6 Fluid Waves

Langmuir Waves (again?)

Equilibrium: $n_{e0} = n_{i0}$, $E_0 = 0$, $B_0 = 0$, $V_{e0} = V_{i0} = 0$

$T_{e0} = \text{const}$, $T_{i0} = \text{const}$; homogeneous.

Linear perturbation: $\vec{k} \parallel \vec{E}_1$, $\vec{k} = k\,\hat{e}_x$, $\vec{E}_1 = E_1\,\hat{e}_x$

$$E = E_0 + E_1, \quad E_1 \sim \hat{E}_1 \exp(ikx - i\omega t)$$

$$\left.\begin{aligned} n_e &= n_{e0} + n_{e1}, \quad n_{e1} \sim \hat{n}_{e1} \exp(ikx - i\omega t) \\ V_e &= V_{e0} + V_{e1}, \quad V_{e1} \sim \hat{V}_{e1} \exp(ikx - i\omega t) \\ T_e &= T_{e0} + T_{e1}, \quad T_{e1} \sim \hat{T}_{e1} \exp(ikx - i\omega t) \end{aligned}\right\}$$

no other perturbed fields

ions do not move

replace the f_{e1} in the Kinetic Theory

Waves
Ogata Korin
(1658-1716)

linearized fluid equation:

$$\begin{cases} \dfrac{\partial n_{e1}}{\partial t} + \nabla\cdot(v_{e1})\, n_{e0} = 0 & : \text{Continuity} \\ m_e n_{e0} \dfrac{\partial v_{e1}}{\partial t} - q_e n_{e0} E_1 + n_{e0} \nabla T_{e1} + T_{e0} \nabla n_{e1} = 0 & : \text{momentum} \\ \nabla_x \cdot \vec{E}_1 = +4\pi q_e n_1 & : \text{poisson} \\ T_{e0}(1-\gamma) \dfrac{\partial n_{e1}}{\partial t} + n_{e0} \dfrac{\partial T_{e1}}{\partial t} = 0 & : \text{Energy} \end{cases}$$

In terms of $\hat{E}_1$, $\hat{n}_{e1}$, $\hat{v}_{e1}$, and $\hat{T}_{e1}$

$$-i\omega \hat{n}_{e1} + ik \hat{v}_{e1} n_{e0} = 0$$

$$-i m_e n_{e0} \omega \hat{v}_{e1} - q_e n_0 \hat{E}_1 + ik n_{e0} \hat{T}_{e1} + ik T_{e0} \hat{n}_{e1} = 0$$

$$ik \hat{E}_1 - 4\pi q_e \hat{n}_{e1} = 0$$

$$T_{e0}(1-\gamma) \hat{n}_{e1} + n_{e0} \hat{T}_{e1} = 0$$

$$\underbrace{\begin{pmatrix} 0 & -i\omega & ikn_{e0} & 0 \\ -q_e n_{e0} & ikT_{e0} & -im_e n_{e0}\omega & ikn_{e0} \\ ik & -4\pi q_e & 0 & 0 \\ 0 & T_{e0}(1-\gamma) & 0 & n_{e0} \end{pmatrix}}_{M(k,\omega)} \begin{pmatrix} \hat{E}_1 \\ \hat{n}_{e1} \\ \hat{V}_{e1} \\ \hat{T}_{e1} \end{pmatrix} = 0$$

For non-trivial solutions of perturbed fields

$$|M(k,\omega)| = 0$$

Determinant

$$|M| = i\omega \begin{vmatrix} -q_e n_{e0} & -im_e n_{e0}\omega & ikn_{e0} \\ ik & 0 & 0 \\ 0 & 0 & n_{e0} \end{vmatrix} + ikn_{e0} \begin{vmatrix} -q_e n_{e0} & ikT_{e0} & ikn_{e0} \\ ik & -4\pi q_e & 0 \\ 0 & T_{e0}(1-\gamma) & n_{e0} \end{vmatrix}$$

$$= i\omega\, n_{e0}\, (-m_e n_{e0}\omega k) + ikn_{e0}\Big[q_e n_{e0}\,(+4\pi q_e)\, n_{e0} - ikT_{e0}\, ikn_{e0} \; ikn_{e0}\, ikT_{e0}(1-\gamma) \Big]$$

$$= n_{e0}^2 k \Big[-i\omega^2 m_e + i\, 4\pi q_e^2 n_{e0} + i k^2 T_{e0}\gamma \Big]$$

$|M| = 0 \Rightarrow$ $$\boxed{\omega^2 = \omega_{pe}^2 + k^2 V_{te}^2 \gamma}$$

charge driven   sound, pressure driven

$\frac{\omega}{k} \ll V_{te}$, $\gamma = 1$, isothermal

$\frac{\omega}{k} \gg V_{te}$, $\gamma = \frac{2+D}{D}$, adiabatic

$D = 1$ for 1D, $\Rightarrow$ $$\boxed{\omega^2 = \omega_{pe}^2 + 3k^2 V_{te}^2}$$

consistant with our previous kinetic result.

$\boxed{\text{Electromagnetic waves}}$

Equilibrium: homogeneous,

$$n_{s0} = \text{Const}, \quad T_{s0} = \text{Const}$$

$$E_{s0} = 0, \quad B_{s0} = 0, \quad V_{s0} = 0$$

perturbation:

$$k \cdot E_1 = 0 \qquad k \cdot B_1 = 0$$

Linearized Fluid - Maxwell eq:

$$\nabla \cdot E_1 = 4\pi \sum_s q_s n_{s1} = 0$$

$$\nabla \times E_1 = -\frac{1}{c} \frac{\partial B_1}{\partial t}$$

$$\nabla \times B_1 = \frac{4\pi}{c} \sum_s q_s n_{s0} V_{s1} + \frac{1}{c} \frac{\partial E_1}{\partial t}$$

$$m n_0 \frac{\partial V_1}{\partial t} = -\nabla T_1 n_0 - \nabla n_1 T_0 + q n_0 E_1$$

$$\frac{\partial n_1}{\partial t} + n_0 \nabla \cdot V_1 = 0$$

$$T_0 (1-\gamma) \frac{\partial n_1}{\partial t} + n_0 \frac{\partial T_1}{\partial t} = 0$$

There are waves with

$$n_1 = 0$$

$$P_1 = 0$$

$$k \cdot v_1 = 0$$

Linearized Eqs reduces to

$$\begin{cases} \nabla \times E_1 = -\frac{1}{c}\frac{\partial B_1}{\partial t} \\ \nabla \times B_1 = \frac{4\pi}{c}\sum_s q_s n_{s0} V_{s1} + \frac{1}{c}\frac{\partial E_1}{\partial t} \\ m n_0 \frac{\partial v_1}{\partial t} = -q n_0 E_1 \end{cases}$$

Assume: $\omega \gtrsim \omega_{pe} \gg \omega_{pi}$, ignore ion motion:

$$\begin{cases} i k \times \hat{E}_1 = \frac{i\omega}{c}\hat{B}_1 & ① \\ i k \times \hat{B}_1 = \frac{4\pi q_e n_{e0}}{c}\hat{V}_{s1} - \frac{i\omega}{c}\hat{E}_1 & ② \\ -m n_{e0} i\omega \hat{V}_1 = +q_e n_{e0}\hat{E}_1 & ③ \end{cases}$$

③ + ②: $$i k \times \hat{B}_1 = \left(-\frac{4\pi q_e^2 n_{e0}}{i m_e \omega c} - \frac{i\omega}{c}\right)\hat{E}_1 \quad \cdots ④$$

$k \times (1) \Rightarrow$

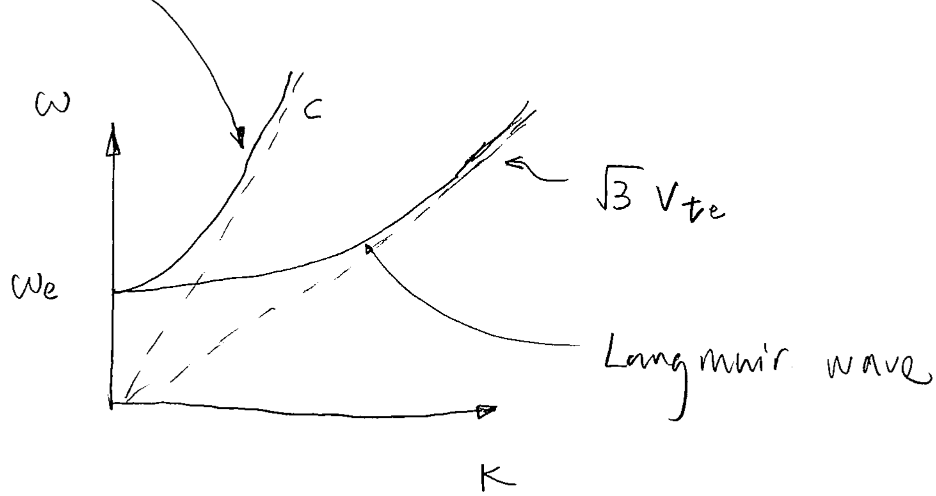
$$i\, \underbrace{k\times(k\times \hat{E}_1)}_{-\hat{E}_1} = \frac{\omega i}{c}\left(+ \frac{4\pi q_e^2 n_{e0}}{mc\omega} - \frac{\omega}{c}\right)\hat{E}_1$$

$$\Rightarrow \quad k^2 = \frac{\omega}{c^2}\left(-\frac{\omega_{pe}^2}{\omega} + \omega\right)$$

$$\omega^2 = k^2c^2 + \omega_{pe}^2$$

electron plasma oscillation

electromagnetic wave in vacuum

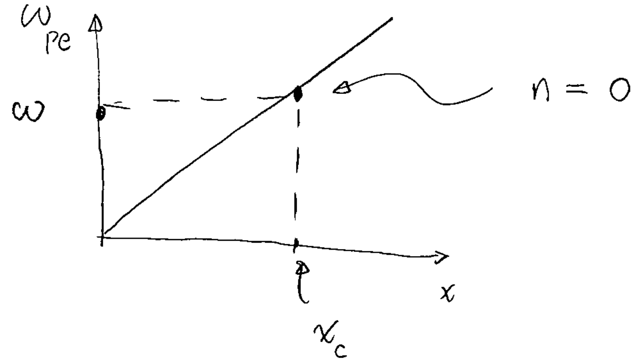


index of refraction

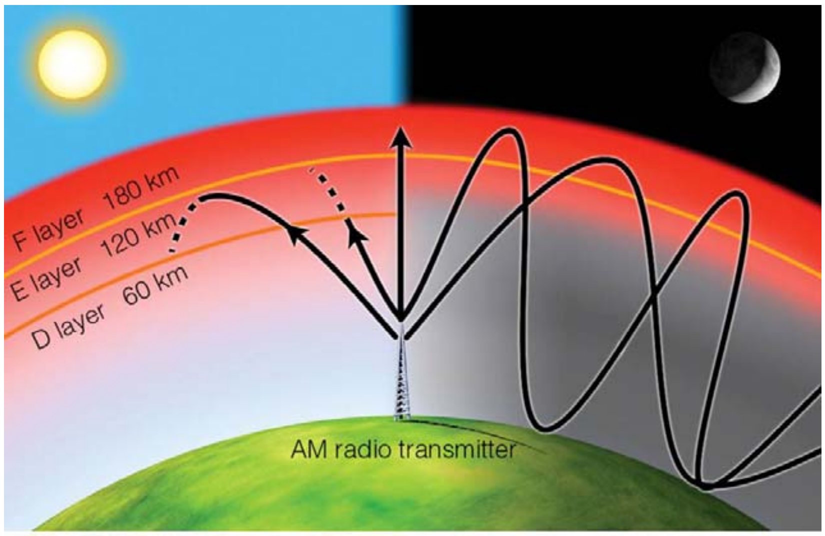
$$n \equiv \frac{c}{\frac{\omega}{k}} = \frac{\sqrt{\omega^2 - \omega_{pe}^2}}{\omega} = \sqrt{1 - \omega_{pe}^2/\omega}$$

phase velocity

$n = 0$ when $\omega = \omega_{pe}$

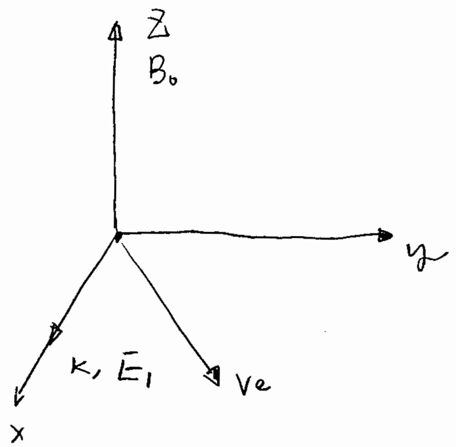


Laser is reflected back at $x = x_c$ where $\omega = \omega_{pe}$.


F layer 180 km
E layer 120 km
D layer 60 km
AM radio transmitter

$\boxed{\text{upper hybrid waves}}$

Langmuir waves in magnetized plasma



electrostatic, $B_1 = 0$

Longitudinal: $\hat{k} \cdot \hat{E}_1 = 1$

perpendicular: $\hat{k} \cdot \hat{B}_0 = 0$

$E_0 = 0, \qquad V_{s0} = 0$

Assume, $T_s = 0$ for simplicity

ions do not move.

$\vec{k} = k\,\vec{e}_x, \quad \vec{E}_1 = E_1\,\vec{e}_x, \quad \vec{V}_1 = V_{1x}\,\vec{e}_x + V_{1y}\,\vec{e}_y$

<u>Linearized fluid eqs for (n_1, V_1, E_1)</u>:

$$
\begin{cases}
\dfrac{\partial n_1}{\partial t} + (\nabla \cdot V_1)\, n_0 = 0 \\[2ex]
m n_0 \dfrac{\partial V_1}{\partial t} = q n_0 E_1 + q n_0 \dfrac{V_1 \times B_0}{c} \\[2ex]
\nabla \cdot E_1 = 4\pi q\, n_1
\end{cases}
$$

in terms of $\hat{n}_1$, $\hat{V}_1$, $\hat{E}_1$

$$
\begin{cases}
-i\omega \hat{n}_1 + i k \cdot \hat{V}_{1x} n_0 = 0 & \text{(1)} \\
-i\omega m n_0 \hat{V}_{1x} = q n_0 \hat{E}_{1x} + \frac{q n_0}{c} \hat{V}_{1y} B_0 & \text{(2)} \\
-i\omega m n_0 \hat{V}_{1y} = -\frac{q n_0}{c} \hat{V}_{1x} B_0 & \text{(3)} \\
i k \hat{E}_1 = +4\pi q n_1 & \text{(4)}
\end{cases}
$$

(1) + (4) $\Rightarrow$ $\hat{E}_1 = \dfrac{4\pi q n_0 \hat{V}_{1x}}{i\omega}$

(3) $\Rightarrow$ $\hat{V}_{1y} = \dfrac{qB}{i\omega m c} \hat{V}_{1x} = \dfrac{\Omega_e}{i\omega} \hat{V}_{1x}$

(2) :

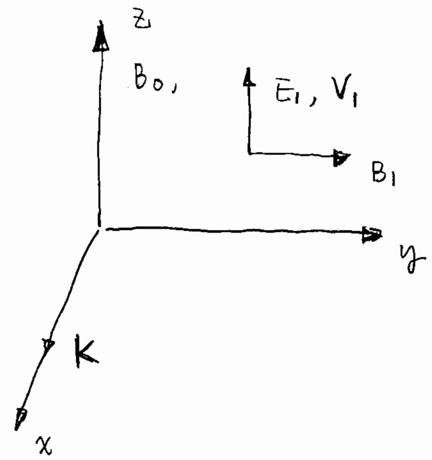
$$-i\omega m n_0 \hat{V}_{1x} = \frac{4\pi q^2 n_0^2}{i\omega} \hat{V}_{1x} + \frac{q n_0}{c} \frac{\Omega_e}{i\omega} B_0 \hat{V}_{1x}$$

$$\left(\omega^2 - \frac{4\pi q^2 n_0}{m} - \Omega^2\right) \hat{V}_{1x} = 0$$

$$\omega^2 = \omega_{pe}^2 + \Omega_e^2 \equiv \omega_{UH}^2$$

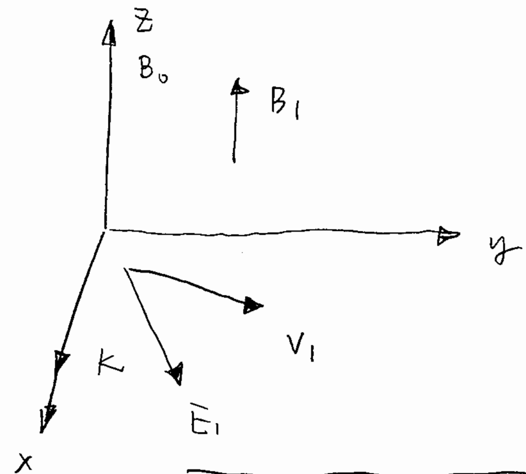
upper hybrid frequency

Perpendicular E&M waves in magnetized plasmas



Ordinary mode

O



eXtraordinary mode

X

O Mode

:

$V_1 \times B_0 = 0$,

Same as the unmagnetized case

$$\omega^2 = \omega_{pe}^2 + k^2 c^2$$

$\boxed{\text{X - Mode}}$

$$\vec{E}_1 = E_{1x}\,\hat{e}_x + E_{1y}\,\hat{e}_y \,, \qquad \vec{B}_1 = B_1\,\hat{e}_z$$

$$\vec{V}_1 = V_{1x}\,\hat{e}_x + V_{1y}\,\hat{e}_y \qquad \vec{K} = K\,\hat{e}_x$$

Momentum eq:

$$-i\omega m V_{1x} = qE_{1x} + \frac{q}{c}V_{1y}B_0 \qquad (1)$$

$$-i\omega m V_{1y} = qE_{1y} - \frac{q}{c}V_{1x}B_0 \qquad (2)$$

Maxwell's Eq:

$$\nabla\times E = -\frac{1}{c}\frac{\partial B}{\partial t}: \quad iKE_{1y} = \frac{i\omega}{c}B_1 \qquad (3)$$

$$\nabla\times B = \frac{4\pi}{c}J + \frac{1}{c}\frac{\partial E}{\partial t}: \begin{cases} -iKB_1 = \dfrac{4\pi q n_0}{c}V_{1y} - \dfrac{i\omega}{c}E_{1y} & (4) \\ 0 = \dfrac{4\pi q n_0}{c}V_{1x} - \dfrac{i\omega}{c}E_{1x} & (5) \end{cases}$$

$$(5) \Rightarrow \quad V_{1x} = \frac{i\omega E_{1x}}{4\pi q n_0}$$

$$(4) \Rightarrow \quad V_{1y} = \frac{i\omega E_{1y}}{4\pi q n_0} - \frac{iKcB_1}{4\pi q n_0} \underset{(3)}{=} \frac{i\left(\omega - \frac{K^2c^2}{\omega}\right)E_{1y}}{4\pi q n_0}$$

⑤, ④ $\Rightarrow$ ①, ②

$$\frac{\omega^2 m}{4\pi q^2 n_0} E_{1x} = E_{1x} + \frac{B_0}{c} \frac{i}{4\pi q n_0} \left(\omega - \frac{k^2c^2}{\omega}\right) E_{1y}$$

$$\frac{\omega^2 m}{4\pi q^2 n_0} - \frac{k^2c^2 m}{4\pi q^2 n_0} E_{1y} = E_{1y} - \frac{i\omega B_0}{4\pi q n_0 c} E_{1x}$$

$$\begin{pmatrix} \frac{\omega^2}{\omega_{pe}^2} - 1 & -\frac{i\Omega_e}{\omega_{pe}^2}\left(\omega - \frac{k^2c^2}{\omega}\right) \\ \frac{i\omega\Omega_e}{\omega_{pe}^2} & \frac{\omega^2 - k^2c^2}{\omega_{pe}^2} - 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = 0$$

$$|D| = 0 \Rightarrow \left(1 - \frac{\omega^2}{\omega_{pe}^2}\right)\left(1 - \frac{\omega^2 - k^2c^2}{\omega_{pe}^2}\right)$$

$$- \frac{\Omega_e^2 (\omega^2 - k^2c^2)}{\omega_{pe}^4} = 0$$

X: $\omega_{pe}^4 \Rightarrow$

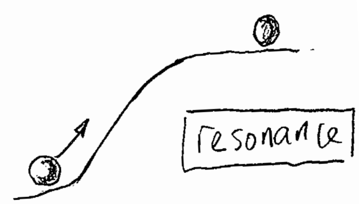
$$(\omega_{pe}^2 - \omega^2)(\omega_{pe}^2 - \omega^2 + k^2c^2) = \Omega_e^2\omega^2 - \Omega_e^2 k^2 c^2$$

$$(\omega_{pe}^2 - \omega^2)^2 + (\omega_{pe}^2 - \omega^2) k^2 c^2 = \Omega_e^2\omega^2 - \Omega_e^2 k^2 c^2$$

$$(\underbrace{\omega_{pe}^2 + \Omega_e^2}_{\omega_{UH}^2} - \omega^2) k^2 c^2 = \Omega_e^2\omega^2 - (\omega_{pe}^2 - \omega^2)^2$$

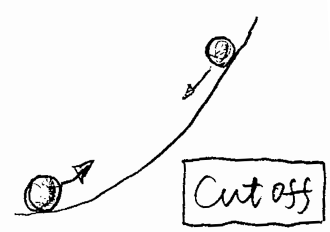
$$k^2c^2 = \frac{\Omega_e^2\omega^2 - (\omega_{pe}^2 - \omega^2)^2}{\omega_{UH}^2 - \omega^2}$$

$$n^2 = \frac{k^2c^2}{\omega^2} = \frac{(\omega^2 - \omega_{pe}^2)^2 - \omega^2\Omega_e^2}{\omega^2(\omega^2 - \omega_{UH}^2)}$$



$n = \infty$, resonance, absorption $\quad \omega = 0, \quad \omega = \omega_{UH}$

$n = 0$, cut off, reflected back.



$$\omega^4 - (2\omega_{pe}^2 + \Omega_e^2)\omega^2 + \omega_{pe}^2 = 0$$

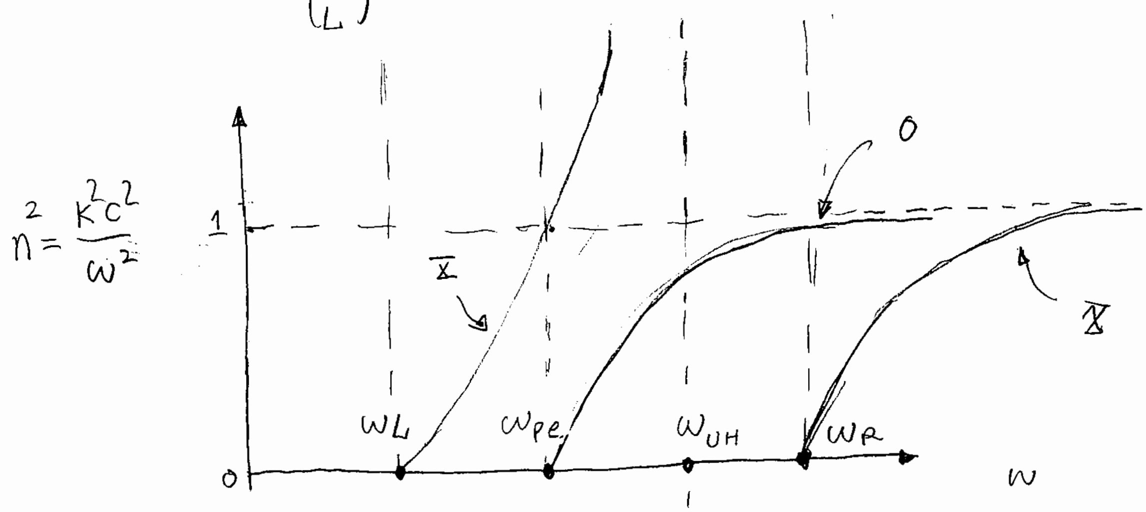
$$\omega^2 = \frac{2\omega_{pe}^2 + \Omega_e^2 \pm \sqrt{(2\omega_{pe}^2 + \Omega_e)^2 - 4\,\omega_{pe}^2}}{2}$$

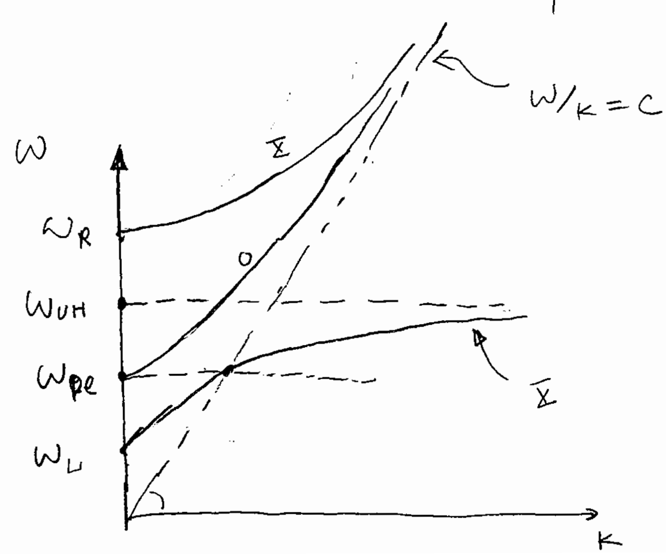
$$= \omega_{pe}^2 + \frac{\Omega_e^2}{2} \pm \sqrt{\frac{\Omega_e^4}{4} + \omega_{pe}^2\,\Omega_e^2}$$

$$= \left(\sqrt{\omega_{pe}^2 + \Omega_e^2/4} \pm \frac{\Omega_e}{2}\right)^2$$

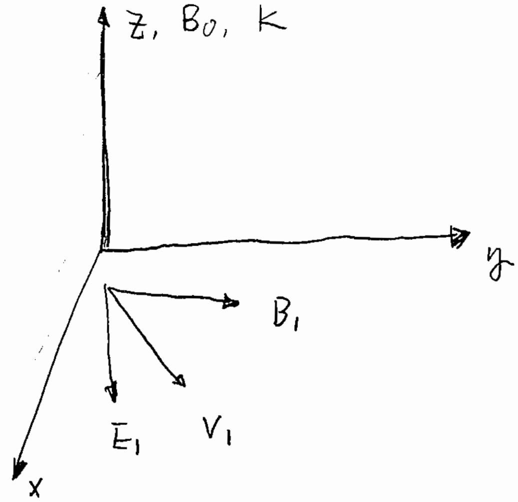
$\omega = \omega_L$ or ω_R

$$\omega_{\binom{R}{L}} \equiv \sqrt{\omega_{pe}^2 + \Omega_e^2/4} \pm \frac{\Omega_e}{2}$$





$\boxed{\text{parallel E\&M waves in magnetized plasmas}}$



$$\vec{k} = k\,\hat{e}_z$$

$$E_1 = E_{1x}\,\hat{e}_x + E_{1y}\,\hat{e}_y$$

$$V_1 = V_{1x}\,\hat{e}_x + V_{1y}\,\hat{e}_y$$

$$B_1 = B_{1x}\,\hat{e}_x + B_{1y}\,\hat{e}_y$$

$$\begin{pmatrix} \dfrac{\omega^2 - k^2c^2}{\omega_{pe}^2} - 1 & -\dfrac{i\Omega_e}{\omega_{pe}^2}\left(\omega - \dfrac{k^2c^2}{\omega}\right) \\[2ex] \dfrac{i\Omega_e}{\omega_{pe}^2}\left(\omega - \dfrac{k^2c^2}{\omega}\right) & \dfrac{\omega^2 - k^2c^2}{\omega_{pe}^2} - 1 \end{pmatrix} \begin{pmatrix} E_x \\[2ex] E_y \end{pmatrix} = 0$$

anti-symmetric

$\boxed{\text{compare with x-mode}}$

$$\left(1 + \frac{k^2c^2}{\omega_{pe}^2} - \frac{\omega^2}{\omega_{pe}^2}\right)^2 = \frac{\Omega_e^2}{\omega_{pe}^4}\left(\omega - \frac{k^2c^2}{\omega}\right)^2$$

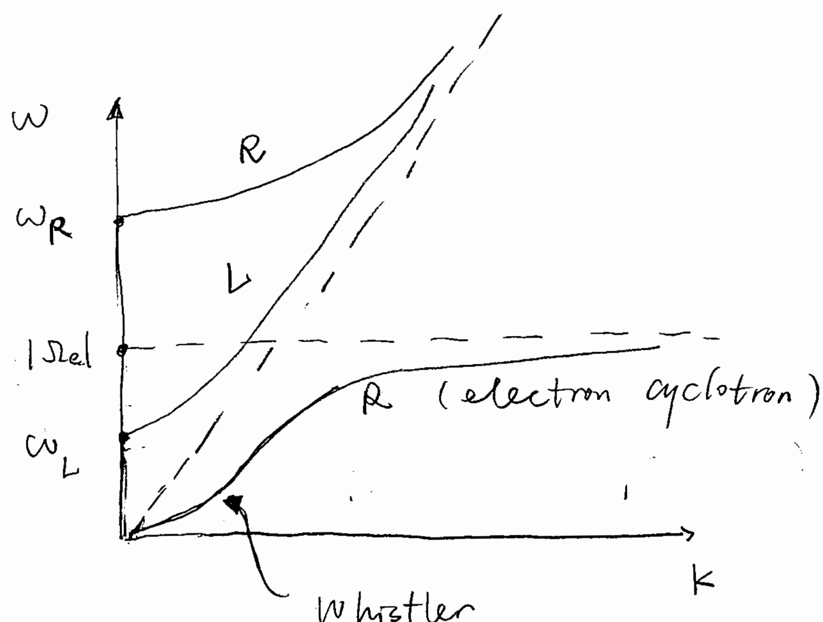
$$1+\frac{k^2c^2}{\omega_{pe}^2}-\frac{\omega^2}{\omega_{pe}^2}=\pm\left[\frac{\Omega_e\,\omega}{\omega_{pe}^2}-\frac{\Omega_e\,\omega}{\omega_{pe}^2}\frac{k^2c^2}{\omega^2}\right]$$

$$\frac{\omega^2}{\omega_{pe}^2}\left(\frac{k^2c^2}{\omega^2}-1\right)\pm\frac{\Omega_e\,\omega}{\omega_{pe}^2}\left(1-\frac{k^2c^2}{\omega^2}\right)=-1$$

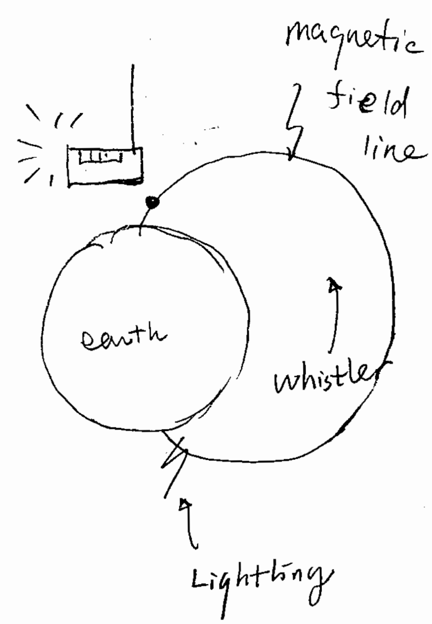
$$n^2=\frac{k^2c^2}{\omega^2}=1-\frac{\omega_{pe}^2}{\omega^2\pm\Omega_e\,\omega} \qquad \Omega_e<0$$

(+) : Right circularly polarized, R-wave

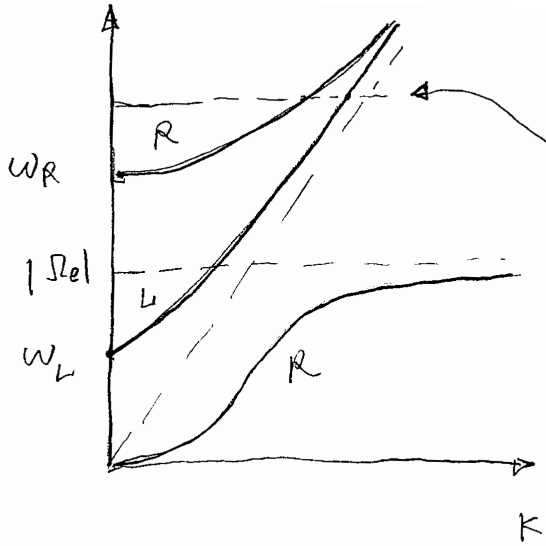
(−) : Left circularly polarized, L-wave



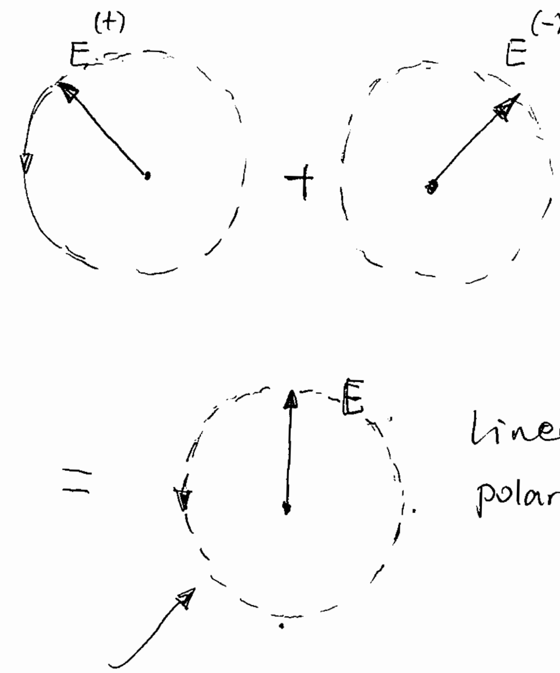
$\left(\frac{d\omega}{dk}>0\right)$



(Barkhausen 1919
Storey 1953)



$\omega^{(+)} > \omega^{(-)}$



linearly polarized

slowly rotating $\propto \omega_{pe}$

Faraday rotation,

- Can be used to measure plasma density.

Zoo of Fluid Waves

	unmagnetized $B_0 = 0$	magnetized $B_0 \neq 0$
high ω, $B_1 = 0$	Langmuir	upper hybrid
low ω, $B_1 = 0$	ion accoustic	ion waves $\begin{cases} \text{ion accoustic} \\ \text{ion cyclotron} \\ \text{lower hybrid} \end{cases}$
high ω, $B_1 \neq 0$	E & M	perpendicular $\begin{cases} O \\ X \end{cases}$ parallel $\begin{cases} R \\ L \end{cases}$
low ω, $B_1 \neq 0$		Alfván $\begin{cases} \text{Fast} \\ \text{slow} \\ \text{shear Alfván} \end{cases}$

	0	1
$\hat{k} \cdot \hat{B}_0$	perpendicular	parallel
$\hat{k} \cdot \hat{E}_1$	transverse	longitudinal