrow \quad u = -\frac{1}{\bar{x}},$$

Solve (16) for $u(\bar{x})$

Numerically



(14) $\Longrightarrow$

$$\left.\frac{\partial B_{1x}}{\partial x}\right|_{-\delta}^{\delta} = -\frac{4\pi\omega\rho_0}{B_{0y}'\,k}\int_{-\delta}^{\delta}\frac{1}{x}\frac{\partial^2 v_{1x}}{\partial x^2}\,dx$$

$$= -\frac{4\pi\omega\rho_0}{B_{0y}'\,k\,\delta^2}\,\frac{\omega\,\overline{B_{1x}}}{B_{0y}'\,k\,\delta}\int_{-\infty}^{\infty}\frac{1}{\bar{x}}\frac{\partial^2 u}{\partial \bar{x}^2}\,d\bar{x}$$

$$\Delta' = \frac{1}{\bar{B}_{1x}} \frac{\partial B_{1x}}{\partial x}\Big|_{-\delta}^{\delta} = + \frac{4\pi \gamma^2 \rho_0}{(B'_{0y} k)^2 \delta^3} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{X} \frac{\partial^2 u}{\partial X^2} dX}_{\text{real number } \alpha}$$

matching condition

$\alpha = 2.12$

$$\Rightarrow \quad \frac{\Delta'}{\alpha} = \frac{4\pi \, \rho_0 \, \gamma^2}{(B'_{0y} k)^2 \, \dfrac{\gamma^{3/4} \rho_0^{3/4} \eta^{3/4} c^{3/2}}{(B'_{0y} k)^{3/2}}}$$

$$= \frac{4\pi \, \rho_0^{1/4} \, \gamma^{5/4}}{(B'_{0y} k)^{1/2} \, \eta^{3/4} \, c^{3/2}}$$

$$\gamma = \left(\frac{\Delta'}{\alpha}\right)^{4/5} \frac{(B'_{0y} k)^{\frac{2}{5}} \, \eta^{3/5} \, c^{\frac{6}{5}}}{(4\pi)^{4/5} \rho_0^{1/5}}$$

$$= \left(\frac{\Delta' a}{\alpha}\right)^{4/5} \frac{(B'_{0y} k a)^{\frac{2}{5}} \quad \eta^{3/5} c^{6/5}}{\rho_0^{1/5} \, 4\pi^{1/5} \quad 4\pi^{3/5} a^{6/5}}$$

$$= \left(\frac{\Delta' a}{\alpha}\right)^{4/5} \omega_A^{2/5} \, \omega_R^{3/5}$$