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Ideal MHD waves and instabilities

Around an equilibrium : $\vec{B}_0$, P_0, ρ_0.

Look at the case without equilibrium flow, $\vec{V}_0 = 0$.

Linearize the Ideal MHD Eqs. :

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 V_1) = 0 \quad \cdots (1)$$

$$\rho_0 \frac{\partial V_1}{\partial t} = -\nabla P_1 + \frac{(\nabla \times B_1) \times B_0}{4\pi} + \frac{(\nabla \times B_0) \times B_1}{4\pi} \quad \cdots (2)$$

$$\frac{1}{\rho_0^\gamma}\frac{\partial P_1}{\partial t} - \gamma \frac{P_0}{\rho_0^{\gamma+1}} \frac{\partial \rho_1}{\partial t} + V_1 \cdot \nabla \frac{P_0}{\rho_0^\gamma} = 0 \quad (3)$$

$$\frac{\partial B_1}{\partial t} = \nabla \times (V_1 \times B_0) \quad (4)$$

Let $\quad v_1 = \dfrac{\partial \xi(x,t)}{\partial t}$

$$(1) \Rightarrow \qquad \rho_1 = -\nabla\cdot(\rho_0 \xi)$$

$$(3) \Rightarrow \qquad P_1 = -\frac{\gamma P_0}{\rho_0}\,\nabla\cdot(\rho_0\xi) - \rho_0^{\gamma}\,\xi\cdot\nabla\frac{P_0}{\rho_0^{\gamma}}$$

$$= -\gamma P_0\,\nabla\cdot\xi - \xi\cdot\nabla P_0$$

$$(4) \Rightarrow \qquad B_1 = \nabla\times(\xi\times B_0)$$

$$\Downarrow + (2)$$

Drop Subscript "0" for Equilibrium field

$$\rho\,\frac{\partial^2 \xi}{\partial t^2} = F(\xi)$$

$$F(\xi) \equiv \frac{1}{4\pi}(\nabla\times B)\times B_1 + \frac{1}{4\pi}(\nabla\times B_1)\times B + \nabla(\xi\cdot\nabla P + \gamma P\,\nabla\cdot\xi)$$

(5)

Waves in a homogeneous plasma

$B_0 = \text{const}$

$p_0 = \text{const}$

$\rho_0 = \text{const},$

$$(B_1, \xi) \sim e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$B_1 = i k \times (\xi \times B) = i \xi (k \cdot B) - i B (k \cdot \xi)$$

$$(\nabla \times B_1) \times B = i B_1 (k \cdot B) - i k (B_1 \cdot B)$$

$$= -(k \cdot B)^2 \xi + (k \cdot B)(B k) \cdot \xi + k \cdot B (k B) \xi$$
$$- B^2 (k k) \xi$$

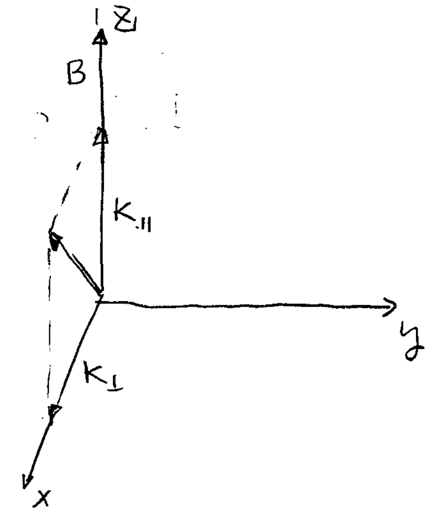
$$\nabla \gamma p \nabla \cdot \xi = -\gamma P (k k) \cdot \xi$$

Eq (5) $\Rightarrow$
$$\boxed{\left\{ \left[\omega^2 - (k \cdot V_A)^2\right] \overleftrightarrow{I} - (V_A^2 + V_s^2)\, \vec{k}\vec{k} + (k \cdot V_A)(\vec{V}_A \vec{k} + \vec{k} \vec{V}_A) \right\} \vec{\xi} = 0} \quad (6)$$

where $\vec{V}_A = \dfrac{\vec{B}}{\sqrt{4\pi \rho}}$, $V_s = \sqrt{\gamma P / \rho}$

Choose $\vec{B} = B\hat{z}$

$$k = \underset{\substack{\parallel \\ K_{\parallel}}}{K_z}\,\hat{z} + \underset{\substack{\parallel \\ K_{\perp}}}{K_x}\,\hat{x}$$



In Matrix form Eq ⑥ is:

$$\begin{pmatrix} \omega^2 - k_x^2(V_A^2+V_s^2) - k_z^2 V_A^2 & 0 & -k_x k_z V_s^2 \\ 0 & \omega^2 - k_z^2 V_A^2 & 0 \\ -k_x k_z V_s^2 & 0 & \omega^2 - k_z^2 V_s^2 \end{pmatrix} \begin{pmatrix} \xi_x \\ \xi_y \\ \xi_z \end{pmatrix} = 0$$

The ξ_y is decoupled!

Det $= 0$

$$\Rightarrow \quad (\omega^2 - k_z^2 V_A^2)\left[\left[\omega^2 - k_\perp^2(V_A^2+V_s^2) - k_\parallel^2 V_A^2\right]\left(\omega^2 - k_\parallel^2 V_s^2\right) - k_\perp^2 k_\parallel^2 V_s^4\right] = 0$$

$$\Rightarrow \quad (\omega^2 - k_z^2 V_A^2)\left[\omega^4 - k^2(V_A^2+V_s^2)\,\omega^2 + k_\parallel^2 k^2 V_A^2 V_s^2\right] = 0$$

$\omega^2 = k_\parallel^2 V_A^2$ Shear Alfvén wave

$$\left(\frac{\omega}{k}\right)^2 = \frac{(V_A^2 + V_s^2) \pm \sqrt{(V_A^2+V_s^2)^2 - 4\left(\frac{k_\parallel}{k}\right)^2 V_A^2 V_s^2}}{2}$$

Fast (+) & Slow (−) magneto-sonic wave

$\boxed{\text{No instability and damping for homogeneous plasmas in ideal MHD}}$

Kinetic theory or Non-ideal MHD is needed for instability or damping associated with velocity-space inhomogeneities, such as temperature anisotropy instability, two-stream instability, and Landau damping.

Energy Principle for General Equilibrium

- For the linearized MHD Eq (5).
- But, we start from the nonlinear energy conservation & the Lagrangian coordinate

$$\begin{cases} \mathcal{E} = \frac{1}{2}\int_V \rho v^2 d^3x + \int_V \left(\frac{P}{\gamma - 1} + \frac{B^2}{8\pi} \right) d^3x \\ \frac{d\mathcal{E}}{dt} = 0 \end{cases}$$

$$x = \underbrace{x(x_0, t)}_{\text{Lagrangian coordinate}} = x_0 + \xi(x_0, t)$$

Section 3 variational principles

(V, ρ, P, B) are functions of ξ

$$V = \left. \frac{\partial \xi}{\partial t} \right|_{x_0} \qquad (10)$$

$$\rho = \rho_0(x_0) / J \qquad J = \mathrm{Det}\left| \bar{\bar{I}} + \frac{\partial \xi_j(x_0, t)}{\partial x_{0i}} \right| \qquad (11)$$

(12)

$$p = P_0(x_0) / J^{\gamma}$$

$$B_i = \frac{B_{0m}}{J}\left[\delta_{mi} + \frac{\partial \xi_m(x_0,t)}{\partial x_{0i}}\right] \qquad (13)$$

$\boxed{\text{Taylor expansion of } \mathcal{E} \text{ in terms of small } \xi}$

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \cdots = \text{const}$$

$$\mathcal{E}_0 = \frac{1}{2}\int_V \left(\frac{P_0}{\gamma-1} + \frac{B_0^2}{8\pi}\right) d^3x = \text{const}$$

$$\mathcal{E}_1 = \frac{1}{2}\int_V \left[\frac{P_1}{\gamma-1} + \frac{B_0\cdot B_1}{4\pi}\right] d^3x$$

$$= \frac{1}{2}\int_V \left[\frac{-\gamma P_0 \nabla\cdot\xi - \xi\cdot\nabla P_0}{\gamma-1} + \frac{B_0\cdot\nabla\times(\xi\times B_0)}{4\pi}\right] d^3x$$

$$\frac{(+\gamma-1)\,\xi\cdot\nabla P_0 - \gamma\nabla\cdot(\xi P_0)}{\gamma-1} \qquad \frac{1}{4\pi}\left(\nabla\cdot(\xi\times B_0\times B_0) + (\xi\times B_0)\cdot\nabla\times B_0\right)$$

$$= \frac{1}{2}\int_V \xi\cdot\underbrace{\left(\nabla P_0 + \frac{B\times\nabla\times B_0}{4\pi}\right)}_{=0} d^3x + \frac{1}{2}\int_V \nabla\cdot\left(\frac{\xi\times B_0\times B_0}{4\pi} - \frac{\gamma\,\xi P_0}{\gamma-1}\right) d^3x$$

$$= 0$$

$$\boxed{\mathcal{E}_2 = \int_V \rho_0 \left(\frac{\partial \xi}{\partial t}\right)^2 d^3x + W_2(\xi, \xi)} \quad (14)$$

linear on both arguments.

The Expression for $W_2(\xi, \xi)$ can be given from Eqs. (10)–(13), but it's complicated. But we can express $W_2(\xi, \xi)$ in terms of $F(\xi)$.

$$\frac{d\mathcal{E}}{dt} = \frac{d(\mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \cdots)}{dt} = \frac{d(\mathcal{E}_2 + \mathcal{E}_3 + \cdots)}{dt} = 0$$

$$\left(\frac{d\mathcal{E}}{dt}\right) = \underset{O(\xi)}{\left(\frac{d\mathcal{E}}{dt}\right)_0} + \underset{O(\xi)}{\left(\frac{d\mathcal{E}}{dt}\right)_1} + \underset{O(\xi^2)}{\left(\frac{d\mathcal{E}}{dt}\right)_2} + \left(\frac{d\mathcal{E}}{dt}\right)_3 + \cdots = 0$$

$$\Rightarrow \left(\frac{d\mathcal{E}}{dt}\right)_i = 0 \qquad (i = 0, 1, 2, \cdots)$$

Note: $\left(\dfrac{d\mathcal{E}}{dt}\right)_i \neq \dfrac{d\mathcal{E}_i}{dt}$,

For example, $\dfrac{d\mathcal{E}_2}{dt}$ Contains $O(\epsilon^3)$, $O(\epsilon^4)$ ⋯ terms

$$\frac{d\mathcal{E}_2}{dt} = \int \rho_0 \left(\frac{\partial \xi}{\partial t}\right) \frac{\partial^2 \xi}{\partial t^2}\, d^3x + W_2\left(\frac{\partial \xi}{\partial t}, \xi\right) + W_2\left(\xi, \frac{\partial \xi}{\partial t}\right)$$

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = \underbrace{F(\xi)}_{\text{Eq.(5)}} + \underbrace{F_2(\xi)}_{O(\xi^2)} + \underbrace{F_3(\xi)}_{O(\xi^3)} + \cdots \qquad (14.1)$$

$$+ \cdots$$

$$\Rightarrow \quad \frac{d\mathcal{E}_2}{dt} = O(\xi^2) + O(\xi^3) + \cdots$$

$$\boxed{\begin{aligned} \left(\frac{d\mathcal{E}}{dt}\right)_2 &= \left(\frac{d\mathcal{E}_2}{dt}\right)_2 \\ &= \int \frac{\partial \xi}{\partial t} F(\xi)\, d^3x + W_2\left(\frac{\partial \xi}{\partial t}, \xi\right) + W_2\left(\xi, \frac{\partial \xi}{\partial t}\right) \\ &= 0 \qquad \boxed{14.5} \end{aligned}}$$

This is true at any t. At $t=0$, we can pick any function $\eta \equiv \dfrac{\partial \xi}{\partial t} \sim O(\xi)$

$$\Rightarrow \quad \boxed{\int_V \eta F(\xi)\, d^3x + W_2(\eta, \xi) + W_2(\xi, \eta) = 0} \quad (15)$$

$\boxed{\text{Eq. (15)} \Rightarrow \text{Two important Results}}$

- Expression for $W_2(\xi, \xi)$ in terms of $F(\xi)$

$$\frac{1}{2}\int_V \xi F(\xi)\, d^3V = -W_2(\xi, \xi) \quad \cdots \; (16)$$

- $F(\xi)$ is self-adjoint:

$$\int_V \eta F(\xi)\, d^3x = \int \xi F(\eta)\, d^3x \quad \cdots \; (17)$$

$\boxed{\text{Eq. (14.5)} \Rightarrow \text{Linear Energy Conservation.}}$

$$\mathcal{E}_\ell \equiv \int_V \frac{1}{2}\rho_0 \left(\frac{\partial \xi}{\partial t}\right)^2 d^3x - \int_V \frac{1}{2}\xi F(\xi)\, d^3x = \text{Const.} \quad \cdots \; (18)$$

where ξ satisfies Eq (5).

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = F(\xi) \quad \cdots \; (5)$$

Note: $\mathcal{E}_\ell \neq \mathcal{E}_2$, The ξ in $\mathcal{E}_2$ satisfies Eq. (14.1)

$$\delta K(\xi, \eta) \equiv \frac{1}{2} \int_V \rho_0 \frac{\partial \xi}{\partial t} \cdot \frac{\partial \eta}{\partial t} \, d^3x$$

$$\delta W(\xi, \eta) \equiv -\frac{1}{2} \int_V \xi \cdot F(\eta) \, d^3x$$

$$\mathcal{E}_p = \delta K(\xi, \xi) + \delta W(\xi, \xi) = \text{const} \quad \text{--- (18)}$$

Eq. (18) $\Longrightarrow$ **Energy Principle**

$$\boxed{\text{Eq. (5) is stable} \iff \delta W(\xi, \xi) > 0 \text{ for any real } \xi \text{ satisfying B.C.}}$$

proof:

(i) $\boxed{\text{Eq. (5) is stable} \Longleftarrow \delta W(\xi, \xi) > 0 \text{ for any real } \xi.}$

otherwise, Eq. (5) has an unstable eigenmode

$$\xi = \xi_x \, e^{-i\omega t} \quad \text{with} \quad \gamma \equiv \operatorname{Im}(\omega) > 0$$

Let $\eta = \frac{1}{2}(\bar{\xi} + \xi) = e^{\gamma t} \operatorname{Re}\left[\xi_x \, e^{-i\omega_r t} \right]$

obviously η is a solution of Eq. (5)

Eq (18) $\Rightarrow$

$$\delta W(\eta,\eta) = \text{const} - \delta K(\eta,\eta)$$

$$\delta K(\eta,\eta) = \frac{1}{2}\int_V \rho_0 \left(\frac{\partial \eta}{\partial t}\right)^2 d^3x = \text{const}\, e^{2\gamma t}$$

grows without bound

$\Rightarrow \quad \delta W(\eta,\eta) < 0$ when t is big enough. ∎

(ii) $\boxed{\text{Eq. (3) has an unstable solution} \Leftarrow \delta W(\xi,\xi) < 0 \text{ for a real } \xi}$

Let $\delta W(\xi_0,\xi_0) < 0$ for a real ξ_0,

Define: $$\gamma_0^2 \equiv \frac{-\delta W(\xi_0,\xi_0)}{\underbrace{\frac{1}{2}\int_V \rho\, \xi_0^2\, d^3x}_{= \delta K(\xi_0,\xi_0)}} > 0 . \quad \cdots (18)$$

let $\xi(t=0) = \xi_0$ and $\frac{\partial \xi}{\partial t}(t=0) = \gamma_0 \xi_0$,

and we now prove

$$\delta K(\xi,\xi) \equiv \frac{1}{2}\int_V \rho\, \xi^2\, d^3x \geq \delta K(\xi_0,\xi_0)\, e^{2\gamma_0 t},$$

which means $\xi(t)$ is an unstable solution with a growth rate $\geq 2\gamma_0$.

$$\frac{d\,\delta K(\xi,\xi)}{dt} = \int_V \xi \cdot \frac{\partial \xi}{\partial t}\, \rho\, d^3x$$

$$\frac{d^2}{dt^2}\delta K(\xi,\xi) = \int_V \left[\rho\left(\frac{\partial \xi}{\partial t}\right)^2 + \xi \cdot F(\xi)\right] d^3x$$

$$= 2\left[\delta K\left(\frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial t}\right) - \delta W(\xi,\xi)\right]$$

Eq (17)

$$= 2\left[2\delta K\left(\frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial t}\right) + \mathcal{E}_2\right],$$

where $\mathcal{E}_2 = \delta K\left(\frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial t}\right) + \delta W(\xi,\xi)$

$$= \left[\delta K\left(\frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial t}\right) + \delta W(\xi,\xi)\right]\Big|_{t=0} \stackrel{\text{Eq. (18)}}{=} 0$$

$$(\dot{\delta K})^2 = \left[\int_V d^3x\, \rho \left(\xi \cdot \frac{\partial \xi}{\partial t}\right)\right]^2 \leq \left[\int_V d^3x\, \rho\left(\xi^2\right)\right]\left[\int_V d^3x\, \rho\left(\frac{\partial \xi}{\partial t}\right)^2\right]$$

$$= \delta K\, \ddot{\delta K}$$

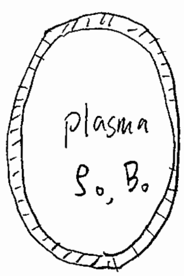
schwarz inequality

$$\Rightarrow \quad \ddot{\delta K}\,\delta K - (\dot{\delta K})^2 \geq 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\dot{\delta K}}{\delta K}\right) \geq 0 \quad \Rightarrow \quad \frac{\dot{\delta K}}{\delta K} \geq \left(\frac{\dot{\delta K}}{\delta K}\right)_{t=0} = 2\gamma_0$$

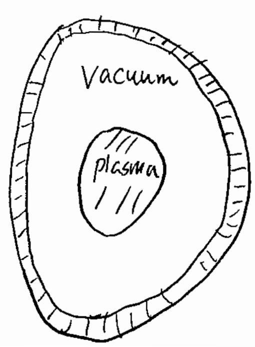
$$\Rightarrow \quad \delta K \geq e^{2\gamma_0 t}\,\delta K\big|_{t=0} \qquad \blacksquare$$

What is the B.C.?



perfectly conducting wall

Including plasma - vacuum - wall B.C. as special cases



Vacuum can be treated as plasma with $\rho_0 = 0$

IF vacuum region exists, the integration domain V in the energy principle can be replaced by the plasma domain only, Because the V in $\int_V d^3x$ of Eq (17) can be replaced by the plasma domain.

Extended Energy principle

was proved by much more complicated methods previously

For the purpose of using harmonic eigenmodes, we note.

$$\boxed{\begin{array}{l} \delta W(\xi,\xi) > 0 \quad \text{for any real } \xi \\ \qquad\quad \Updownarrow \\ \delta W(\eta^*,\eta) > 0 \quad \text{for any complex } \eta \end{array}}$$

proof

(i); $\delta W(\eta^*,\eta) > 0$ for any complex η

$$\Downarrow$$

$\delta W(\xi,\xi) > 0$ for any real ξ ← $\boxed{\text{Obviously}}$

(ii)

$$\delta W(\eta^*,\eta) = \delta W(\eta_R,\eta_R) + \delta W(\eta_I,\eta_I)$$

$$-i\,\delta W(\eta_I,\eta_R) + i\,\delta W(\eta_R,\eta_I)$$

$$= \delta W(\eta_R,\eta_R) + \delta W(\eta_I,\eta_I)$$

self adjointness
$\delta W(\eta_I,\eta_R) = \delta W(\eta_R,\eta_I)$

Therefore, $\delta W(\xi,\xi) > 0$ for any real ξ

$$\Downarrow$$

$\delta W(\eta^*,\eta) > 0$ for any complex η

▨

$$\boxed{\text{Different Forms for } \delta W(\xi^*, \xi)}$$

$$\boxed{\delta W(\xi^*, \xi) = -\frac{1}{2}\int_V \xi^* \cdot \Big[\underbrace{\frac{1}{4\pi}(\nabla\times B)\times B_1}_{①} + \underbrace{\frac{1}{4\pi}(\nabla\times B_1)\times B}_{②} + \underbrace{\nabla(\xi\cdot\nabla P)}_{③} + \underbrace{\nabla(\gamma P\, \nabla\cdot\xi)}_{④} \Big]\, d^3x} \quad \ldots (20)$$

$$② = -\frac{1}{2}\int_V (\nabla\times B_1)\cdot(B\times\xi^*)\,d^3x = \frac{1}{2}\int_V B_1^2\, d^3x - \underbrace{\int_{\partial V} \left[B_1\times(B\times\xi^*)\right]\cdot ds}$$

$$\int_{\partial V} B\cdot ds\,[B_1\cdot\xi^*] \to 0$$

$$-\int_{\partial V} \underbrace{\xi^*\cdot ds}_{\xi_\perp^*\cdot ds}\,[B_1\cdot B]$$

$$\nabla\cdot\left[B_1\times(B\times\xi^*)\right]$$

$$= -B_1\cdot\underbrace{\nabla\times(B\times\xi^*)}_{-B_1^*} + (B\times\xi^*)\cdot\nabla\times B_1$$

Assume $\vec{B}\cdot d\vec{s} = 0$

$$④ = +\frac{1}{2}\int_V \gamma P(\nabla\cdot\xi)^2\, d^3x - \frac{1}{2}\int_{\partial V} [\gamma P\,\nabla\cdot\xi]\, \underbrace{\xi^*\cdot ds}_{\xi_\perp^*\cdot ds}$$

$$① + ③ = -\frac{1}{2}\int_V \xi_\perp^*\cdot\left[\frac{1}{4\pi}(\nabla\times B)\times B_1 + \nabla(\xi\cdot\nabla P)\right] d^3x$$

$$\boxed{b\cdot\left[\frac{j\times B_1}{c} + \nabla\cdot(\xi\cdot\nabla p)\right] = 0}$$

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$$b\cdot\left[\frac{J}{c}\times B_1\right] = -\frac{J\times\vec{B}}{cB}\cdot B_1 = -\frac{\nabla p}{B}\cdot B_1$$

$$= -\frac{\nabla P\cdot[\nabla\times(\xi\times B)]}{B}$$

$$= \frac{\nabla\cdot[\nabla p\times(\xi\times B)]}{B} = \frac{\nabla\cdot(\xi\,\overset{\nearrow 0}{\vec{B}\cdot\nabla p} - \vec{B}\,\xi\cdot\nabla p)}{B}$$

$$= -\frac{\vec{B}}{B}\nabla(\xi\cdot\nabla p)$$

$$(1)+(3) = -\frac{1}{2}\int_V\left[\xi_\perp^*\cdot\frac{1}{4\pi}(\nabla\times B)\times B_1\right]d^3x$$

$$+\frac{1}{2}\int_V \underbrace{(\xi\cdot\nabla p)}_{\xi_\perp\cdot\nabla p}\,\nabla\cdot\xi_\perp^*\,d^3x - \frac{1}{2}\int_{\partial V}[ds\cdot\xi_\perp^*]\underbrace{\xi\cdot\nabla p}_{\xi_\perp\cdot\nabla p}$$

$$\delta W = \delta W_F + B.T.$$

$$\delta W_F = \frac{1}{2}\int_V d^3x\left[\frac{B_1^2}{4\pi} - \xi_\perp^*\cdot\frac{(\nabla\times B)\times B_1}{4\pi} + \gamma P\,|\nabla\cdot\xi|^2 + \xi_\perp\cdot\nabla p\,\nabla\cdot\xi_\perp^*\right]$$

(21)

$$B.T. = \frac{1}{2}\int_{\partial V}\vec{ds}\cdot\xi_\perp^*\left[\frac{B_1\cdot B}{4\pi} - \gamma p\,\nabla\cdot\xi - \xi_\perp\cdot\nabla p\right]$$

(22)

$\boxed{\text{Intuitive Form for } \delta W_F(\xi^*, \xi)}$

$$\delta W_F = \frac{1}{2}\int_V d^3x \left[\frac{B_1^2}{4\pi} - \xi_\perp^* \cdot \frac{J \times B_1}{c} + \gamma p |\nabla\cdot\xi|^2 + \xi_\perp \cdot \nabla p \, \nabla\cdot\xi_\perp^* \right]$$

$\boxed{\text{Eliminate } B_{1\parallel} \text{ using } \vec{\xi}_\perp, \text{ and } \vec{J}_\perp \text{ using } p \text{ and } \vec{B}}$

$$\boxed{B_1^2 = B_{1\perp}^2 + B_{1\parallel}^2}$$

$$B_{1\parallel} = b \cdot \nabla \times (\xi_\perp \times B) = b \cdot \left(-\vec{B}\, \nabla\cdot\xi_\perp + \underbrace{\vec{B}\cdot\nabla\, \xi_\perp}_{(i)} - \underbrace{\xi_\perp \cdot \nabla \vec{B}}_{(ii)} \right)$$

$$(i): \quad b \cdot \left[\vec{B}\cdot\nabla\, \xi_\perp\right] = \vec{B}\cdot\nabla(\overset{0}{b\cdot\xi_\perp}) - \left[\vec{B}\cdot\nabla b\right]\cdot\xi_\perp = -B\,\kappa\cdot\xi_\perp$$

$$(ii): \quad b \cdot \left[\xi_\perp \cdot \nabla \vec{B}\right] = \xi_\perp\cdot\nabla(B) - \overbrace{(\xi_\perp\cdot\nabla\vec{b})\cdot b}^{0}\, B$$

$$\frac{J\times B}{c} = \kappa \frac{B^2}{4\pi} - \nabla_\perp \frac{B^2}{8\pi} = \nabla p \qquad = \xi_\perp \cdot \left[\kappa B - \frac{4\pi \nabla p}{B}\right]$$

$$\Rightarrow \quad B_{1\parallel} = -B\,\nabla\cdot\xi_\perp - 2B\,\kappa\cdot\xi_\perp + \frac{4\pi}{B}\,\xi_\perp\cdot\nabla p$$

$$\boxed{0 = a\cdot\nabla(b\cdot b) = 2(a\cdot\nabla b)\cdot b}$$

$$\boxed{\xi_\perp^* \cdot \left[\frac{J \times B_1}{c}\right]} = \frac{\xi_\perp^*}{c} \cdot \left[J_\perp \times B_{1\parallel} + J_\perp \times B_{1\perp} + J_\parallel \times B_{1\perp} \right]$$

$$= \frac{\xi_\perp^* \cdot J_\perp \times \vec{B}_{1\parallel}}{c} + \xi_\perp^* \cdot \frac{J_\parallel \times \vec{B}_{1\perp}}{c}$$

$$\parallel$$

$$\xi_\perp^* \cdot \frac{\nabla P}{B} B_{1\parallel}$$

$$\frac{B_{1\parallel}^2}{4\pi} - \frac{\xi_\perp^* \cdot \nabla P}{B} B_{1\parallel} = B_{1\parallel} \left(\frac{-B\, \nabla \cdot \xi_\perp^* - 2B\, \kappa \cdot \xi_\perp^*}{4\pi} \right)$$

$$= \frac{\left(-B\, \nabla \cdot \xi_\perp - 2B\, \kappa \cdot \xi_\perp\right)^2}{4\pi} - \frac{1}{B} \xi \cdot \nabla P \left(B\, \nabla \cdot \xi_\perp^* + 2B\, \vec{\kappa} \cdot \xi_\perp^* \right)$$

$$\Rightarrow \quad \frac{B_{1\parallel}^2}{4\pi} - \frac{\xi_\perp^* \cdot J_\perp \times B_{1\parallel}}{B} + \xi \cdot \nabla p \;\; \nabla \cdot \xi_\perp^*$$

$$= \frac{\left(B\, \nabla \cdot \xi_\perp + 2B\, \kappa \cdot \xi_\perp\right)^2}{4\pi} - 2\xi \cdot \nabla P \;\; \vec{\kappa} \cdot \xi_\perp^*$$

$\Rightarrow$

$$\delta W_F(\xi^*, \xi) = \frac{1}{2}\int d^3x \left[\underbrace{\frac{B_{1\perp}^2}{4\pi}}_{a} + \underbrace{\frac{\left[B\,\nabla\cdot\xi_\perp + 2B\,\kappa\cdot\xi_\perp\right]^2}{4\pi}}_{b} + \underbrace{\gamma p\,|\nabla\cdot\xi|^2}_{c} \right.$$

$$\left. - \underbrace{2(\xi_\perp\cdot\nabla p)(\kappa\cdot\xi_\perp^*)}_{d} - \underbrace{\xi_\perp^* \times \vec{J}_{\parallel}\cdot B_{1\perp}}_{e} \right]$$

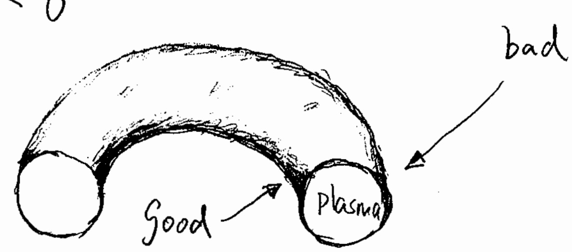
Stabilizing

(a) shear Alfvén, bending

(b) compressional Alfvén, Compressing

(c) Sound, compressing

(d) good curvature, $\nabla p \cdot \kappa < 0$



De-stabilizing

(d) bad curvature, $\nabla p \cdot \kappa > 0$

(e) $\vec{J}_{\parallel}$

$\boxed{\text{Applications of the energy principle}}$

$\boxed{\theta\text{-pinch}}$

$$\left.\begin{aligned} J_{\parallel} &= 0 \\ \kappa &= 0 \end{aligned}\right\} \Longrightarrow \text{stable}$$

$\boxed{z\text{ - pinch}}$

$$\frac{d}{dr}\left(p + \frac{B_\theta^2}{8\pi}\right) + \frac{B_\theta^2}{4\pi r} = 0$$

Stratage: pick $\vec{\xi}$ to minimize $\delta W(\xi^*, \xi)$.

$$\begin{cases} \text{Min}(\delta W) > 0 & \Rightarrow \text{stable} \\ \delta W < 0 \text{ for some } \xi & \Rightarrow \text{unstable} \end{cases}$$

consider $\xi = \vec{\xi}(r)\, e^{i(m\theta + kz)}$

parallel component ξ_θ only appears in $|\nabla\cdot\xi|^2$,

We can set $|\nabla\cdot\xi| = 0$ to minimize δW with respect to ξ_θ,

$$\nabla\cdot\xi = 0 \Rightarrow \xi_\theta = \frac{i}{m}\left[(r\xi_r)' + ik\xi_z\right]$$

$$(m \neq 0)$$

$$B_{1\perp} = \left[\nabla\times(\xi\times B)\right]_\perp = \frac{imB_\theta}{r}\left(\xi_r\hat{e}_r + \xi_z\hat{e}_z\right)$$

$$\vec{K} = -\frac{e_r}{r}$$

$$\vec{J}_\parallel = 0$$

$$\nabla \cdot \xi_\perp + 2\xi_\perp \cdot \kappa = r\left(\frac{\xi_r}{r}\right)' + ik\xi_z$$

$$\delta W_F(\xi^*, \xi) = \frac{1}{2} 2\pi R_0 \int_0^a W(r)\, r\, dr$$

$$W(r) = \frac{m^2 B_\theta^2}{4\pi r^2}(\xi_r^2 + \xi_z^2) + \frac{B_\theta^2}{4\pi}\left| r\left(\frac{\xi_r}{r}\right)' + ik\xi_z \right|^2$$

$$+ \frac{2p'}{r}\xi_r^2$$

Minimize W.R.T ξ_z

$$\Rightarrow \left(\frac{m^2 B_\theta^2}{4\pi r^2} + \frac{2p'}{r}\right)\xi_r^2 + \frac{m^2 B_\theta^2}{4\pi(m^2 + k^2 r^2)}\left|\left(\frac{\xi}{r}\right)'\right|^2$$

$\boxed{\text{Complete the Square}}$

$$a^2\xi_z^2 + (b + ik\xi_z)^2 = a^2\xi_z + b^2 - ibk\xi_z^* + ik\xi_z b^* + k^2\xi_z^2$$

$$= b^2 - \frac{b^2 k^2}{a^2+k^2} + \left[\frac{b^2k^2}{a^2+k^2} - ibk\xi_z^* + ik\xi_z b^* + (k^2+a^2)\xi_z^2\right]$$

$$= b^2\left[1 - \frac{k^2}{a^2+k^2}\right] + \left[\frac{b}{\sqrt{a^2+k^2}} + i\sqrt{k^2+a^2}\,\xi_z\right]^2$$

$$= b^2\left[1 - \frac{k^2}{a^2+k^2}\right] = \frac{b^2a^2}{a^2+k^2} \qquad b = r\left(\frac{\xi_r}{r}\right)'$$

minimize W.R.T. ξ_z

$$a^2 = \frac{m^2}{r^2}$$

$$W(r) = \left(\frac{m^2 B_\theta^2}{4\pi r^2} + \frac{2p'}{r}\right)\xi_r^2$$

minimize
W.R.T. K
$K \to \infty$

(i) $\dfrac{m^2 B_\theta^2}{4\pi} + 2rp' > 0 \Rightarrow$ stable

(ii) $\dfrac{m^2 B_\theta^2}{4\pi} + 2rp' < 0$ for some $0 < r_0 < a$,

$\Rightarrow$ unstable with $\xi_r = \delta(r - r_0)$

$\boxed{\text{Stable Condition (necessary \& sufficient)}}$

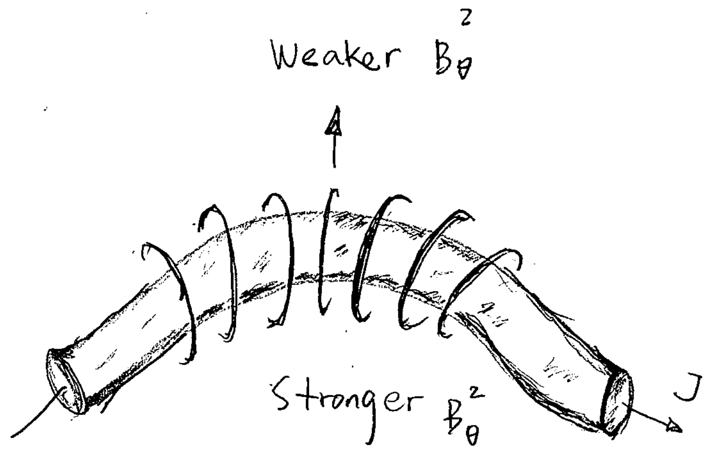
$$\left.\begin{array}{l} \dfrac{m^2 B_\theta^2}{4\pi} + 2rp' > 0 \\ + \\ p' + \left(\dfrac{B_\theta^2}{8\pi}\right)' + \dfrac{B_\theta^2}{4\pi r} = 0 \end{array}\right\} \Rightarrow (m^2 - 2) B_\theta^2 > r\,(B_\theta^2)'$$

Finite current density
$B_\theta 2\pi r = \frac{4\pi}{c}\pi r^2 J_0$

$\boxed{m=1}$: $\quad -B_\theta^2 > r\,(B_\theta^2)'$

Not possible near $r \to 0$, where $B_\theta \sim r$

$\Rightarrow$ $\boxed{\text{z-pinch unstable for } m=1}$


Weaker B_θ^2
Stronger B_θ^2
J