Disastrous Equations

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Science on Saturday

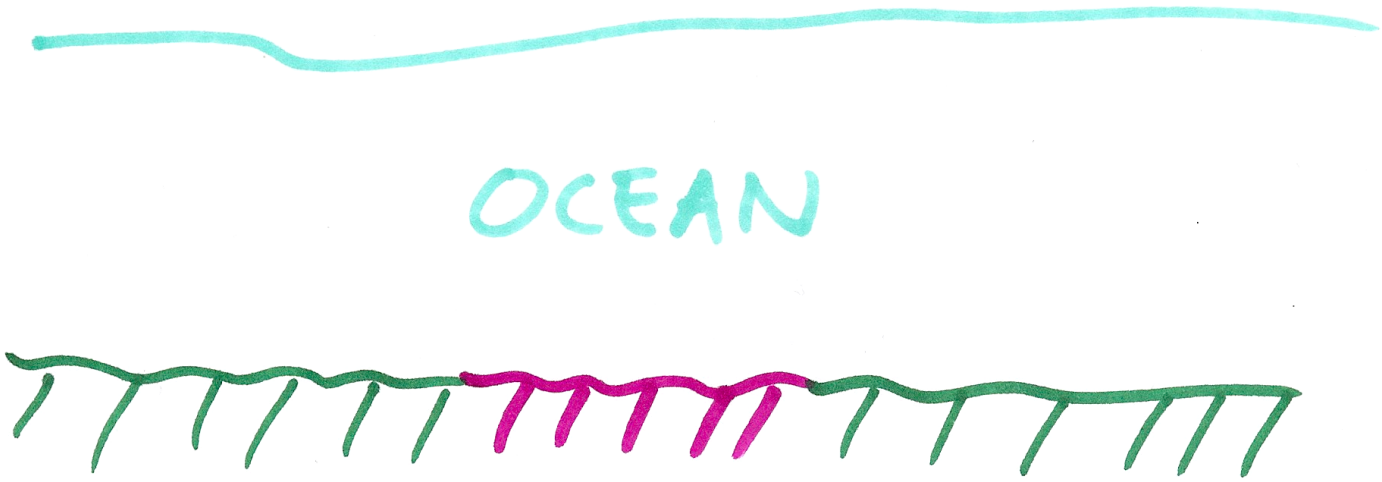
Earthquakes far out in the ocean generate massive water waves called tsunami. When such waves hit coastlines they can cause massive damage.

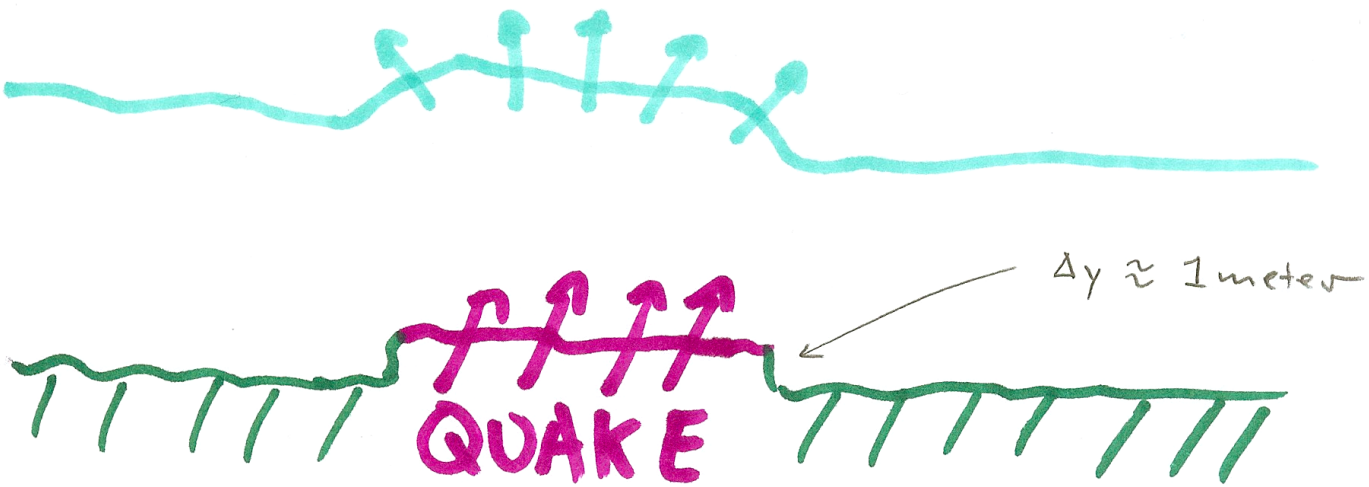
The 2004 Indian Ocean Tsunami: 100' waves. 230,000 deaths.
(photo by David Rydevik)

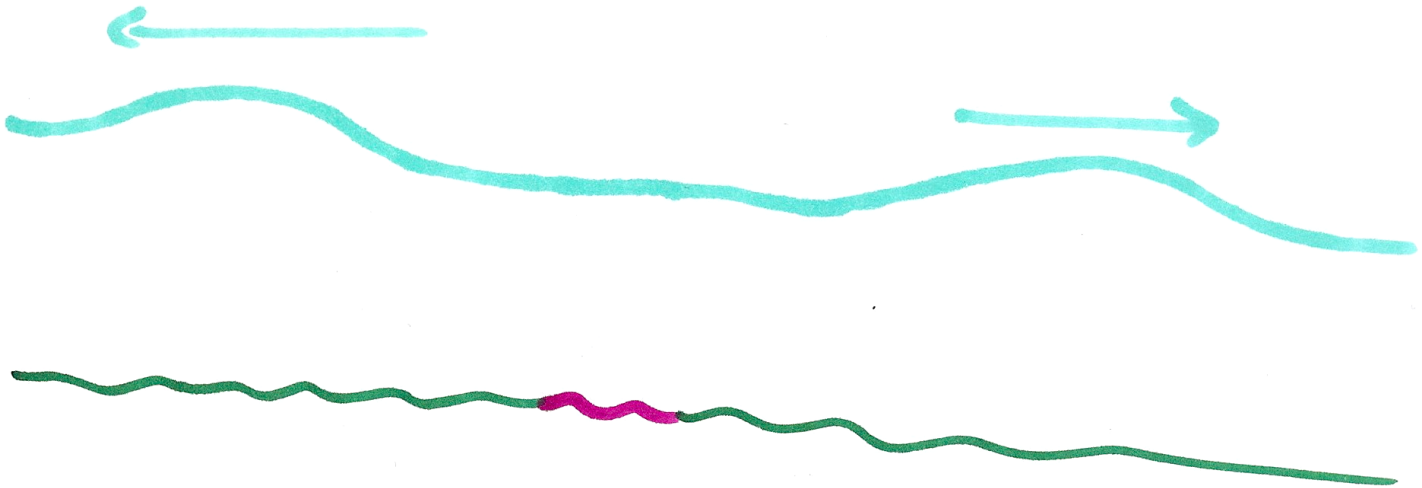
The 2011 Tōhoku Tsunami: 130' waves. 15,000 deaths + Nuclear accidents. (photo from National Geographic)

The 2011 Tōhoku Tsunami: 130' waves. 15,000 deaths + Nuclear accidents. (photo from National Geographic)

Way out in The Ocean




Δy ≈ 1 meter
QUAKE



Tsunami behavior

The waves which leave from the quake zone:

1. are NOT very high, around five feet;
2. are extremely wide, around one hundred miles;
3. move very fast, order of 500 mph;
4. go long distances, like halfway around the world, and do not "disperse" as they travel.

Tsunami behavior

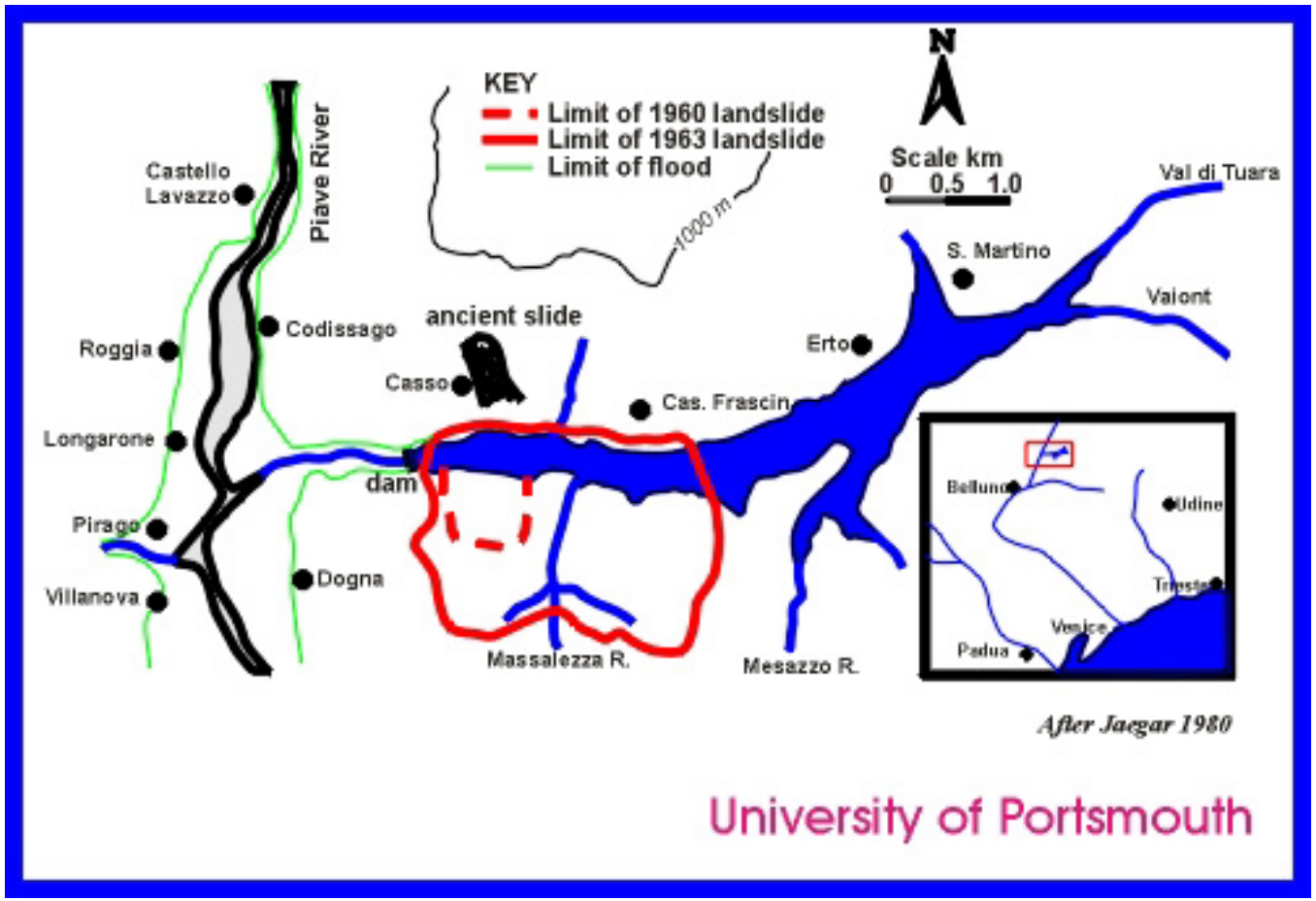
So long as they are in deep water, these waves are basically impossible to observe. They do not cause any problems for ocean-going vessels.

As they reach the shore, they "pile up" and grow to huge proportions; this is the origin of their destructive power.

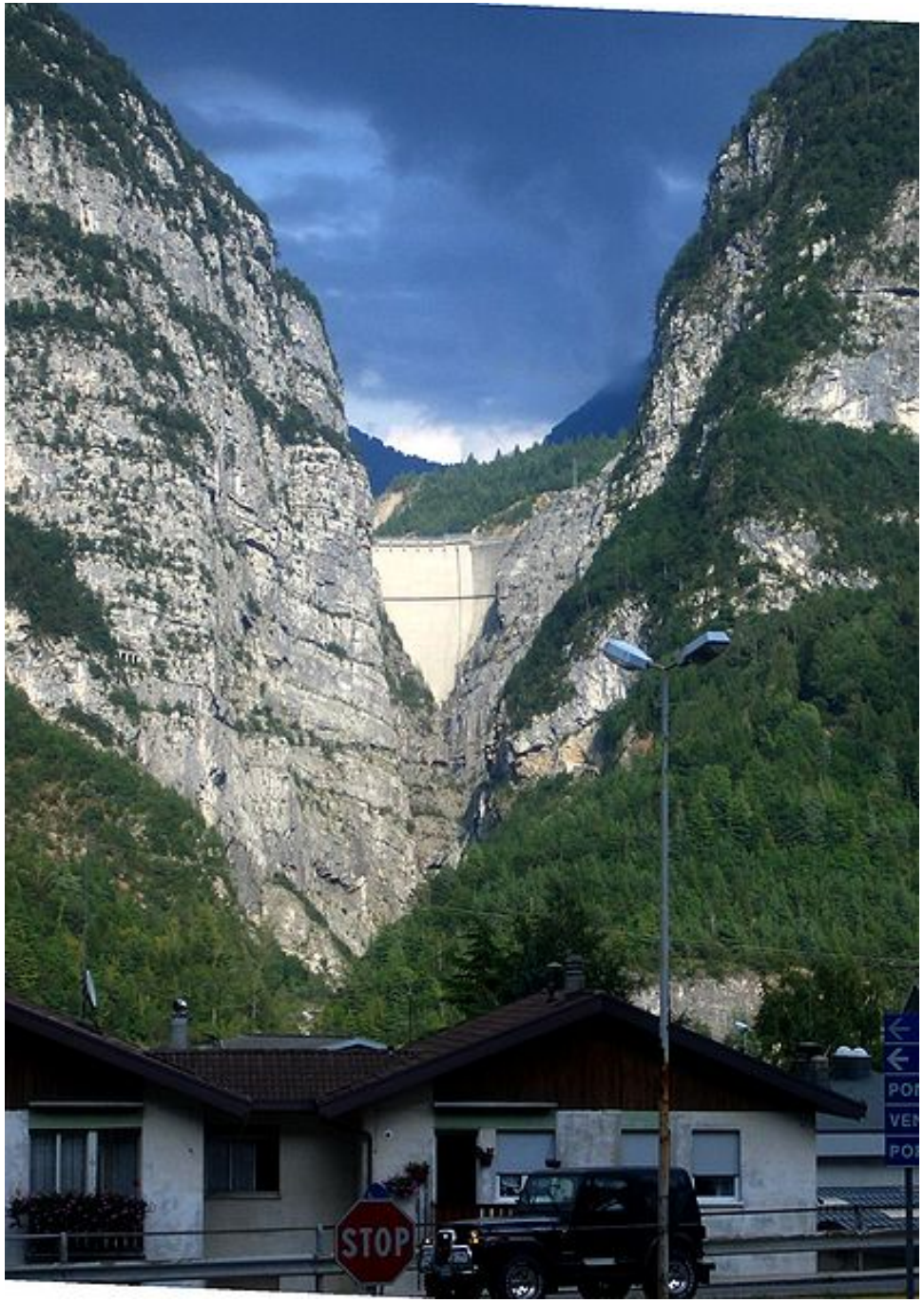
There are other instances of dangerous water waves that are believed to have features in common with earthquake generated tsunami.

(b) HSC operating near super-critical speed in the Marlborough Sound

Late 90s high speed ferry wakes on the English Coast: 15' wave. 1 death.



The 1963 Vajont Dam Disaster: 800' (800 feet!) wave. 2,000 deaths.


STOP

In the time of Zeus...

It was known at least as long ago as 426 BCE that earthquakes caused tsunami. From *History of the Peloponnesian War* by Thucydides:

"...the sea came in at Orobiae...and overflowed most part of the city. ...the people ... perished. And it seemeth unto me that without an earthquake such an accident could never happen."

But only in the last 150 years or so do we have an understanding of their behavior far from shore...

Mathematics has played a crucial role in our understanding of tsunami.

The story begins in Scotland...

The most famous quote in mathematical fluid mechanics

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

John Scott Russell

Wave re-creation

The oceanographic and fluid mechanics communities generally hold that tsunami are instances of Russell's "waves of translation" in the open ocean.

These waves are generally called "solitary waves" nowadays.

In Russell's time, it was hotly debated whether or not such waves actually exist.

Umm, what do you mean by "exist?"

I mean, are there solutions of the equations of motion for water waves which behave like those Russell claims to have observed.

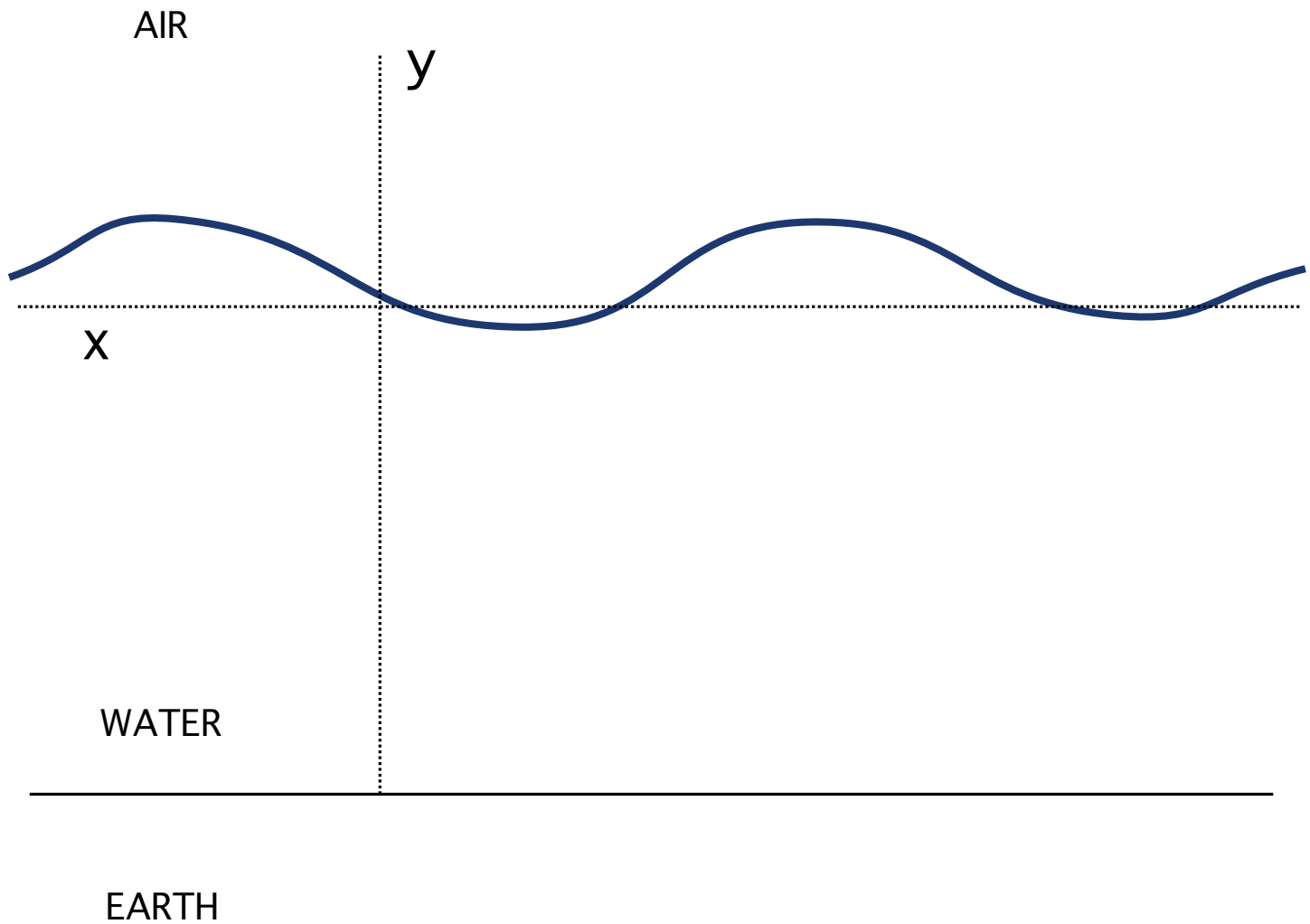
Umm, what are the equations of motion for water waves?

AIR

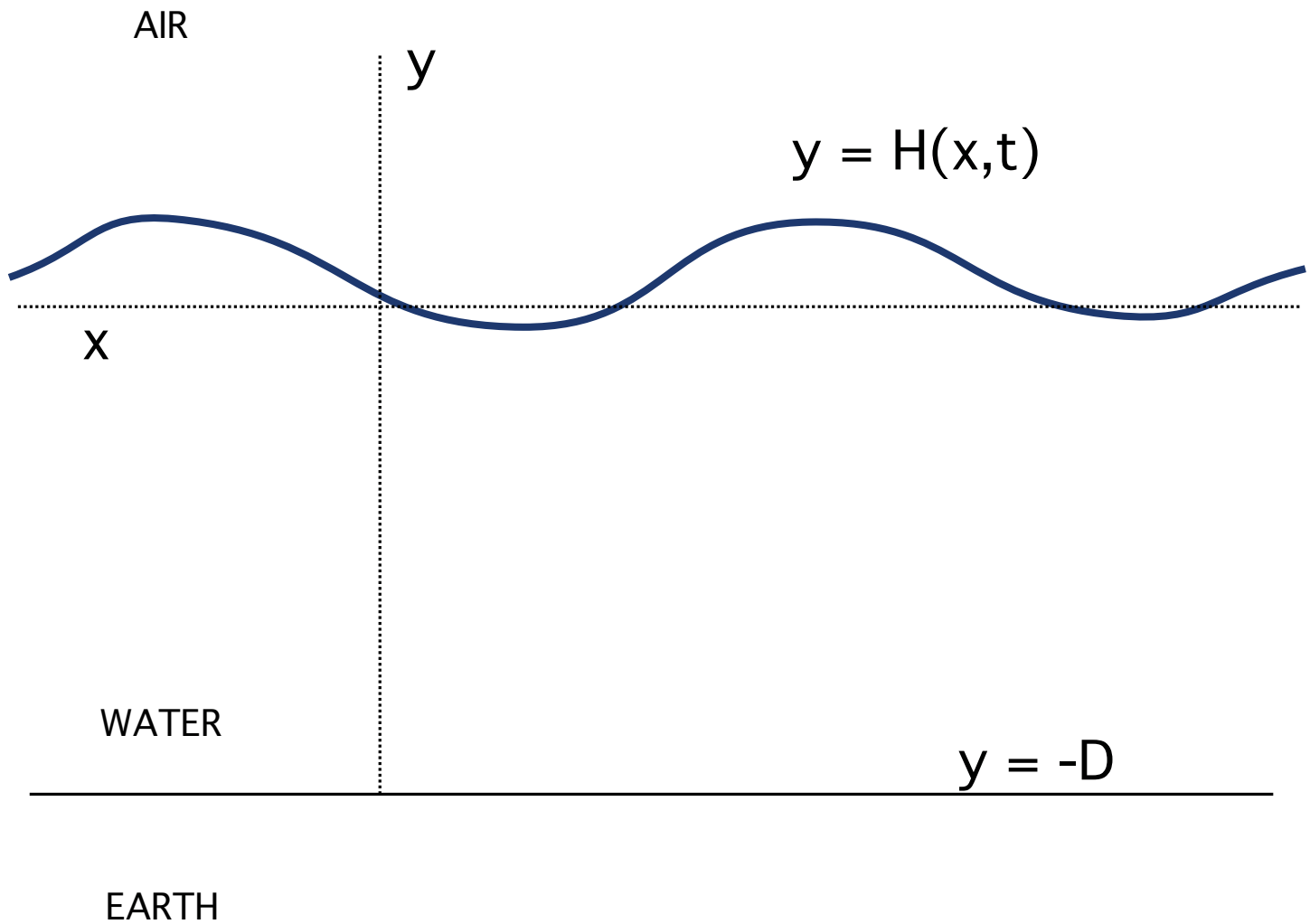
WATER

EARTH

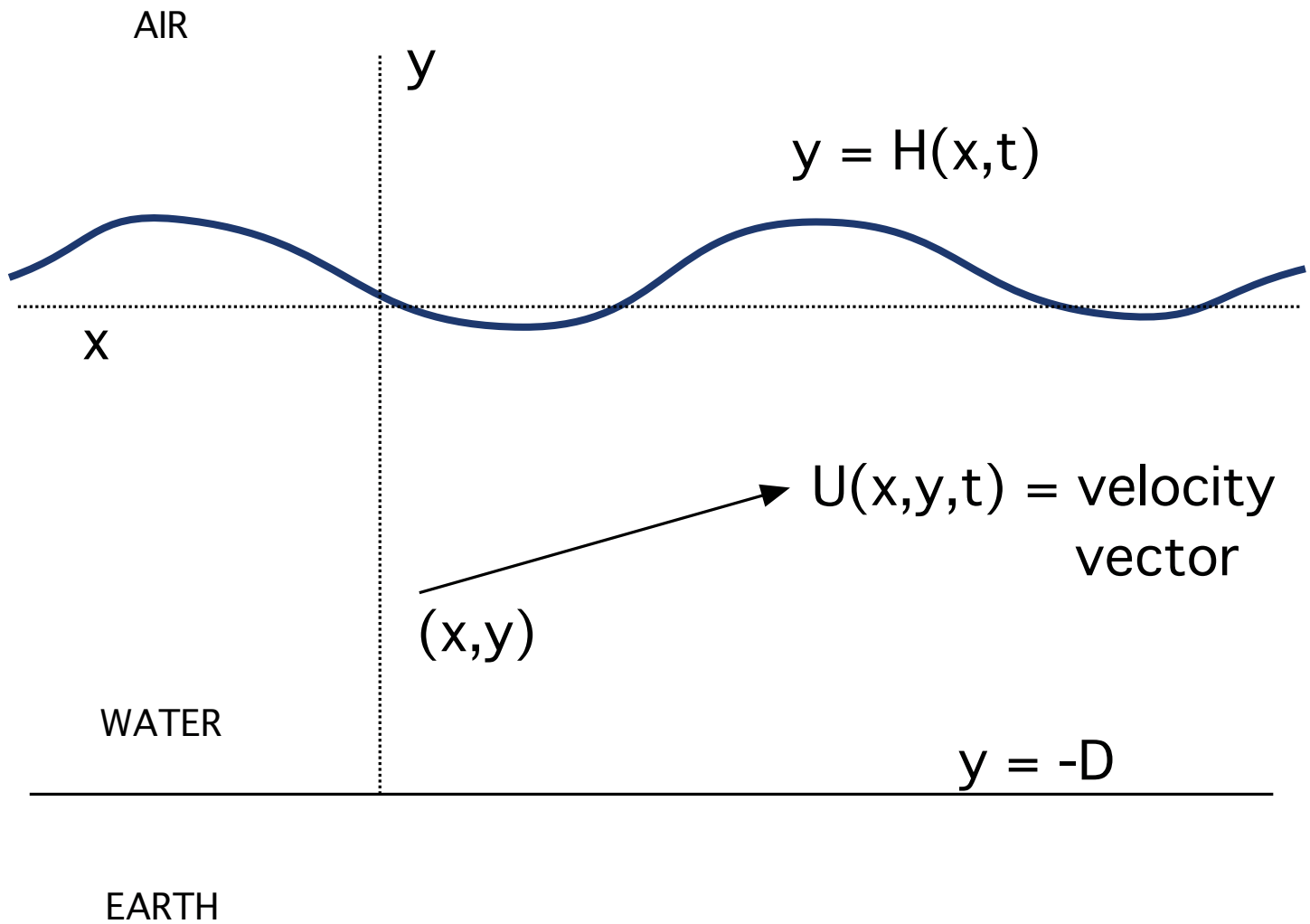
THE OCEAN (IN A 2 D WORLD)



THE OCEAN (IN A 2 D WORLD)

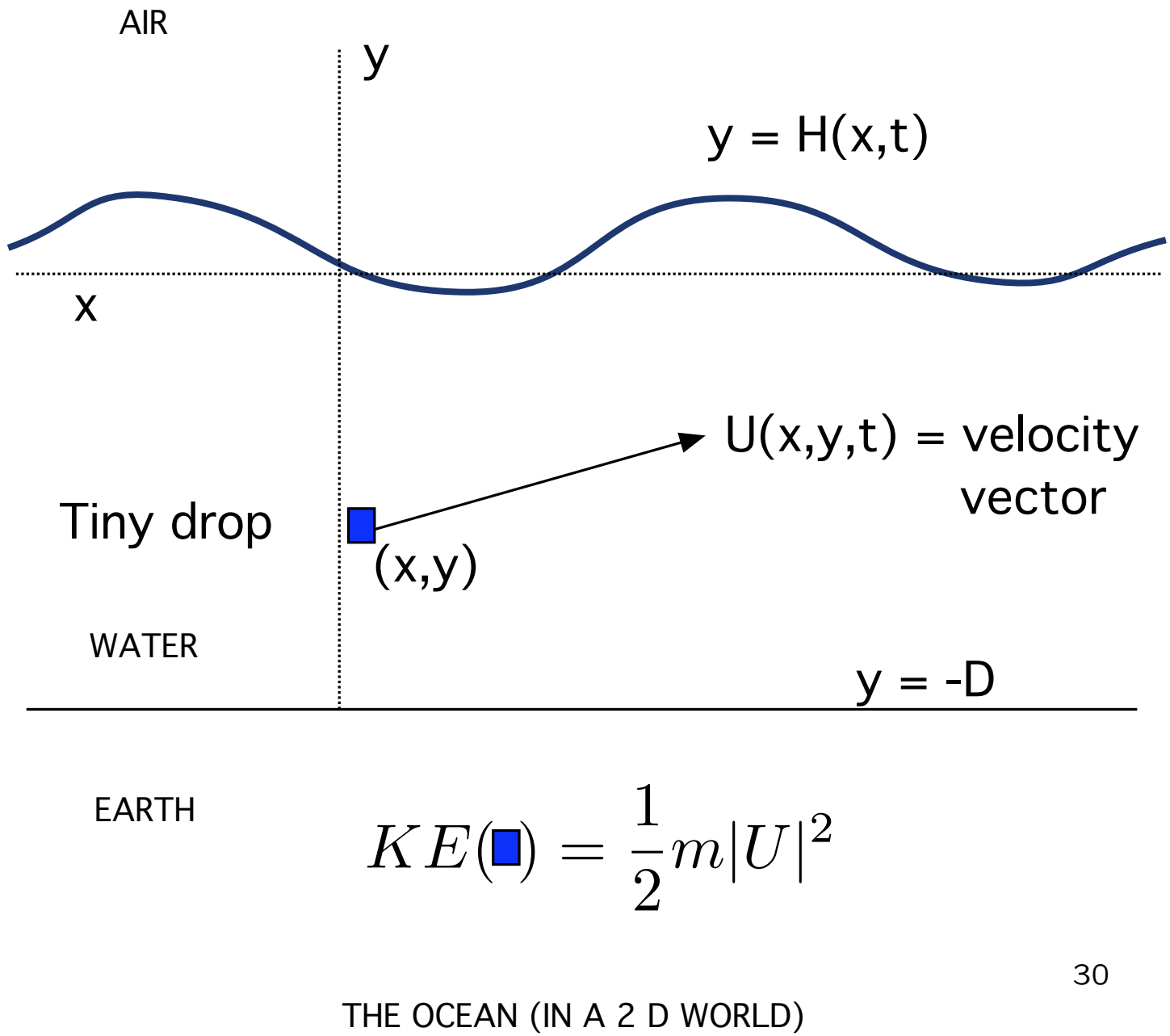


THE OCEAN (IN A 2 D WORLD)



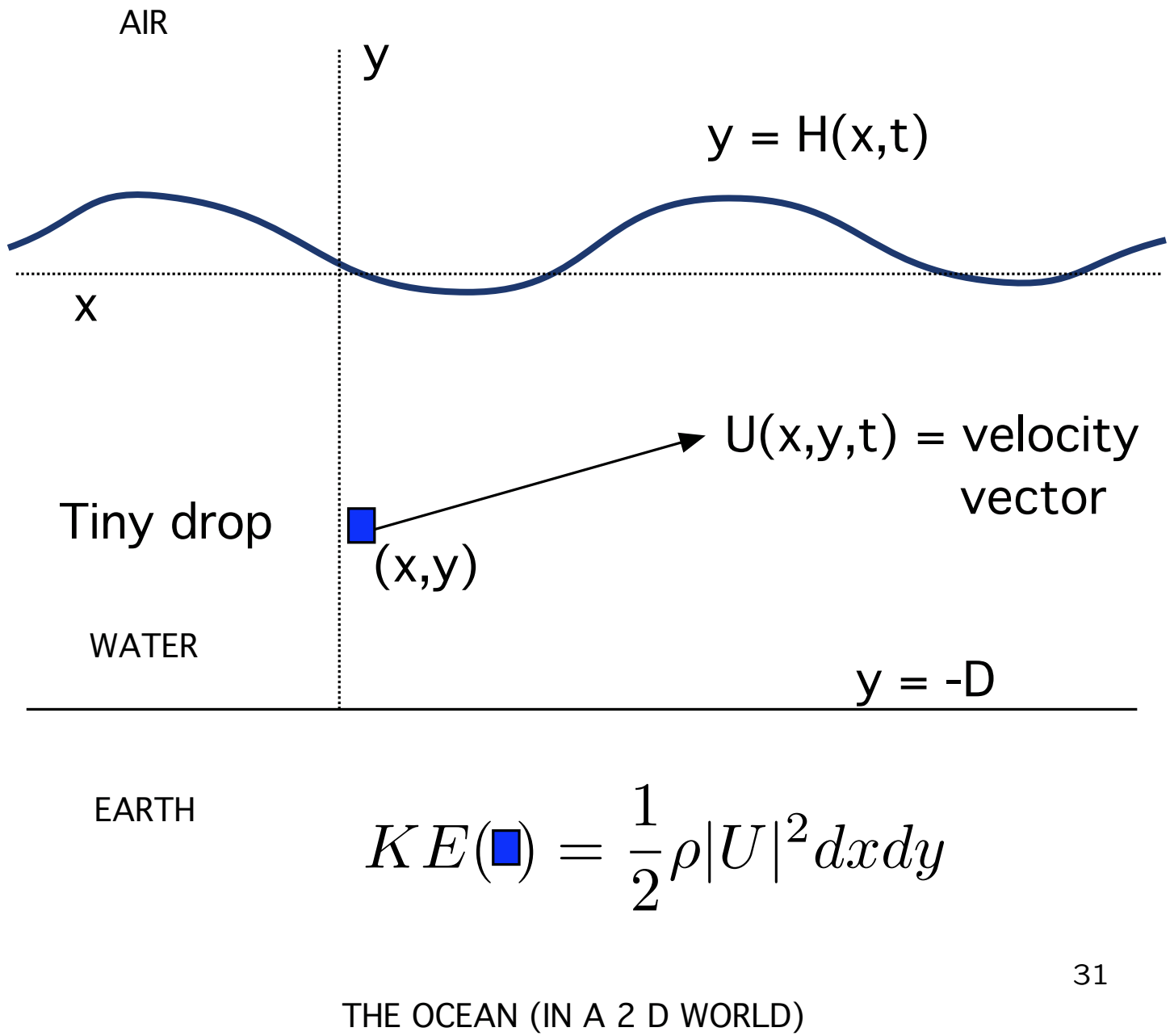
THE OCEAN (IN A 2 D WORLD)

In physics, if you know the Kinetic and Potential energy of a mechanical system, you can write down the equations of motion for that system.



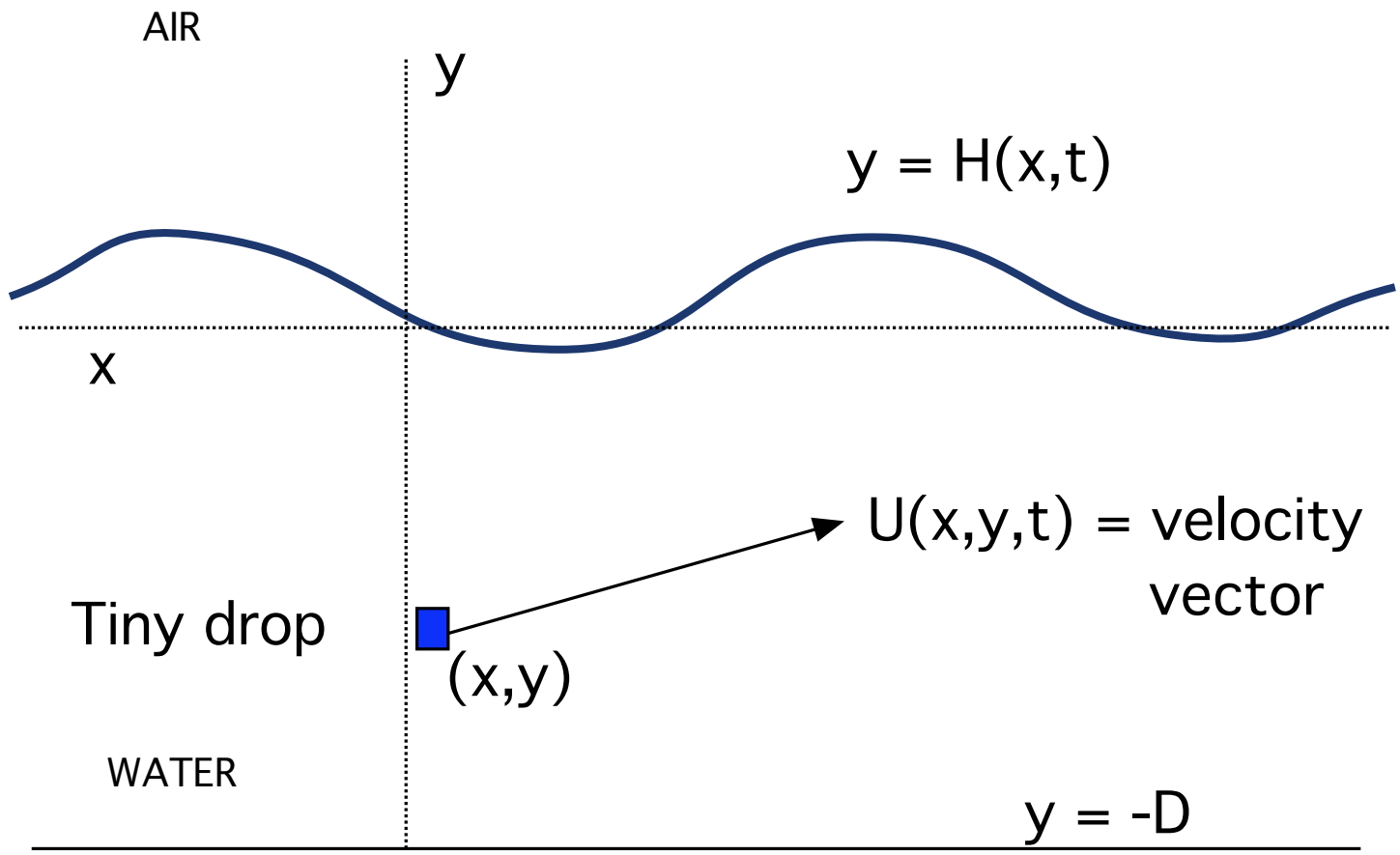
$$KE(\blacksquare) = \frac{1}{2} m |U|^2$$

THE OCEAN (IN A 2 D WORLD)



$$KE(\blacksquare) = \frac{1}{2}\rho|U|^2 dxdy$$

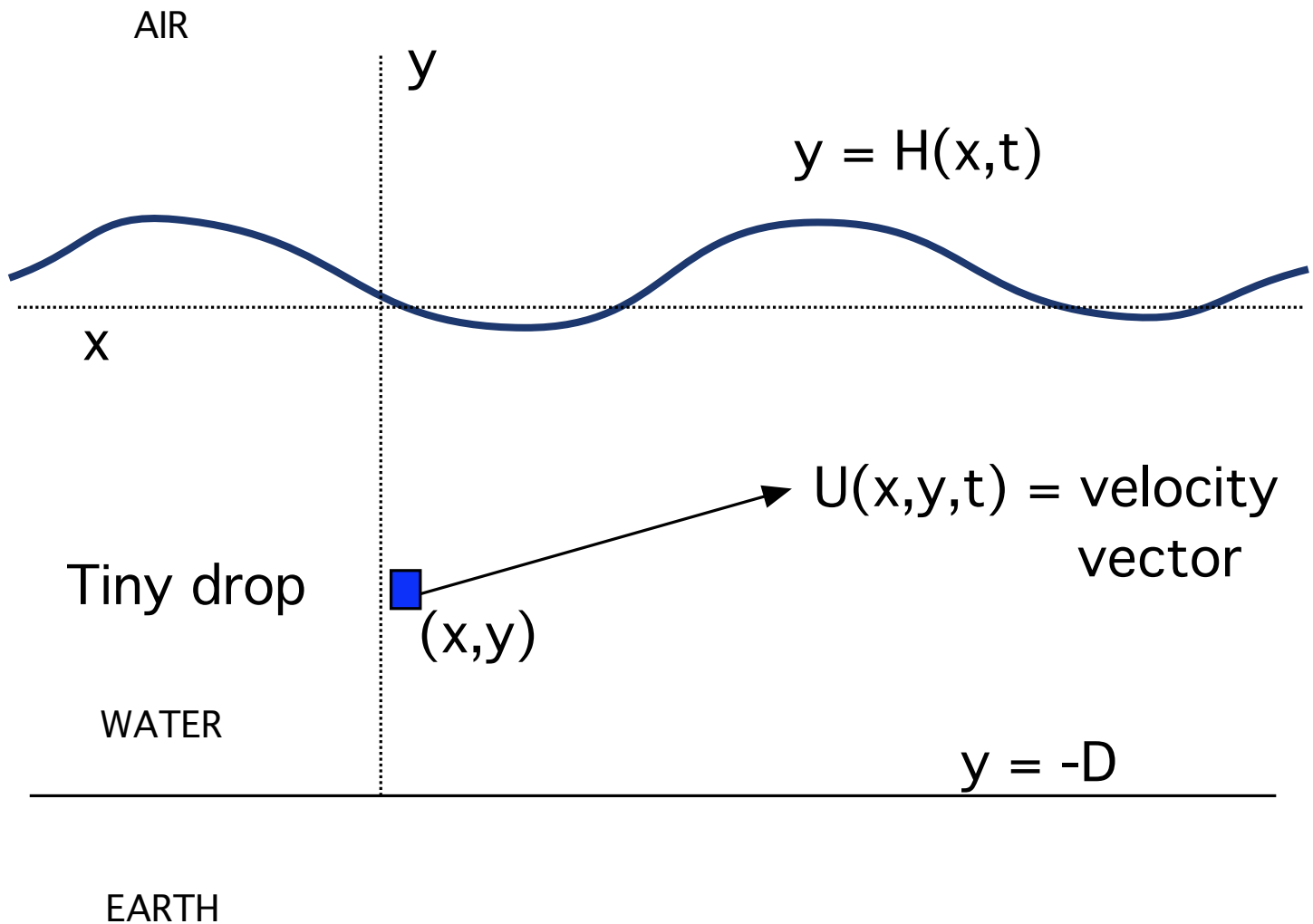
THE OCEAN (IN A 2 D WORLD)



EARTH

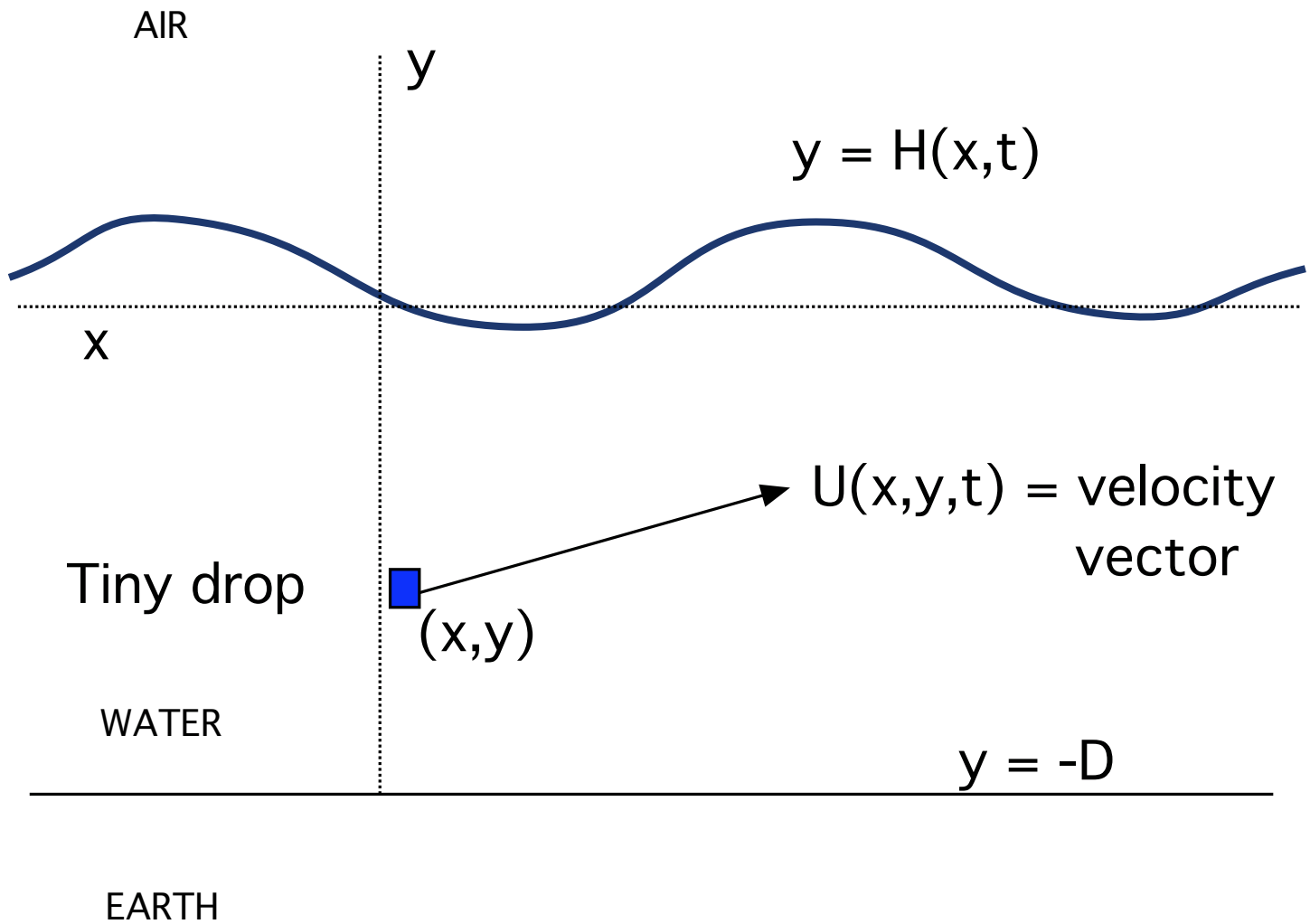
$$\text{Total } KE = \int_{\mathbf{R}} \int_{-D}^{H(x,t)} \frac{1}{2}\rho |U|^2 dx dy$$

THE OCEAN (IN A 2 D WORLD)



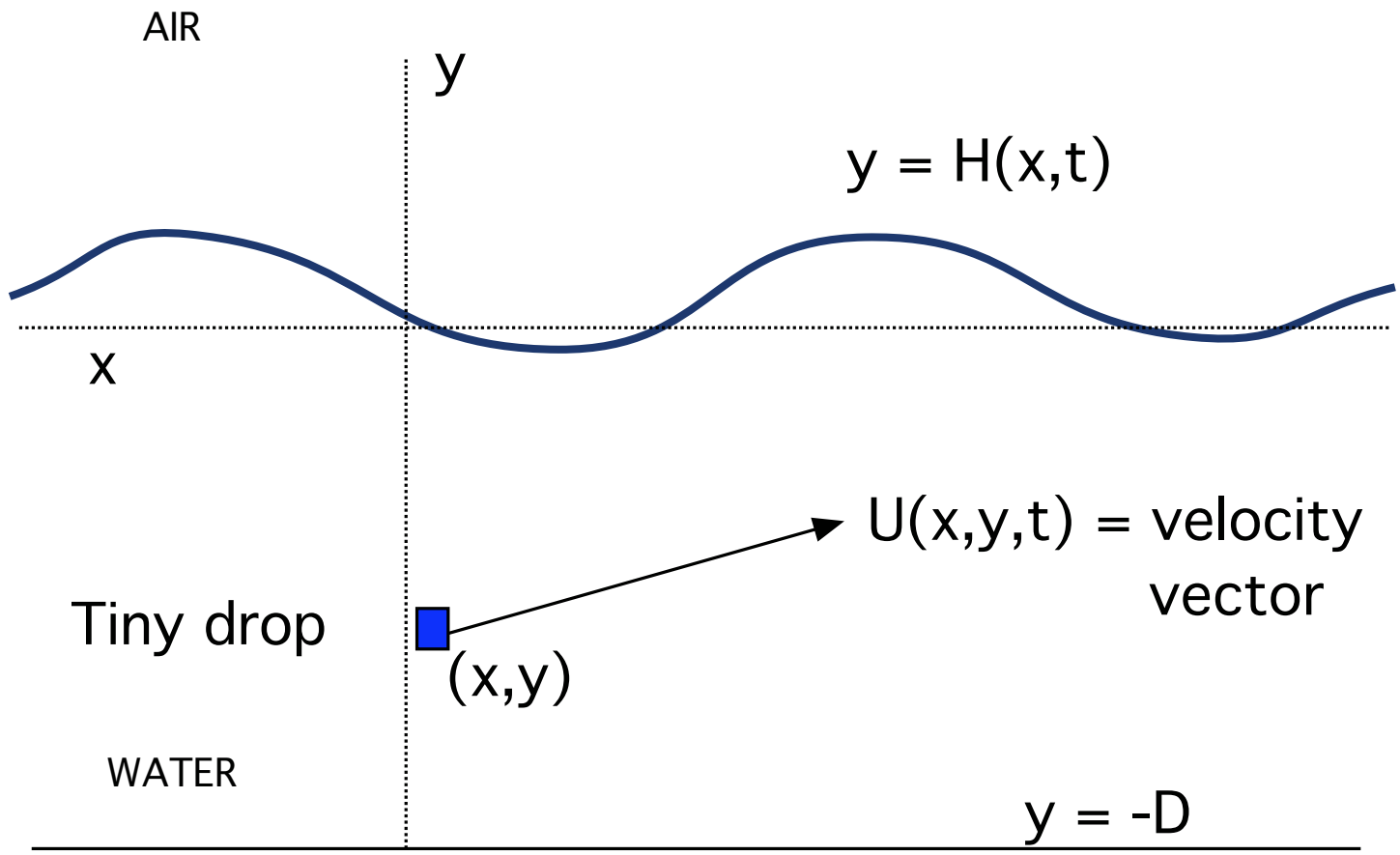
$$PE(\blacksquare) = mg(y + D)$$

THE OCEAN (IN A 2 D WORLD)



$$PE(\blacksquare) = \rho g(y + D)dxdy$$

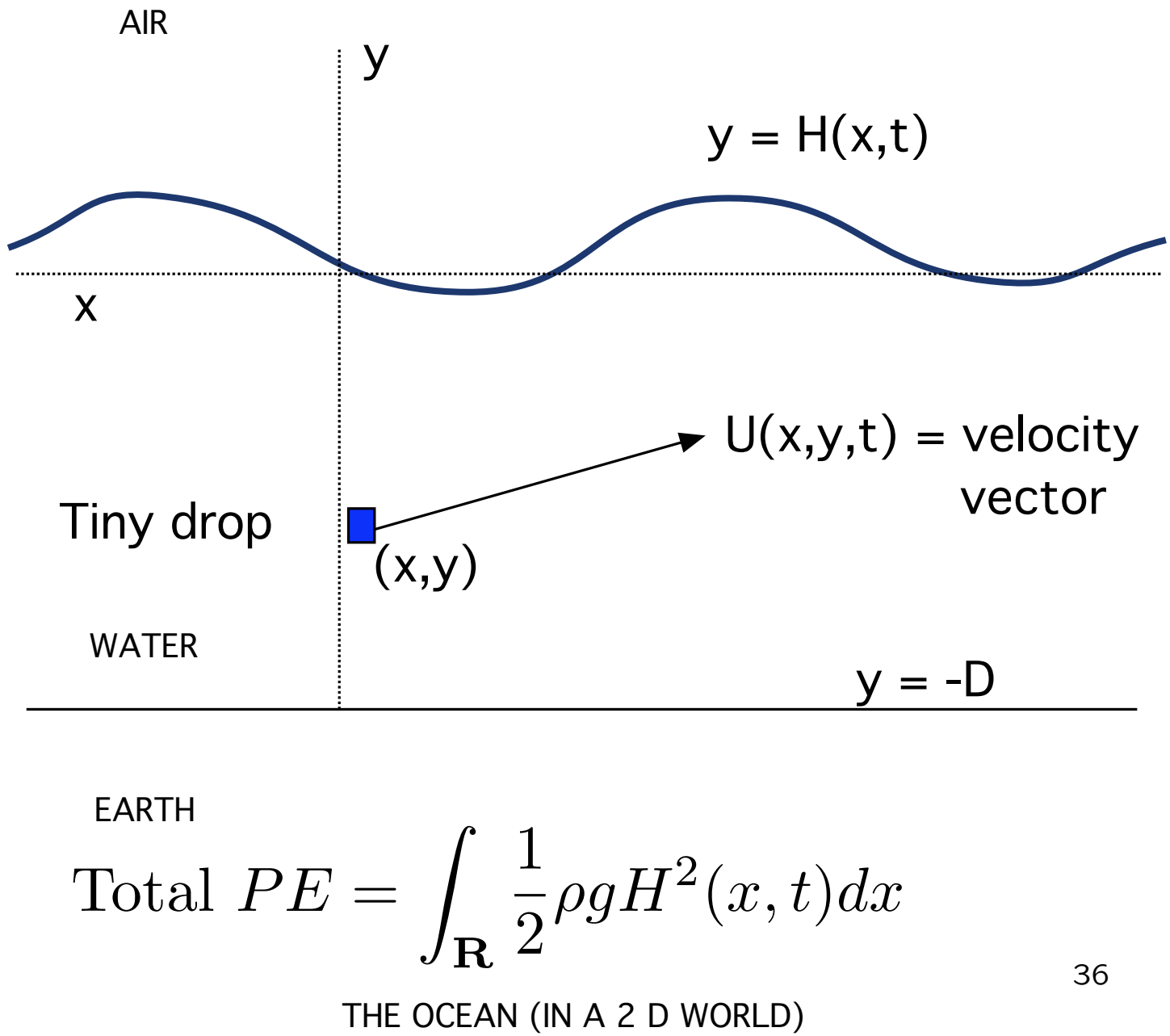
THE OCEAN (IN A 2 D WORLD)



EARTH

$$\text{Total } PE = \int_{\mathbf{R}} \int_{-D}^{H(x,t)} \rho g (y + D) dx dy$$

THE OCEAN (IN A 2 D WORLD)



$$\text{Total } PE = \int_{\mathbf{R}} \frac{1}{2}\rho g H^2(x,t)dx$$

THE OCEAN (IN A 2 D WORLD)

$$KE = \int_{\mathbf{R}} \int_{-D}^{H(x,t)} \frac{1}{2}\rho |U(x,y,t)|^2 dy dx.$$

$$PE = \int_{\mathbf{R}} \frac{1}{2}\rho g H^2(x,t) dx.$$

TURN THE PHYSICS CRANK!

This strategy for getting the equations of motion is due to Zakharov in 1968.

OH NOESSS!

$$\partial_t H = G(H)\psi$$
$$\partial_t \psi = -gH - \frac{1}{2}|\partial_x \psi|^2 + \frac{(G(H)\psi + \partial_x H \partial_x \psi)^2}{2\left(1 + |\partial_x H|^2\right)}$$

$H(x,t) =$ surface elevation.

$\psi(x,t) = \Phi(x, H(x,t), t)$ where $\nabla\Phi(x,y,t) = U =$ fluid velocity.

$G(H) =$ "Dirichlet to Neumann" map for the fluid domain. (People spend their whole research careers studying these things.)

$$\partial_t H = G(H)\psi$$
$$\partial_t \psi = -gH - \frac{1}{2}\left|\partial_x \psi\right|^2 + \frac{\left(G(H)\psi + \partial_x H \partial_x \psi\right)^2}{2\left(1 + \left|\partial_x H\right|^2\right)}$$

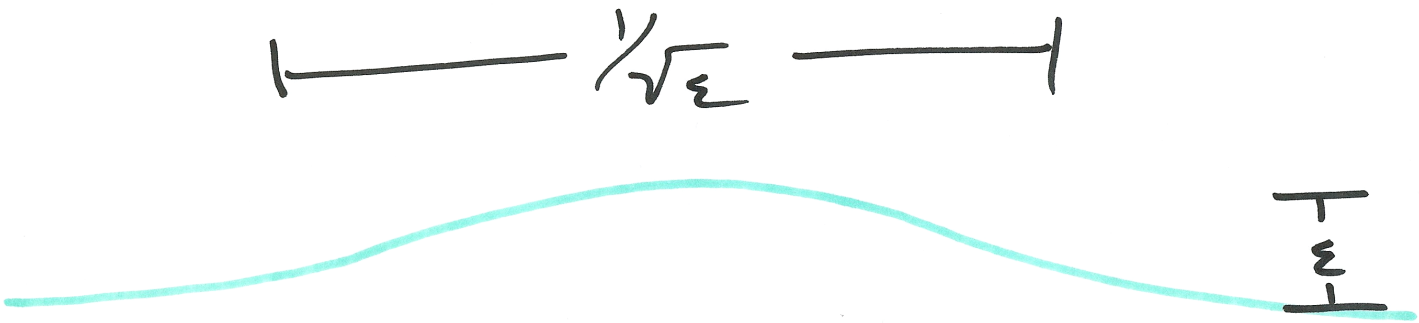
These equations are horrible, but they apply to BASICALLY ANY SORT OF SURFACE WAVE.

We are not interested in any old wave, we are interested in Russell's wave of translation.

The observed wave of translation:

1. is NOT VERY high;
2. is VERY wide.

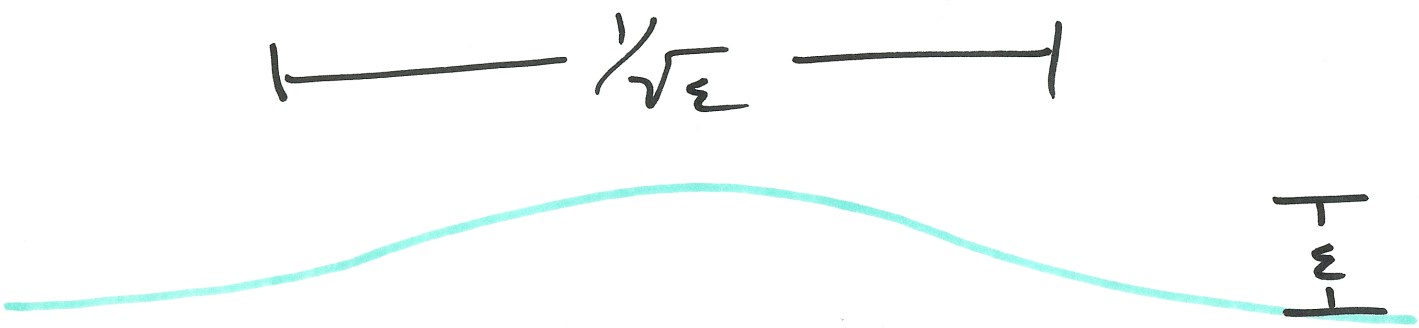
Russell did experiments in the 19th century that demonstrated that if the the amplitude of the wave was given by ϵ, then its wavelength is $1/\sqrt{\epsilon}$.



So fix $0 < \epsilon \ll 1$ and set:

$$H(x,t) = \epsilon u(\sqrt{\epsilon}x, \epsilon t)$$

and a similar expression for ψ. (This was an idea of Joseph Valentin Boussinesq in 1872.)



If you make the assumption

$$H(x,t) = \epsilon u(\sqrt{\epsilon}x, \epsilon t)$$

then this system:

$$\partial_t H = G(H)\psi$$
$$\partial_t \psi = -gH - \frac{1}{2}|\partial_x \psi|^2 + \frac{(G(H)\psi + \partial_x H \partial_x \psi)^2}{2\left(1 + |\partial_x H|^2\right)}$$

(after omitting "small terms") becomes...

The Korteweg-de Vries equation*

$$\partial_t u = -c_0 \partial_x u + \alpha \partial_x^3 u + \beta u \partial_x u$$

Here

$$c_0 = \sqrt{gD}$$

and α and β are complicated expressions made out of g, D and some other things.

(* It was first derived by Boussinesq, but K and d-V did it in 1895 to more acclaim...)

$$\partial_t u = -c_0 \partial_x u + \alpha \partial_x^3 u + \beta u \partial_x u$$

In a rough sense, the debate in the old days over the existence of Russell's wave of translation hinged on correctly assigning the values of α and β. (No one argued with c_0.)

Many people thought that $\beta = 0$.

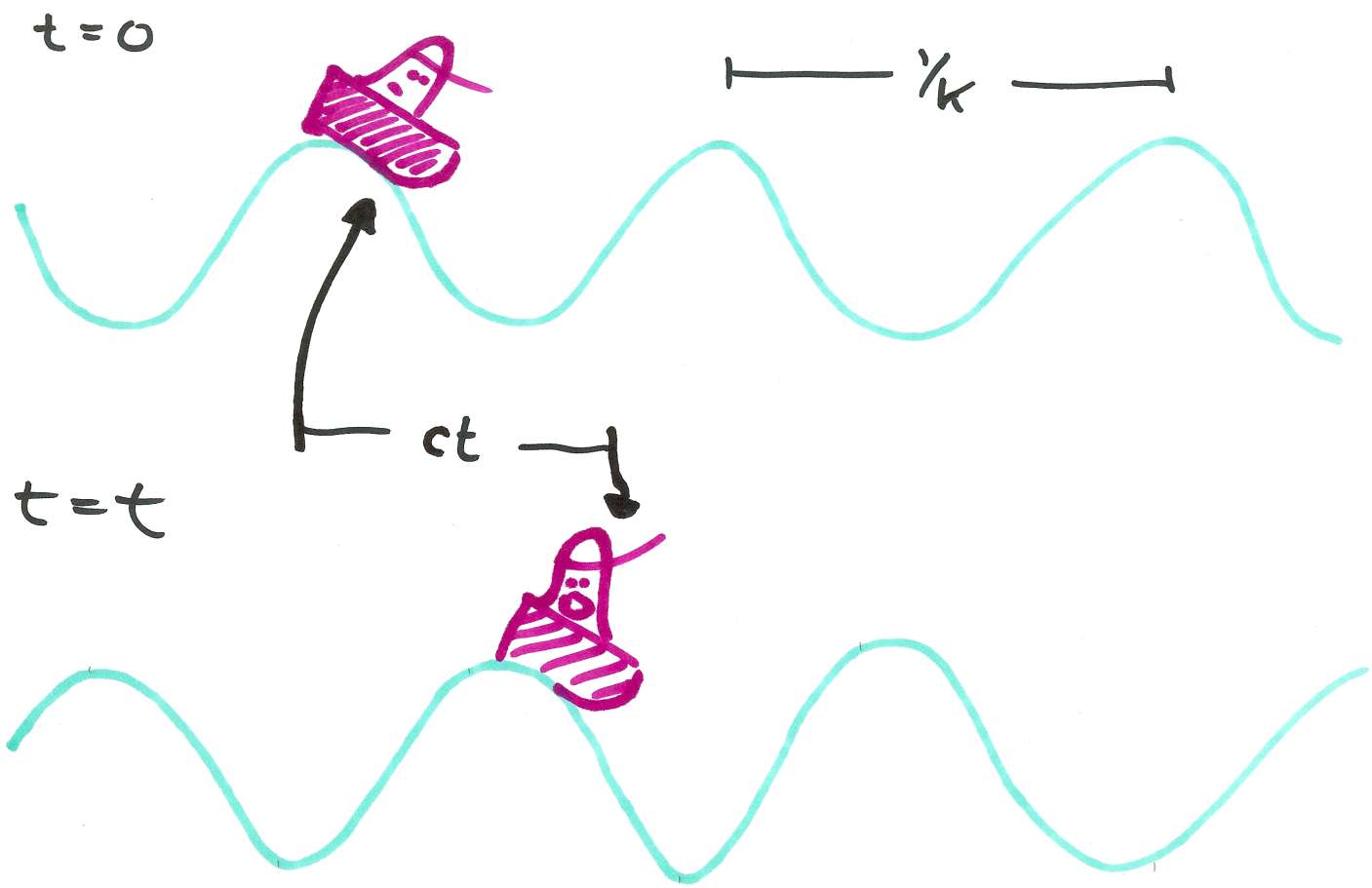
So they'd have:

$$\partial_t u = -c_0 \partial_x u + \alpha \partial_x^3 u$$

Let's look for a special "wave solution" of the form

$$u(x,t) = \sin\left[k\left(x - ct\right)\right].$$

Notice: this is traveling sine wave with spatial frequency k and speed $c = c(k)$.


t=0
1/k
ct
t=t

$$\partial_t u = -c_0 \partial_x u + \alpha \partial_x^3 u$$

$$u(x,t) = \sin\left[k\left(x - ct\right)\right].$$

$$\partial_t u = -ck \cos[k(x - ct)]$$

$$\partial_x u = k \cos[k(x - ct)]$$

$$\partial_x^3 u = -k^3 \cos[k(x - ct)]$$

$$\partial_t u = -c_0 \partial_x u + \alpha \partial_x^3 u$$

$$u(x,t) = \sin\left[k\left(x - ct\right)\right].$$

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$$\partial_x u = k \cos[k(x - ct)]$$

$$\partial_x^3 u = -k^3 \cos[k(x - ct)]$$

$$-ck \cos[k(x - ct)] = -c_0 k \cos[k(x - ct)] - \alpha k^3 \cos[k(x - ct)]$$

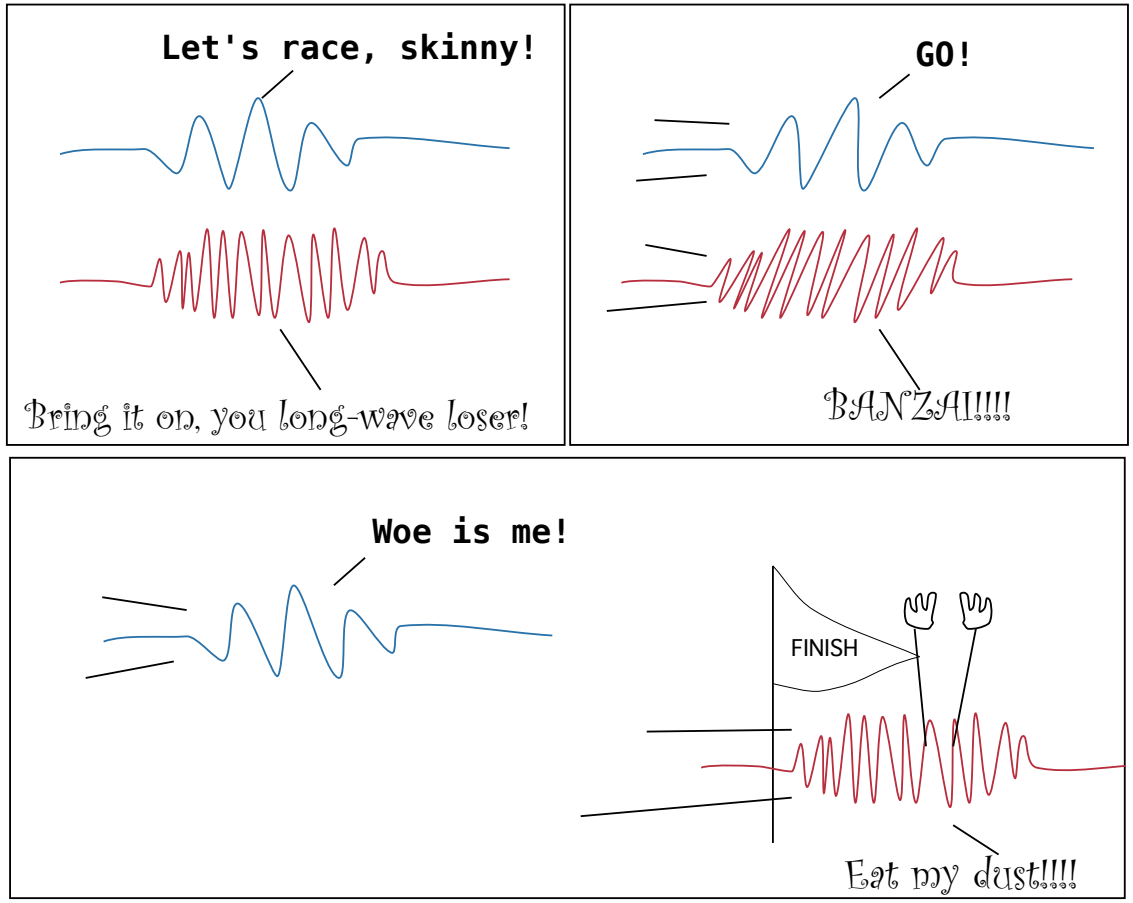
$$-ck\cos[k(x-ct)] = -c_0 k\cos[k(x-ct)] - \alpha k^3\cos[k(x-ct)]$$

$$c = c_0 + \alpha k^2$$

Important: The wavespeed depends on wavelength!

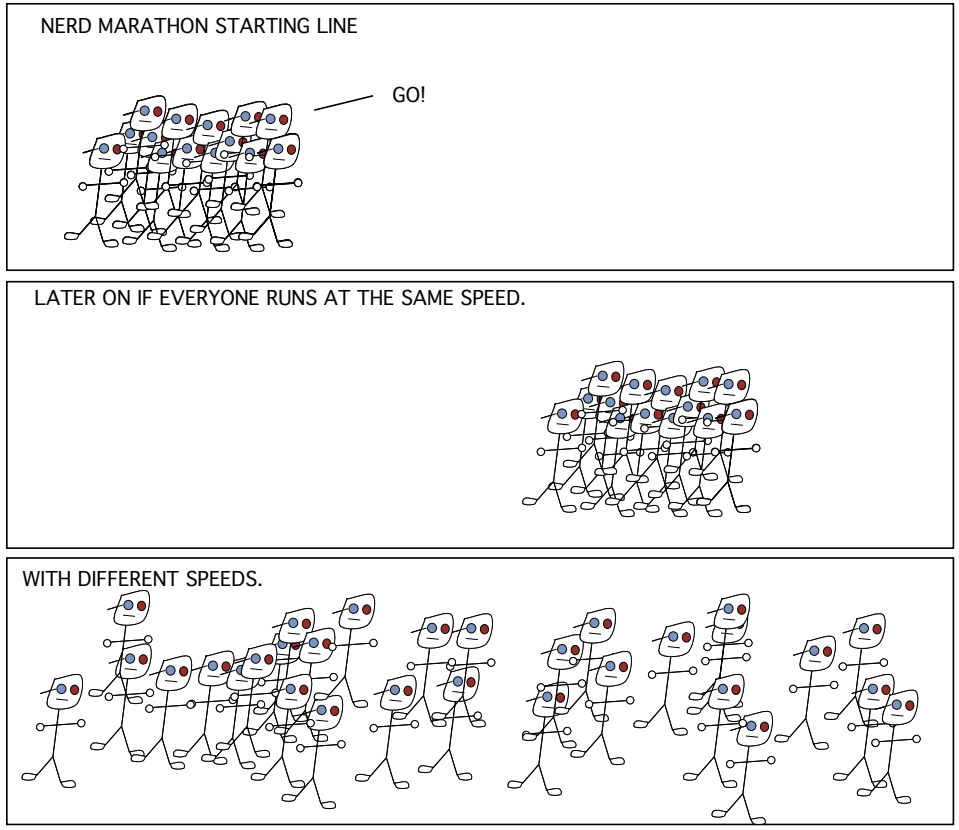
Note that it is α that causes this. If $\alpha = 0$, all waves move at the same speed, c_0.

This is called dispersion:

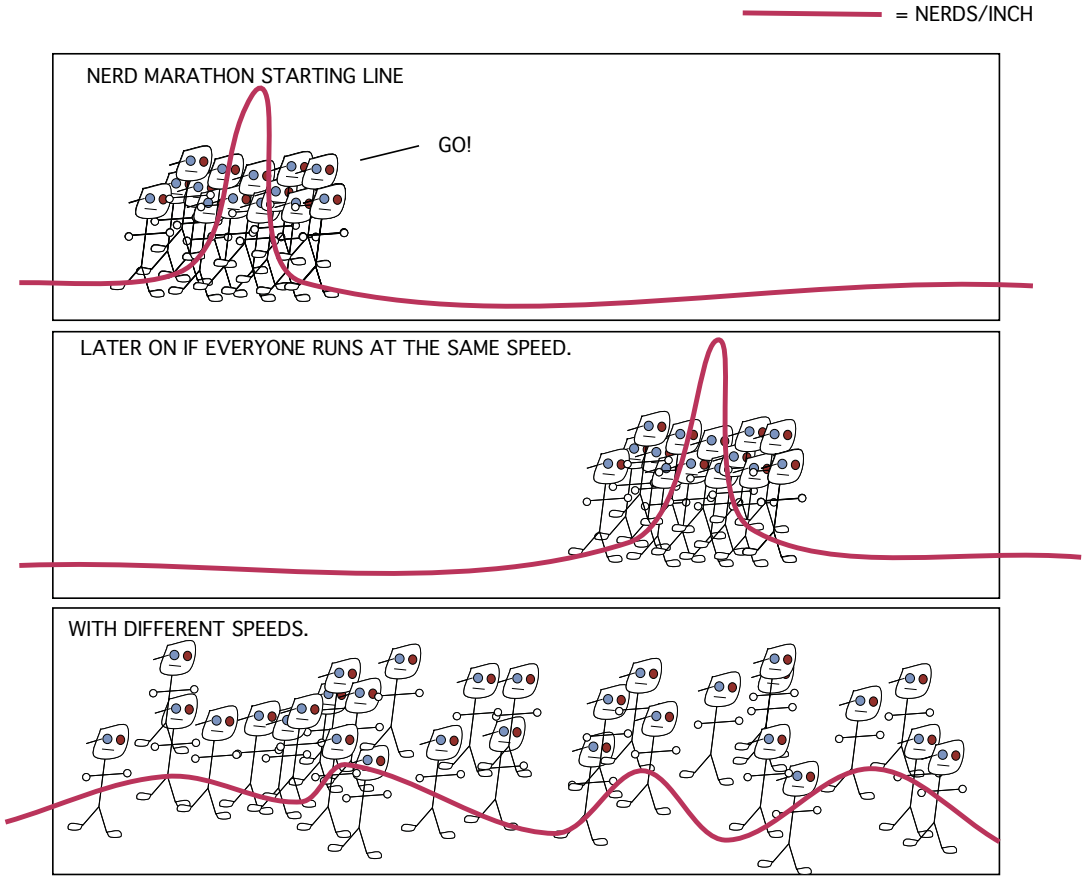


Wavespeed depends on wavelength!

Dispersion smooths things out



Dispersion smooths things out



One can uses this idea to show a "dispersive estimate" for solutions of

$$\partial_t u = -c_0 \partial_x u + \alpha \partial_x^3 u$$

which are initially sort of localized:

$$AMPLITUDE := \sup_{x \in \mathbf{R}} |u(x,t)| \leq t^{-1/3} \int_{-\infty}^{\infty} |u(x,0)| \, dx$$

The point: there's no way such a wave is making it from the middle of the ocean to shore.

Nonlinearity

Our model equation for water waves:

$$\partial_t u = -c_0 \partial_x u + \alpha \partial_x^3 u + \beta u \partial_x u$$

has both dispersive and nonlinear effects.

What does that nonlinear term do?

Famously, Burgers' equation

$$\partial_t u = u \partial_x u$$

forms "shocks" in finite time.

Shocking behavior

Recall that solutions of

$$\partial_t u = -c_0 \partial_x u$$

move at speed c_0.

For Burgers' equation, $\partial_t u = u \partial_x u$ we could look at this like

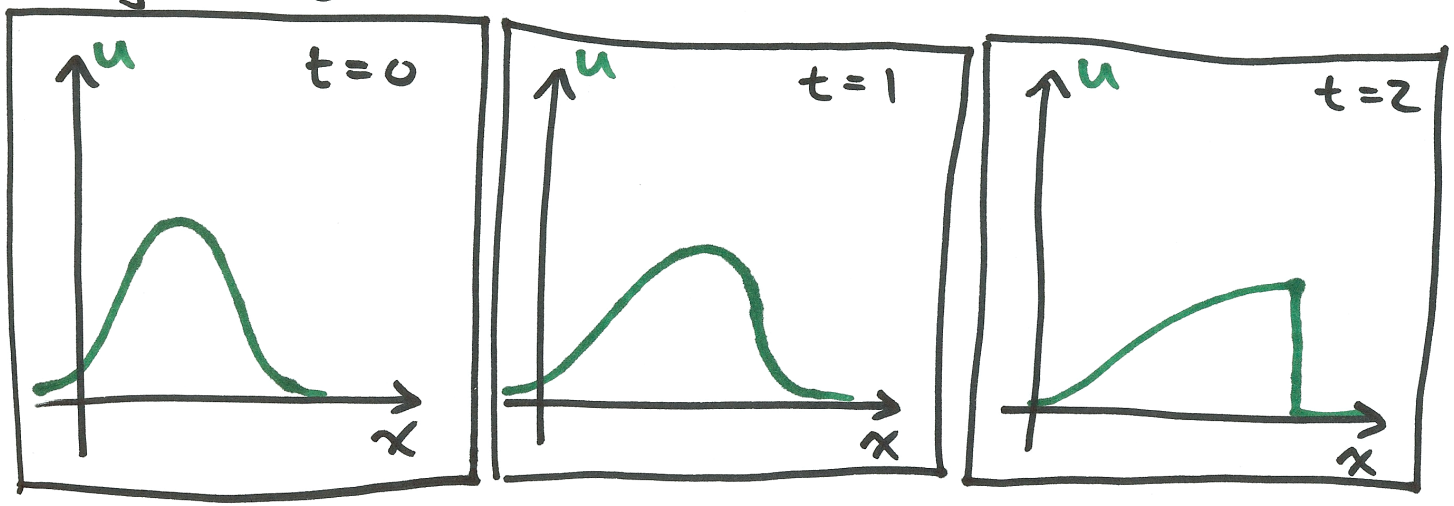
$$\partial_t u = -c(u) \partial_x u.$$

where $c(u) = -u$.

That is: where u is large, the speed is large. where u is small, the speed is small.

The top of the wave goes faster than the bottom.

BurgerTime



Though this is harder to quantify, if the nonlinear effects are dominant, then one would expect waves generated in the middle of the ocean to "break" long before they hit shore.

A disastrous balance

In the KdV equation,

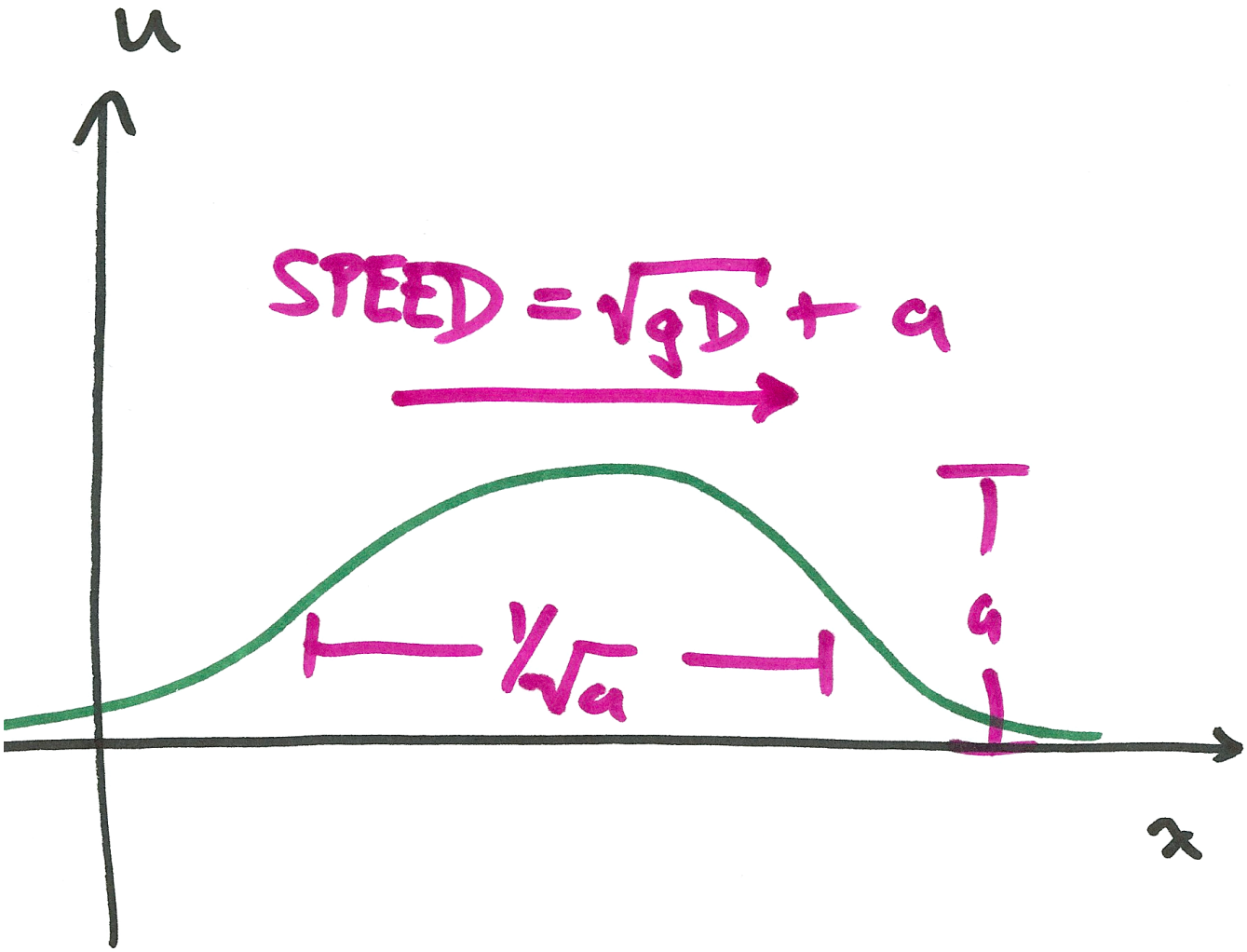
$$\partial_t u = -c_0 \partial_x u + \alpha \partial_x^3 u + \beta u \partial_x u$$

the dispersive effects and nonlinear effects are, in some sense, perfectly matched. Some waves neither break, nor disperse.

There is a special solution of this equation called a *SOLITON*:

$$u(x,t) = a\,\mathrm{sech}^2\left(\sqrt{a}(x - (c_0 + a)t)\right)$$

$$a = AMPLITUDE > 0$$


u
SPEED = √gD + a
1/√a
a
x

Observations:

$$u(x,t) = \frac{1}{2} a \operatorname{sech}^2 \left(\frac{\sqrt{a}}{2} (x - (c_0 + a)t) \right)$$

1. Higher waves go faster.

2. Higher waves are narrower.

3. This wave NEVER changes its shape, no matter how far it travels.

Take-away message

Dispersive and nonlinear effects balance in the equations of motion for surface water waves. The result is a class of wave which never changes shape and can travel enormous distances: **the tsunami**.

Final remarks

The KdV equation is only an approximate model for water waves, but huge progress has been made in the late 20th century in (a) rigorously connecting KdV to water waves (eg Schneider and Wayne 2000), (b) showing that solitary waves exist in water waves (eg Beale 1979) and (c) understanding the interaction of the wave with the coast (too many to list.)

Final remarks

The KdV equation itself turns out to be a MATHEMATICALLY remarkable equation. The initial value problem can solved for basically any initial data explicitly by use of something called "inverse scattering." The method connects it seemingly disparate branches of mathematics like random matrix theory and algebraic geometry.

Final remarks

In addition to water waves, many interesting physical phenomena are modeled by nonlinear dispersive PDE! So are:

(a) Light pulses in fiber optics.

(b) Sound waves in the air.

(c) Vibrations in atomic lattices.

(d) Plasmas in the sun.

(e) Bose-Einstein condensates.

State of the art:

Two Fields medals have been awarded to people who have substantially contributed to understanding the interplay of dispersion and nonlinearity. (Bourgain '94 and Tao '06.)

There are many nonlinear dispersive PDE and in 2011 there are many **many MANY** different ways of treating a given nonlinear dispersive problem. (4132 Matches of Korteweg de Vries in MathSciNet. 6536 for nonlinear Schrödinger. 4 for Koretweg.)

Thanks!