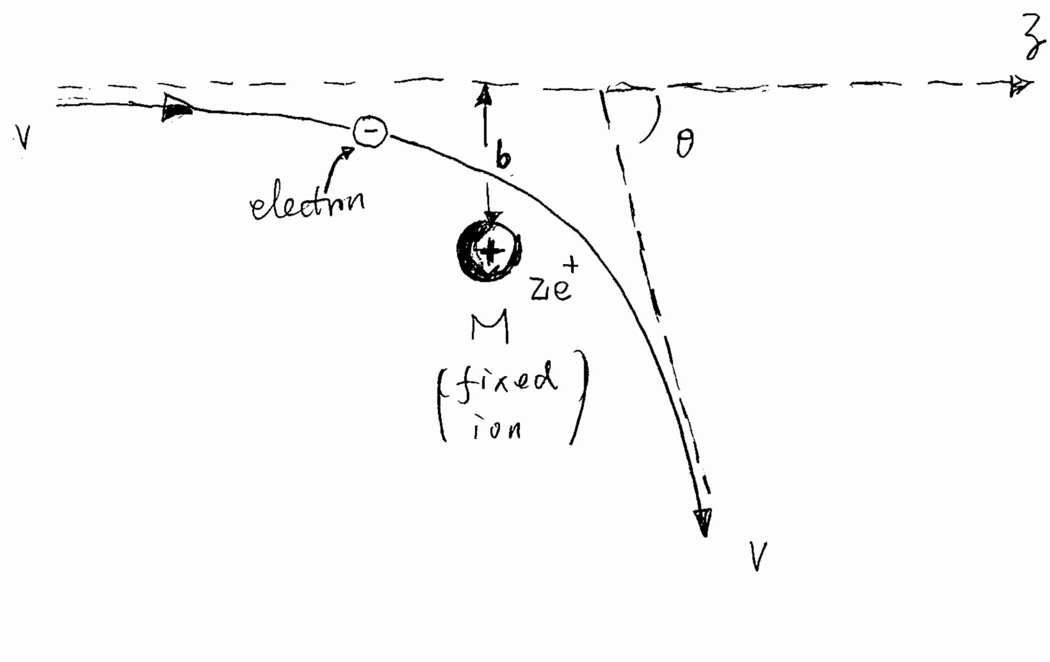
Collisions in Fully ionized Plasmas

Random walk of velocity v due to short-scale length coulomb Collision

$$\left(\frac{\partial f}{\partial t}\right)_c = \frac{\partial}{\partial v}\left[-\lim_{\Delta t \to 0} \frac{\langle \Delta v \rangle}{\Delta t} f\right] + \frac{\partial}{\partial v}\frac{\partial}{\partial v} : \left[\lim_{\Delta t \to 0} \frac{\langle \Delta v \Delta v \rangle}{2 \Delta t} f\right]$$

Consider a <u>Lorentz model</u>.

electrons scattering off heavy ions.



$$\operatorname{tg}\frac{\theta}{2} = \frac{Ze^2}{m_e v^2 b} = \frac{b_0}{b}$$

$$b_0 = \frac{Ze^2}{m_e v^2}$$

$$\boxed{\sin\theta = \frac{2\operatorname{tg}\frac{\theta}{2}}{1+\operatorname{tg}^2\frac{\theta}{2}} = \frac{2\operatorname{ctg}\frac{\theta}{2}}{1+\operatorname{ctg}^2\frac{\theta}{2}}}$$

$$\langle \Delta V_x \rangle = 0$$

$$\langle \Delta V_y \rangle = 0$$

$$\langle (\Delta V_x)^2 \rangle = \langle (\Delta V_y)^2 \rangle = \frac{1}{2}\langle (\Delta V_\perp)^2 \rangle \neq 0$$

$$(\Delta V_\perp)^2 = v^2 \sin^2\theta = \frac{v^2 \, 4 \, b^2/b_0^2}{(1 + b^2/b_0^2)^2}.$$

For every collision

$$\lim_{\Delta t \to 0} \frac{\langle (\Delta V_\perp)^2 \rangle}{\Delta t} = 2\pi n_i v \int_0^{b_{max}} (\Delta V_\perp)^2 \, b \, db$$

simultaneous collision with many ions.

$$= 4\pi n_i v^3 b_0^2 \int_0^{b_{max}} \frac{b^2/b_0^2}{(1 + b^2/b_0^2)^2} \, d\, b^2/b_0^2$$

$$\int \frac{x}{(1+x)^2} dx = \frac{1}{1+x} + \ln(1+x)$$

$$= 4\pi n_i v^3 b_0^2 \left[\frac{1}{1 + b_{max}^2/b_0^2} + \ln\left[1 + \frac{b_{max}^2}{b_0^2}\right] - 1 \right]$$

$$\boxed{\frac{b_{max}}{b_0} \gg 1} \;\Rightarrow\; \simeq 8\pi n_i v^3 b_0^2 \ln \underbrace{\frac{b_{max}}{b_0}}_{\Lambda}$$

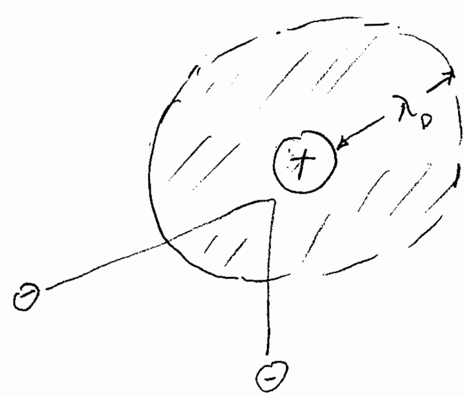
$\boxed{\text{Diverge as } b_{max} \to \infty}$

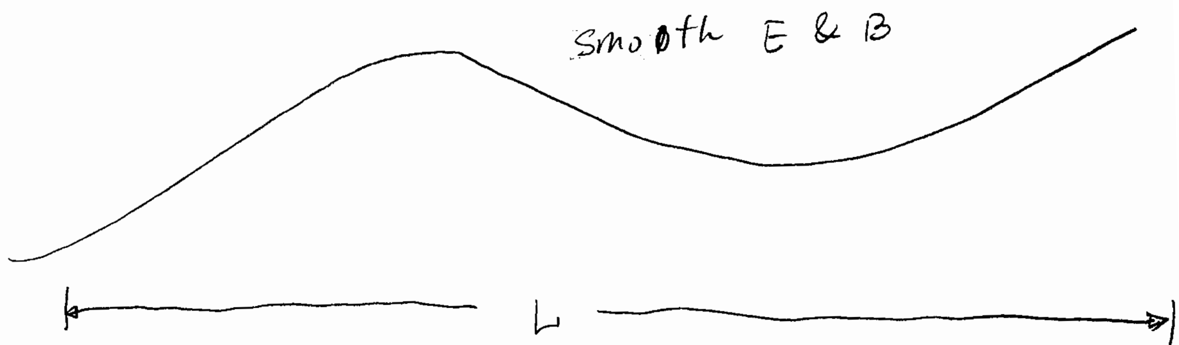
$$\lim_{\Delta t \to 0} \frac{\langle (\Delta v_\perp)^2 \rangle}{\Delta t} = 2 \nu_{ei} v^2$$

$$\nu_{ei} \cong 4\pi n_i v^2 b_0^2 \ln \Lambda$$

$$= \frac{4\pi n_i Z^2 e^4 \ln \Lambda}{m_e^2 v^3}$$

Normally pick $b_{max} = \lambda_D$





$L \gg \lambda_D$

Note:

$$\frac{\langle (\Delta v_\perp)^2 \rangle}{\Delta t} = 4\pi n_i v^3 b_0^2 \underbrace{\int_0^{\lambda_0^2} \frac{x}{(1+x)^2}\, dx}_{\displaystyle \int_0^{b_0^2} \frac{x}{(1+x)^2}\, dx + \int_{b_0^2}^{\lambda_0^2} \frac{x}{(1+x)^2}\, dx}$$

$$\underbrace{\int_0^{b_0^2} \frac{x}{(1+x)^2}\, dx}_{\text{large angle scattering}} \ll \underbrace{\int_{b_0^2}^{\lambda_0^2} \frac{x}{(1+x)^2}\, dx}_{\text{small angle scattering}}$$

At $\Delta t \sim \frac{1}{\nu_{ei}}$, $\langle (\Delta v_\perp)^2 \rangle \sim v^2$,

$\Rightarrow$ large angle diffusion. But this is mostly due to many small angle scatterings, instead of a few large angle scatterings.

Energy conservation:

$$(V+\Delta V_{\parallel})^2 + (\Delta V_{\perp})^2 = V^2$$

initial velocity

$$\Rightarrow \quad V\,\Delta V_{\parallel} = -\frac{1}{2}(\Delta V_{\perp})^2$$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\langle \Delta V_{\parallel} \rangle}{\Delta t} = -4\pi n_i V^2 b_0^2 \ln \Lambda$$

$$= -\nu_{ei} V$$

also, $\Delta V_{\parallel}^2 \ll \Delta V_{\perp}^2$

$$\langle \Delta v_x \Delta v_y \rangle = 0, \cdots$$

$$\lim_{\Delta t \to 0} \frac{\langle \Delta \vec{v} \Delta \vec{v} \rangle}{\Delta t} = \lim_{\Delta t \to 0} \begin{pmatrix} \frac{1}{2}\frac{\langle (\Delta v_\perp)^2 \rangle}{\Delta t} & 0 & 0 \\ 0 & \frac{1}{2}\frac{\langle (\Delta v_\perp)^2 \rangle}{\Delta t} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\boxed{\langle \Delta v_\parallel^2 \rangle \text{ is higher order}}$

$$= + \nu_{ei} \left(\overleftrightarrow{I} v^2 - \vec{v}\vec{v} \right)$$

$$\Longrightarrow$$

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = \frac{4\pi n_i Z^2 e^4 \ln\Lambda}{m^2} \left[\frac{\partial}{\partial \vec{v}} \cdot \left(\frac{\vec{v} f}{v^3} \right) + \underbrace{\frac{1}{2} \frac{\partial}{\partial \vec{v}} \frac{\partial}{\partial \vec{v}} : \left(\frac{\overleftrightarrow{I} v^2 - \vec{v}\vec{v}}{v^3} f \right)}_{\boxed{\frac{\partial}{\partial \vec{v}} \cdot \left[\frac{\partial}{\partial \vec{v}} \cdot \left(\frac{\overleftrightarrow{I} v^2 - \vec{v}\vec{v}}{v^3} \right) f + \frac{\partial}{\partial \vec{v}} f \cdot \left(\frac{\overleftrightarrow{I} v^2 - \vec{v}\vec{v}}{v^3} \right) \right]}} \right]$$

$$= \frac{4\pi n_i Z^2 e^4 \ln\Lambda}{m^2} \frac{\partial}{\partial \vec{v}} \cdot \left[\frac{1}{2} \frac{\partial f}{\partial \vec{v}} \cdot \left(\frac{\overleftrightarrow{I} v^2 - \vec{v}\vec{v}}{v^3} \right) \right]$$

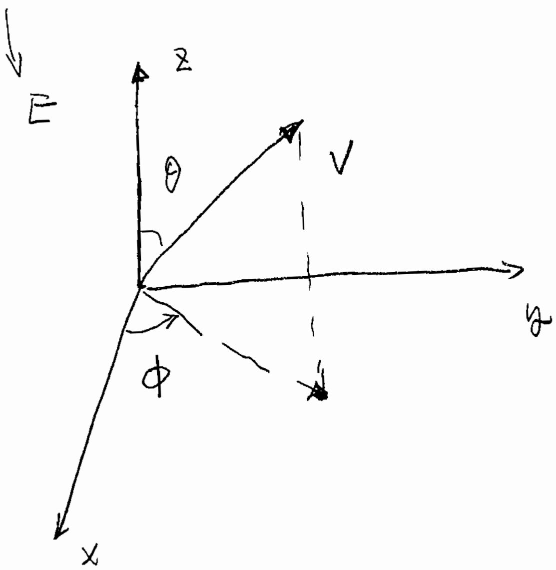
$$\boxed{\left[\frac{1}{2} \frac{\partial}{\partial \vec{v}} \cdot \left(\frac{\overleftrightarrow{I} v^2 - \vec{v}\vec{v}}{v^3} \right) = - \frac{\vec{v}}{v^3} \right]}$$

$$\left(\frac{\partial f}{\partial t}\right)_{Collision} = -\frac{\partial}{\partial v} \cdot \Big[\underbrace{\overleftrightarrow{D}_W \cdot \frac{\partial f}{\partial v}}_{J_v} \Big]$$

$$\overleftrightarrow{D}_W = \frac{1}{2} \frac{4\pi n_i Z^2 e^4 \ln\Lambda}{m^2} \frac{\left(\vec{v}\vec{v} - v^2 \overleftrightarrow{I}\right)}{v^3}$$

Plasma Resistivity (Lorentz - Model)

applied electrical field



Assume:

<u>steady state:</u>

$$\frac{q}{m}\vec{E}\cdot\frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t}\right)_{\text{collision}}$$

$$\left(\frac{\partial f}{\partial t}\right)_{\text{collision}} = \frac{\nu_{ei} v^3}{2}\,\frac{\partial}{\partial v}\cdot\left[\frac{\partial f}{\partial v}\cdot\frac{\overleftrightarrow{I}v^2-\vec{v}\vec{v}}{v^3}\right]$$

In the Spherical coordinates:

$$\nabla_{\vec{v}} f = \hat{v}\,\frac{\partial f}{\partial v} + \hat{\theta}\,\frac{1}{v}\frac{\partial f}{\partial \theta} + \hat{\phi}\,\frac{1}{v\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\left(\overleftrightarrow{I}v^2 - \vec{v}\vec{v}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & v^2 & 0 \\ 0 & 0 & v^2 \end{pmatrix}$$

$$\Rightarrow \quad \frac{\partial f}{\partial \vec{v}} \cdot \frac{\bar{\bar{I}} v^2 - \vec{v}\vec{v}}{v^3}$$

$$= \hat{\theta} \frac{1}{v^2} \frac{\partial f}{\partial \theta} + \hat{\phi} \frac{1}{v^2 \sin\theta} \frac{\partial f}{\partial \phi}$$

$$\nabla_v \cdot \left[\frac{\partial f}{\partial \vec{v}} \cdot \frac{\bar{\bar{I}} v^2 - \vec{v}\vec{v}}{v^3} \right]$$

$$= \frac{1}{v^3 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right)$$

$$+ \frac{1}{v^3 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}$$

$$\Rightarrow \quad \left(\frac{\partial f}{\partial t} \right)_{collision} = \frac{\nu_{ei}}{2} \left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 f}{\partial \phi^2} \right\}$$

$$\frac{q}{m} E \cdot \frac{\partial f}{\partial v} = \left(\frac{\partial f}{\partial t} \right)_{collision}$$

Assume: E is weak, (no run-away electron)

$$f = f_0 + f_1 + \cdots$$

Leading order: $$\left(\frac{\partial f_0}{\partial t}\right)_{collision} = 0$$

take $$f_0 = \frac{n_e}{(2\pi \frac{T}{m})^{3/2}} \exp\left(-\frac{v^2 m}{2T}\right)$$

Maxwellian

Next order:

$$-\frac{q\vec{v}}{T} f_0 \cdot E = \left(\frac{\partial f_1}{\partial t}\right)_{collision}$$

$$-\frac{q E v \cos\theta}{T} f_0 = \frac{\nu_{ei}}{2} \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial f_1}{\partial \theta}$$
$$+ \frac{\nu_{ei}}{2} \frac{1}{\sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}$$

$$\Rightarrow \quad f_1 = \cos\theta \frac{1}{\nu_{ei}} \frac{q}{T} E v f_0$$

$$\vec{j} = \int (f_0 + f_1)\, \vec{v}\, d^3v$$

$$= \int f_1\, \vec{v}\, d^3v$$

$$\vec{v} = v \sin\theta \cos\phi\, \hat{x} + v \sin\theta \sin\phi\, \hat{y} + v \cos\theta\, \hat{z}$$

$$\vec{j} = q \int v \cos^2\theta\, \frac{1}{\nu_{ei}} \frac{q}{T} E\, v\, f_0\, d^3v\ \hat{z}$$

$$= E\, \frac{q^2}{T}\, \frac{m_e^2}{4\pi n_i Z^2 e^4 \ln\Lambda}\, \frac{n_e}{(2\pi\, T/m)^{3/2}} \times$$

$d^3v = v^2 \sin\theta\, dv\, d\theta\, d\phi$

$$2\pi \underbrace{\int_0^\infty dv\, v^7 \exp\left(-\frac{v^2}{2T/m}\right)}_{3\left(\frac{2T}{m}\right)^4} \underbrace{\int_0^\pi d\theta \sin\theta \cos^2\theta}_{2/3}\ \hat{z}$$

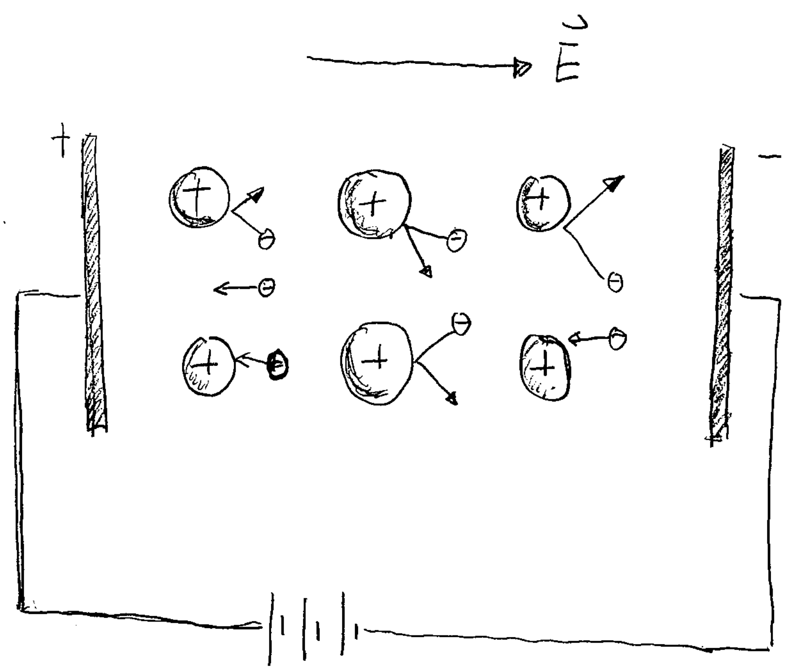
$$= \frac{32}{\sqrt{2\pi}}\, \frac{q^2 n_e}{m_e}\, \frac{m_e^2}{4\pi n_i Z^2 e^4 \ln\Lambda} \left(\frac{\pi}{m_e}\right)^{3/2} E\, \hat{z}$$

$\Longrightarrow$

$$\sigma = \frac{32}{\sqrt{2\pi}} \frac{n_e e^2}{m_e \bar{\nu}_{ei}} \quad , \quad \eta = \frac{\sqrt{2\pi}}{32} \frac{m_e \bar{\nu}_{ei}}{n_e e^2}$$

$$\bar{\nu}_{ei} \equiv \frac{m_e^2 \, V_{te}^2}{4\pi n_i Z_i^2 e^4 \ln\Lambda}$$

$$V_{te}^2 \equiv T / m_e$$



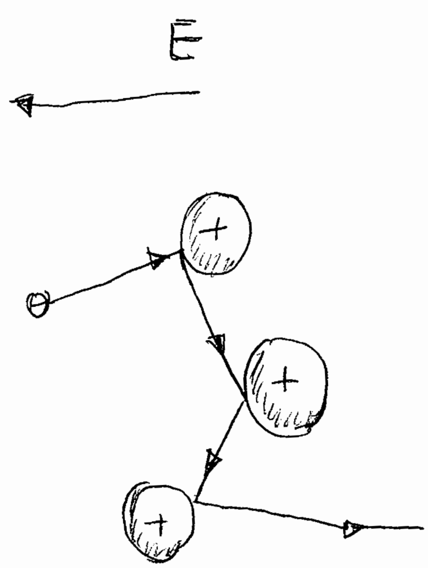
Run away electron

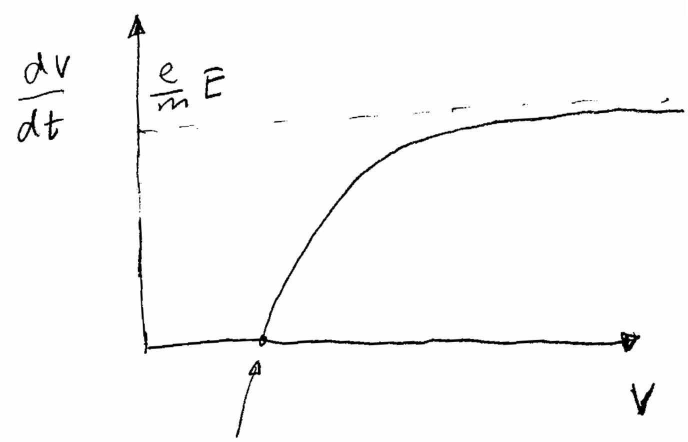
$$\frac{dV}{dt} = -\nu_{ei} V + \frac{e}{m} E.$$

$$\nu_{ei} = \frac{4\pi n_i Z^2 e^4 \ln\Lambda}{m_e^2 V^3}$$

IF $\quad -\nu_{ei} V + \frac{eE}{m} > 0 \quad$ at $t = 0$

then $\quad \frac{dv}{dt} > 0 \quad$ for all $t > 0$





$$V_c = \sqrt{\frac{4\pi n_i Z^2 \ln\Lambda e^3}{m_e E}}$$

For the previous calculation on Restivity to be valid, we require

$$\frac{f_1}{f_0} \ll 1 \quad \text{for} \quad |v| < v_t$$

$$\Rightarrow \quad \frac{1}{\nu_{ei}} \frac{e}{T} E v_t \ll 1$$

$$\Rightarrow \quad v_t \ll v_c \, .$$