Magnetostatics and Faraday's Law Review

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1 Things to know

- Relevant Maxwell equations for Magnetostatics:

  $\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}_{free}, \quad \mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$

- Vector potential: $\nabla \times \mathbf{A} = \mathbf{B}$ ($\mathbf{H}$ is not necessarily divergence-free!)

- Scalar Magnetic potential, when $\mathbf{J} = 0: \quad \mathbf{H} = -\nabla\Phi_M$

- Boundary conditions derived from Maxwell equations (Jackson p.16–18)

  $(\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n}_{1\to2} = 0 \qquad \mathbf{n}_{1\to2} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_{free}$

  where $\mathbf{K}$ is a surface current

  ie. $\mathbf{B}_\perp$ and $\mathbf{H}_\parallel$ are continuous without any free surface currents

- Energy in magnetic fields; forces. (Jackson p.212–215): $U = \frac{1}{8\pi}\int_V \mathbf{B} \cdot \mathbf{H} dv$

- Mutual- and self-inductance relates time-varrying currents in one circuit to induced currents in another; constant for a given geometry.

2 Tools for Magnetostatics Problems

- Amperian current loop: given the current distribution, exploit the symmetry of the geometry using Stoke's law. (ie. analog to using Gauss' law in electrostatics)

- Biot-Savart Law: $d\mathbf{H} = \frac{I}{c}\frac{d\mathbf{l}\times\mathbf{x}}{x^3}$

- Expansion of vector potential in orthonormal basis functions:

  $A = \frac{1}{c}\int_V \frac{\mathbf{J}(\mathbf{x}')d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}$

  Expand the denominator into a set of suitable basis functions just as in the electrostatic case (sinusoids, hyperbolic sinusoids, legendre polynomials, spherical harmonics ...)

- Can also use Green function, just as in electrostatics.

3 Multipole Expansions

- Take moments of the current distribution. (Jackson, p.184–186)
- Useful equations for a Dipole:
  $\mathbf{m} = \frac{I}{c}\oint \mathbf{x} \times d\mathbf{l}, \quad \mathbf{A}(\mathbf{x}) = \frac{1}{c}\frac{\mathbf{m}\times\mathbf{x}}{x^3}, \quad \mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}) \quad U = -\mathbf{m}\cdot\mathbf{B}$

4 Magnetization

- In general, the magnetization is a nonlinear function of the applied $\mathbf{H}$ field.
- In linear materials, ($\mathbf{B} = \mu\mathbf{H}$), where μ is the permeability tensor, which is simply a scalar constant if the material is isotropic and uniform. For ferromagnetic materials, this doesn't work!
- Jackson works through some examples in p.194–201.

5 Faraday's Law

- Expressions of Faraday's Law:
  $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \mathscr{E} = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{L}{c}\frac{dI}{dt}$
  $\frac{d\Phi}{dt} = \int_A \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{a} + \oint_C (\mathbf{B}\times\mathbf{v})\cdot d\mathbf{a}$
  Note that the 2nd term above is just (negative) the magnetic force per unit charge