$\boxed{1}$ Magnetohydrodynamics (MHD)

Liouville Eq.
(6N Dimensional phase space)

$\xrightarrow{\text{BBGKY}}$ $\boxed{\text{Boltzmann Eq}}$

$\Big\downarrow$ Braginskii ('65)

$\boxed{\text{Fluid Eqs (with transport)}}$

$\swarrow$

$\boxed{\text{MHD Eqs}}$

$\boxed{\text{8 Eqs for 8 components}}$

(Mass) Continuity : $$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho V) = 0 \qquad (1)$$

Momentum : $$\rho \frac{\partial V}{\partial t} + \rho\, V\cdot\nabla V = -\nabla p + \frac{(\nabla\times B)\times B}{4\pi} \qquad (2)$$

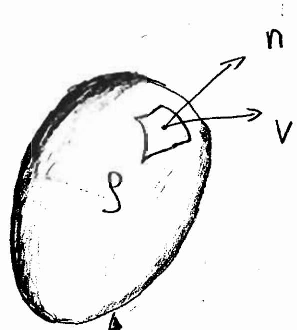
Energy : $$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0 \qquad (\gamma = 5/3) \qquad (3)$$

(magnetic) Field : $$\frac{\partial B}{\partial t} = \nabla\times(V\times B) + \frac{\eta c^2}{4\pi}\nabla^2 B \qquad (4)$$

resistivity

Continuity Eq:

Heuristic Derivation



fixed volume V
with surface $S = \partial V$

Change $\quad$ Net gain

$$\frac{\partial}{\partial t}\left[\int_V \rho \, dx^3\right] = -\int_S \rho v \cdot d\vec{s}$$

Stokes's Theorem $\longrightarrow$ $\parallel$

$$-\int_V \nabla \cdot (\rho v)\, d^3x$$

V is arbitray $\Rightarrow$ $\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v \cdot \nabla \longrightarrow \Big\Downarrow$$

Time derivative along fluid element

$$\frac{d\rho}{dt} + \rho \, \nabla \cdot v = 0$$

Momentum Eq. Heuristic derivation

$$\left\{\begin{array}{l} \rho \dfrac{dv}{dt} = -\nabla p + \dfrac{j \times B}{c} \\[2ex] \dfrac{4\pi}{c} j = \nabla \times B \end{array}\right.$$

(No Displacement current $\frac{1}{c}\frac{\partial E}{\partial t}$)

$$\Rightarrow \quad \rho \frac{dv}{dt} = -\nabla p + \frac{(\nabla \times B) \times B}{4\pi}$$

$$= -\nabla p - \nabla \frac{B^2}{8\pi} + \frac{B \cdot \nabla B}{4\pi}$$

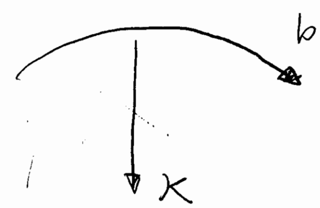
?

$$\nabla (B \cdot B) = 2 \left[B \times (\nabla \times B) + B \cdot \nabla B \right]$$

magnetic pressure (∥ & ⊥)

Not really

$$\vec{B} \cdot \nabla \vec{B} = B \vec{b} \cdot \nabla (\vec{b} B) = B^2 \underbrace{\vec{b} \cdot \nabla \vec{b}}_{\kappa \text{ curvature}} + \vec{b}\, \vec{b} \cdot \nabla \frac{B^2}{2}$$



$$\frac{j \times B}{c} = \frac{(\nabla \times B) \times B}{4\pi} = \kappa \frac{B^2}{4\pi} - \nabla_\perp \frac{B^2}{8\pi}$$

Bending force

pressure force

Momentum Eq. in Conservative form

$$\frac{\partial (\rho V)}{\partial t} + \nabla \cdot T = 0$$

magnetic pressure

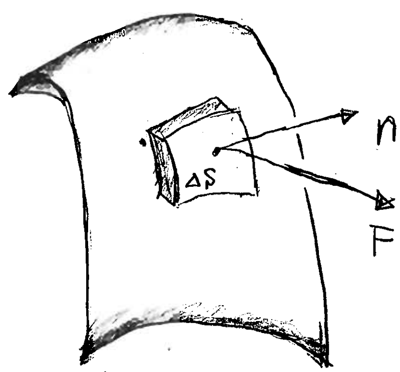
$$T = \rho \vec{V}\vec{V} + \left(p + \frac{B^2}{8\pi}\right)\overline{\overline{I}} - \frac{\vec{B}\vec{B}}{4\pi}$$

stress tensor

Reynolds Stress

Magnetic stress

why the name "stress"?



Force

$$\frac{\vec{F}}{\Delta S} = T \cdot n$$

Area

$\boxed{\text{proof:}}$

$$\nabla \cdot T = \frac{\partial}{\partial x_i}\left[\rho V_i V_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi}\right]$$

$$\frac{\partial}{\partial x_i}\left[\rho v_i v_j\right] = \rho_{,i} v_i v_j + \rho v_{i,i} v_j + \rho v_i v_{j,i}$$

$$= v\,(v\cdot\nabla)\rho + \rho v\, \nabla\cdot v + \rho\, v\cdot\nabla v$$

$$\frac{\partial}{\partial x_i}\left[\left(p+\frac{B^2}{8\pi}\right)\delta_{ij}\right] = \nabla\left(p+\frac{B^2}{8\pi}\right)$$

$$\frac{\partial}{\partial x_i}\frac{B_i B_j}{4\pi} = \frac{1}{4\pi}\left[B_{i,i} B_j + B_i B_{j,i}\right]$$

$$= \underbrace{\frac{(\nabla\cdot B)B}{4\pi}}_{0} + \frac{1}{4\pi}(B\cdot\nabla)B$$

0, continuity Eq.

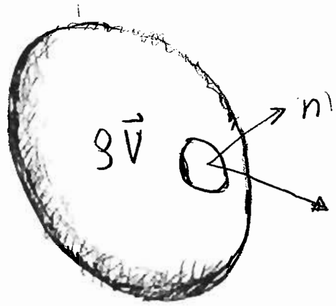
$$\frac{\partial(\rho v)}{\partial t} + \nabla\cdot T = \underline{v\rho_{,t}} + \rho v_{,t} + \underline{v(v\cdot\nabla)\rho} + \underline{\rho v\nabla\cdot v}$$

$$+ \rho\, v\cdot\nabla v + \nabla\left(p+\frac{B^2}{8\pi}\right) - \frac{1}{4\pi}(B\cdot\nabla)B$$

$$= \rho\frac{\partial v}{\partial t} + \rho\, v\cdot\nabla v + \nabla\left(p+\frac{B^2}{8\pi}\right) - \frac{1}{4\pi}(B\cdot\nabla)B$$

$$= \rho\frac{\partial v}{\partial t} + \rho\, v\cdot\nabla v + \nabla p - \frac{(\nabla\times B)\times B}{4\pi}$$

$$= 0 \qquad \blacksquare$$



$T\cdot n$: flux of momentum

global conservative form

$$\frac{\partial}{\partial t}\int_V \rho\vec{v}\, d^3x = -\int_S T\cdot n\, ds$$

Stokes's Theorem

$$= -\int_V \nabla\cdot T\, d^3x$$

$$\Updownarrow$$

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla\cdot T = 0$$

local conservative form

Stokes's Theorem in general form:

$$\int_V d\omega = \int_{\partial V} \omega$$

$d\omega$: (P+1)-form

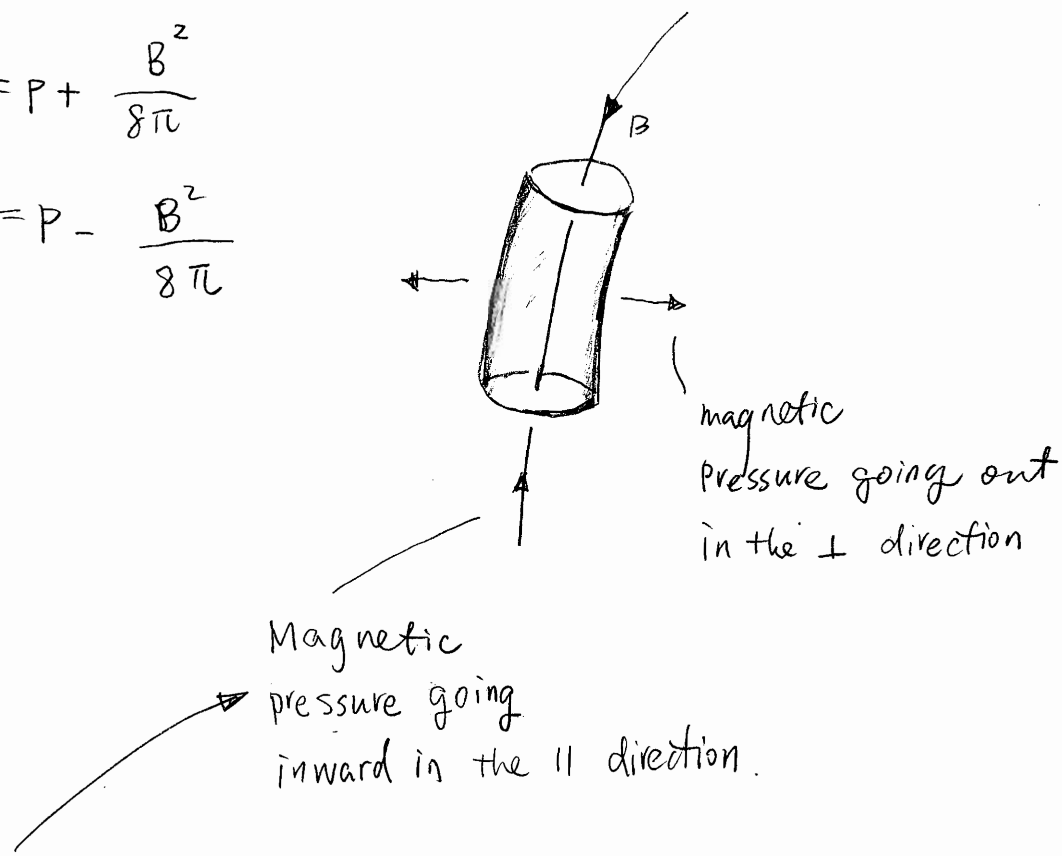
ω: P-form

Question: Take a Cartisian coordinate, that $B \parallel \hat{e}_z$

$$p = \begin{pmatrix} p_\perp & & \\ & p_\perp & \\ & & p_\parallel \end{pmatrix}$$

$$p_\perp = p + \frac{B^2}{8\pi}$$

$$p_\parallel = p - \frac{B^2}{8\pi}$$



This is wrong! Why?

Energy Eq:

Heuristic Derivation:

Consider a (drop) fluid element; Assume No heat flow, (adiabatic)

$$d\mathcal{E} = -p\,dV + T\,\cancelto{0}{ds}$$

with $\quad pV = NT$

$$\mathcal{E} = \frac{3}{2} NT$$

$$\left.\begin{aligned} d\mathcal{E} &= -p\,dV \\ d\mathcal{E} &= \frac{3}{2} N\,dT \end{aligned}\right\} \Rightarrow \frac{\frac{3}{2} N\,dT}{NT} = \frac{-p\,dV}{NT}$$

$$\Downarrow$$

$$\frac{3}{2}\frac{dT}{T} = -\frac{dV}{V}$$

$$pV = NT \Rightarrow \frac{dp}{p} = \frac{dT}{T} - \frac{dV}{V}$$

$$\frac{dp}{p} = -\frac{5}{3}\frac{dV}{V}$$

$$= +\frac{5}{3}\frac{d\rho}{\rho}$$

$\rho V = \text{const}$

$$\Downarrow$$

$$p = \rho^{5/3}\,\text{const}$$

$$\Downarrow$$

$$\frac{d}{dt}\left(p/\rho^{\gamma}\right) = 0$$

$$\gamma = 5/3$$

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \quad \Longrightarrow \quad \frac{1}{\rho^\gamma}\left[\frac{dp}{dt}\right] - \frac{\gamma p}{\rho^{\gamma+1}} \underbrace{\frac{d}{dt}\rho}_{\substack{\parallel \ \leftarrow \ \text{Continuity} \\ -\rho \nabla\cdot v}} = 0$$

$$\Downarrow$$

$$\frac{dp}{dt} + \gamma p \nabla\cdot v = 0$$

<u>With finite resistivity?</u>

$$\frac{dp}{dt} + \gamma p \nabla\cdot v = (\gamma - 1)\eta j^2 .$$

Magnetic Field Eq.

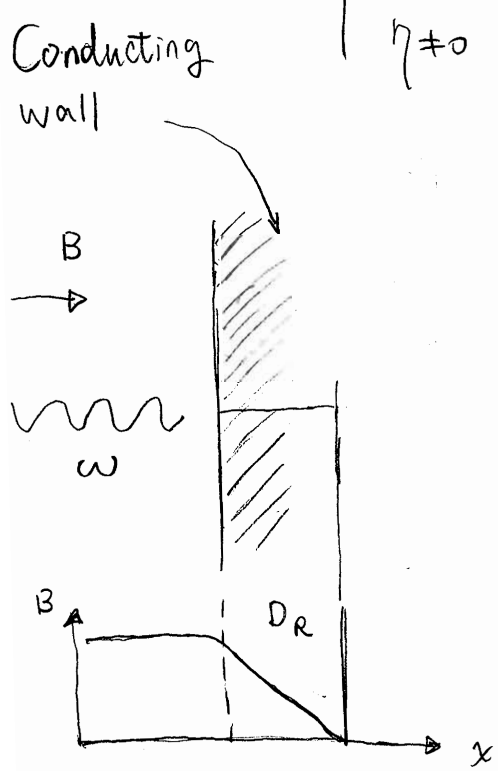
$$\begin{cases} \dfrac{\partial B}{\partial t} = -c\nabla \times E \\ E + \dfrac{V\times B}{c} = \eta j \end{cases} \Rightarrow \frac{\partial B}{\partial t} = \nabla\times(V\times B) - c\eta\nabla\times j$$

$$\nabla\times j = \frac{c}{4\pi}\nabla\times(\nabla\times B) = \frac{c}{4\pi}\left[\nabla(\nabla\cdot B) - \nabla^2 B\right]$$

$$\Rightarrow \quad \frac{\partial \vec{B}}{\partial t} = \nabla\times(V\times B) + \boxed{\frac{c^2\eta}{4\pi}\nabla^2\vec{B}}$$

Diffusion

$$\begin{cases} \eta = 0 & \text{— ideal MHD} \\ \eta \neq 0 & \text{— resistive MHD} \end{cases}$$

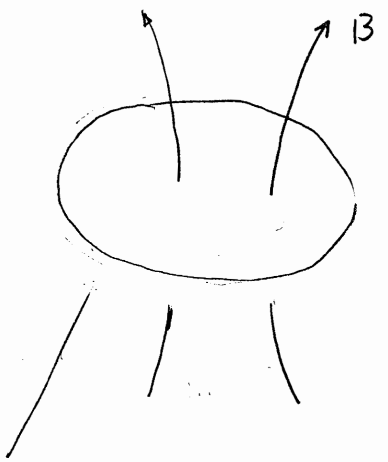


Skin Depth

$$\frac{\partial \vec{B}}{\partial t} \sim \frac{c^2\eta}{4\pi}\nabla^2\vec{B}$$

$$\omega B \sim \frac{c^2\eta}{4\pi}\frac{B}{D_R^2} \Rightarrow D_R = \sqrt{\frac{c^2\eta}{4\pi\omega}}$$

Frozen-in Law: For ideal MHD

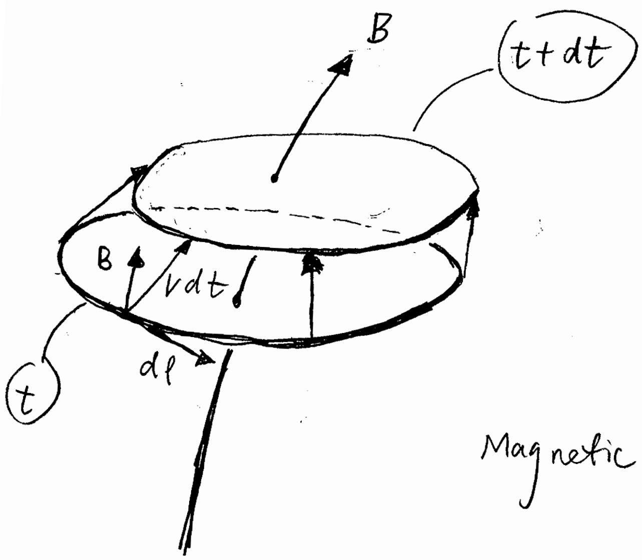


$$\int_S B\cdot ds \equiv \psi$$

$$\boxed{\frac{d\psi}{dt} = 0}$$

Fluid-element ring

proof:



$$\frac{d\psi}{dt} = \int_S \frac{\partial B}{\partial t}\cdot d\vec{s} + \int_{\partial S} B\cdot[V\times dl]$$

$$= \int_S \nabla\times(V\times B)\cdot d\vec{s} + \int_{\partial S}(B\times V)\cdot dl$$

Magnetic eq.

$$= \int_{\partial S}(V\times B)\cdot dl + \int_{\partial S}(B\times V)\cdot dl$$

$$= 0$$

Energy Eq. in conservative form

$$\frac{\partial W}{\partial t} + \nabla \cdot \vec{S} = 0$$

with $$W = \frac{1}{2}\rho v^2 + \frac{B^2}{8\pi} + \frac{P}{\gamma - 1}$$

$$\vec{S} = \left(\frac{1}{2}\rho v^2 + \frac{P\gamma}{\gamma-1}\right)\vec{V} + \frac{B \times (V \times B)}{4\pi}$$

$\boxed{\text{proof}}$

$$\frac{\partial W}{\partial t} = \underset{①}{\frac{1}{2}\rho_{,t} V^2} + \underset{②}{\rho \vec{V}\cdot\vec{V}_{,t}} + \underset{③}{\frac{\vec{B}\cdot\vec{B}_{,t}}{4\pi}} + \underset{④}{\frac{1}{\gamma-1}\frac{\partial P}{\partial t}}$$

$$\nabla \cdot \vec{S} = \underset{⑤}{\frac{1}{2}\rho V^2\, \nabla \cdot V} + \underset{⑥}{\vec{V} \cdot \nabla\rho\, \frac{V^2}{2}} + \underset{⑦}{\rho \vec{V} \cdot \underbrace{\frac{\nabla (V\cdot V)}{2}}_{V\times\nabla\times V + V\cdot\nabla \vec{V}}}$$

$$+ \left(\underset{⑧}{\nabla P \cdot \vec{V}} + \underset{⑨}{P\, \nabla \cdot \vec{V}}\right)\frac{\gamma}{\gamma-1} + \frac{\nabla \cdot (B \times (V \times B))}{4\pi}$$

$$\frac{\nabla \cdot (B \times (V \times B))}{4\pi} \rightarrow \underset{⑩}{\frac{-B \cdot \nabla \times (V \times B)}{4\pi}} + \underset{⑪}{\frac{(V \times B)\cdot \nabla \times B}{4\pi}}$$

① + ⑤ + ⑥ = 0 : continuity Eq.

③ + ⑩ = 0 : magnetic Eq.

$$(2) + (7) + (11) = \underbrace{-\, V \cdot \nabla p}_{(12)} \quad \Leftarrow \text{Momentum Eq.}$$

$$(4) + (8) + (9) + (12) = 0 \quad \Leftarrow \frac{dp}{dt} + \gamma p \nabla \cdot V = 0$$

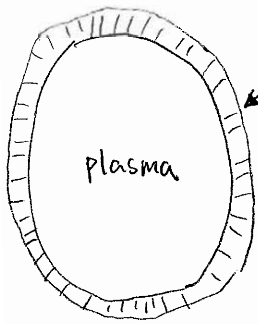
$$\Rightarrow \quad \frac{\partial W}{\partial t} + \nabla \cdot \vec{S} = 0 \qquad \blacksquare$$

Where is the electric field energy in W ?

Ordered out because $E = \frac{V \times B}{c}$, $\frac{V}{c} \ll 1$.

If keep $\frac{E^2}{8\pi}$ in W, need to keep the displacement current $\frac{1}{c}\frac{\partial E}{\partial t}$

Boundary Conditions



perfectly conducting wall

$$\vec{n} \times \vec{E} = 0$$
$$\vec{n} \cdot \vec{B} = 0$$

on the plasma-wall interface

no magnetic charge or current

$$E + \frac{V \times B}{C} = 0 \xRightarrow{n\times} n \times E + \frac{V}{c}(n \cdot B) - \frac{B}{c}(n \cdot V) = 0$$

$$\Rightarrow \quad n \cdot V = 0 \quad \text{on the interface}$$

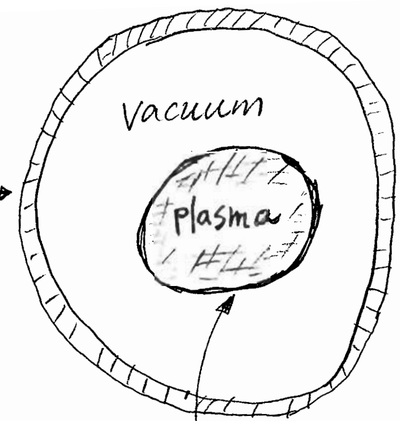
$$\left.\begin{aligned} \vec{n} \times \vec{E} &= 0 \\ \vec{n} \cdot \vec{B} &= 0 \\ \vec{n} \cdot \vec{v} &= 0 \end{aligned}\right\} \quad \text{Not independent, need only two.}$$

other B.Cs

$$n \cdot \left(E + \frac{V \times B}{c}\right) = 0, \quad \text{etc.}$$

Plasma-Vacuum Boundary

perfectly conducting wall



Free boundary

$$[n\cdot B]=0$$

$$[n\times B]=\frac{4\pi}{c}K$$

surface current density

$$\left[P+\frac{B^2}{8\pi}\right]=0$$

$$[Q]=Q_{vacuum}-Q_{plasma}\Big|_{interface}$$

assume $B\cdot n\big|_{interface}=0$

Question:

Does ideal MHD ($\eta=0$) imply $n\cdot B\big|_{interface}=0$?

Characteristic space time scale length in ideal MHD :

$$E + \frac{V \times B}{c} = 0 \quad \Rightarrow \quad E \sim \frac{V}{c} B$$

$$\rho \frac{\partial V}{\partial t} + \rho V \cdot \nabla V = -\nabla p + \frac{(\nabla \times B) \times B}{4\pi}$$

$$\Downarrow$$

$$\rho \omega v \sim \frac{k B^2}{4\pi} \quad \Rightarrow \quad \frac{\omega}{k} v \sim \frac{B^2}{4\pi \rho} \sim V_A^2$$

For most Laboratory & space plasmas, $V_A \ll c$

$$\frac{4\pi}{c} j + \underbrace{\frac{1}{c} \frac{\partial E}{\partial t}} = \underbrace{\nabla \times B}$$

$$\frac{\frac{1}{c}\frac{\partial E}{\partial t}}{\nabla \times B} \sim \frac{\frac{1}{c}\omega E}{k B} \sim \frac{\omega}{k} \frac{V}{c^2} \sim \frac{V_A^2}{c^2} \ll 1$$

$\Rightarrow$ Displacement current can be ignored.

put in higher order

$\boxed{\text{How about } \nabla\cdot B = 0}$?

$$\frac{\partial}{\partial t}(\nabla\cdot B) = \nabla\cdot\left(\frac{\partial B}{\partial t}\right) = -\nabla\cdot(c\,\nabla\times E) = 0$$

$\nabla\cdot B = 0$ is an initial condition.

$\boxed{\nabla\cdot E = ?}$

In MHD, $\nabla\cdot j \simeq \nabla\times \frac{c\,\nabla\times B}{4\pi} = 0$

Then, should $\frac{\partial}{\partial t}\nabla\cdot E = 0$ by conservation of charge?

$$\frac{\partial \rho}{\partial t} + \nabla\cdot j = 0$$

$\nabla\cdot E = 4\pi\rho \longrightarrow \Updownarrow \longleftarrow \nabla\times B = \frac{4\pi j}{c} + \frac{1}{c}\frac{\partial E}{\partial t}$

$$\underbrace{\frac{\partial}{\partial t}\nabla\cdot E}_{①} + c\nabla\cdot\nabla\times B - \underbrace{\nabla\cdot\frac{\partial E}{\partial t}}_{②} = 0$$

Droping $\frac{1}{c}\frac{\partial E}{\partial t}$ $\Longleftrightarrow$ Droping $\nabla\cdot E$ in MHD Eq.

But, $\frac{\partial E}{\partial t} \neq 0$ and $\nabla\cdot E \neq 0$ in MHD Eq.

① & ② cancel in the next order.

It is a true statement that

Ideal MHD conserves charge only to $O(\frac{V_A^2}{c^2})$ by ignoring the displacement current.

If $\frac{V_A^2}{c^2} \sim 1$ or exact charge conservation is impoutant, $\frac{1}{c}\frac{\partial E}{\partial t}$ should be kept in the momentum Eq.

$\boxed{2}$ Rigorous derivation of MHD From Boltzmann – Maxwell Eq

$$\frac{\partial f_i}{\partial t} + v \cdot \nabla f_i + \frac{q_i}{m_i}\left(E + \frac{v \times B}{c}\right)\frac{\partial f_i}{\partial v} = \left(\frac{\partial f_i}{\partial t}\right)_c \quad \cdots$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$$\nabla \times B = \frac{4\pi}{c}\sum_i q_i \int f_i \, v \, d^3v + \frac{1}{c}\frac{\partial E}{\partial t}$$

$$\nabla \cdot E = 4\pi \sum_i q_i \int f_i \, d^3v$$

$$\nabla \cdot B = 0$$

$$\left(\frac{\partial f_i}{\partial t}\right)_c = \sum_j C_{ij}$$

Take moments of the kinetic equation

$$\int Q_i \left[\text{Boltzman Eq.} \right] d^3v$$

$Q_1 = 1 \implies$ Continuity Eq.

$Q_2 = mv \implies$ momentum Eq   <u>for each species</u>

$Q_3 = \frac{1}{2}mv^2 \implies$ Energy Eq.

Expressed in terms of Macroscopic quantities:

$$n(x,t) = \int f\, d^3v$$

$$\vec{V}(x,t) = \frac{1}{n}\int f\, v\, d^3v$$

$$p(x,t) = \frac{1}{3} m \int f v^2\, d^3v$$

and

$$\Pi = \vec{\vec{P}} - p\vec{\vec{I}}, \qquad \vec{\vec{P}} = \int m f\, vv\, d^3v$$

$$h = \int \frac{1}{2} m v^2 \vec{v}\, d^3v$$

$$\vec{R} = \int m v C\, d^3v,$$

$$Q = \int \frac{1}{2} m v^2 c\, d^3v$$

MHD scaling:

$$\begin{cases} \omega \sim \dfrac{\partial}{\partial t} \sim \dfrac{V_{Ti}}{L} \\ k \sim \dfrac{1}{L} \\ V_{Ti} \sim V_A \equiv \dfrac{B}{\sqrt{4\pi \rho}} \end{cases}$$

Assume:

① High collisionality: $\boxed{\begin{matrix} \omega \tau_{ii} \ll 1 \\ \omega \tau_{ee} \ll 1 \end{matrix}} \Rightarrow \Pi \to 0$

② low velocity: $V_{Ti} \sim V_A \ll c \Rightarrow \frac{1}{c}\frac{\partial E}{\partial t} \to 0$

③ Low frequency: $\boxed{\omega \ll \omega_c} \Rightarrow L \gg \rho$

cyclotron freq. gyro-radius

$\sum q_i n_i = 0 \Leftarrow \boxed{\omega \ll \omega_p}$

plasma freq.

④ Low resistivety:

$$\boxed{\eta J \ll \frac{V \times B}{c}}, \quad \eta = \frac{m_e}{n e^2 \tau_{ei}}$$

under these conditions $\Longrightarrow$ MHD Eq.

> However, ① is never true for Fusion plasmas, which are basically collisionless.

But, MHD seems to work very well. There must be $\boxed{\text{another Reason}}$ that MHD Eqs are correct for collisionless plasmas.

$\boxed{\text{Gyrokinetic Theory}}$

Other Names:
Guiding Center Plasma model;
Drift-kinetic Theory.

Gyrokinetic ordering:

$\frac{\rho}{L} \ll 1$

$\omega \ll \Omega$

$\boxed{\nabla \cdot V = 0}$

$\boxed{\text{Collisionless MHD}}$

$\approx$

$$\boxed{\text{MHD}} - \left[\text{Momentum Eq}\right]_{\parallel} + \boxed{\nabla \cdot V = 0}$$

In collisionless MHD, the fast space-time scale of gyromotion provides the mechanism for isotropic pressure in the perpendicular direction.