Initial Vlasov Distributions for Simulation of Charged Particle Beams with High Space-Charge Intensity*

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Work carried out in collaboration with:

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Taksahi Kikuchi, Utsunomiya University, Japan

Subset of material presented is contained in an extensive review article recently submitted:

Lund, Kikuchi, and Davidson, "Generation of initial Vlasov distributions for simulation of charged particle beams with high space-charge intensity," *Phys. Rev. Special Topics – Accelerators and Beams*, submitted 2007.

Acknowledge help from:

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Jean-Luc Vay LBNL

in support of the WARP code used in beam simulations

Overview:

Concept of a plasma equilibrium proves highly useful:

- Simpler characterization of waves and instabilities

For beams with space-charge strength comparable to applied focusing, the plasma equilibrium concept can prove complicated:

- Finite system size
- Applied focus (analog of neutralizing species) periodically oscillates
- Beams typically formed far from expected equilibrium form

Desire equilibrium (or near-equilibrium) Vlasov “ansatz” type distributions to analyze accelerator beams with intense space-charge:

- Improve specifications over present applied in simulations

Are system equilibria necessary for reliable beam transport?

First look at a simple beam system where plasma physics can be applied for guidance -- continuous focusing.

Transverse Vlasov-Poisson model for a coasting, single species beam with electrostatic self-fields propagating in a uniform, axisymmetric focusing field

$\mathbf{x}_\perp, \ \mathbf{x}'_\perp$	transverse particle coordinate, angle
$q, \ m$	charge, mass
$\gamma_b, \ \beta_b$	axial relativistic factors
$f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$	single particle distribution
$H_\perp(\mathbf{x}_\perp, \mathbf{x}_\perp, s)$	single particle Hamiltonian

Vlasov Equation:

$$\frac{d}{ds} f_\perp = \left\{ \frac{\partial}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \cdot \frac{\partial}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp} \cdot \frac{\partial}{\partial \mathbf{x}'_\perp} \right\} f_\perp = 0$$

Particle (Characteristic) Equations of Motion:

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \qquad \frac{d}{ds}\mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

Hamiltonian:

$$H_\perp = \frac{1}{2}\mathbf{x}'^{\,2}_\perp + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2 c^2}\phi \qquad k_{\beta 0}^2 = \text{const}$$

Poisson Equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = -\frac{q}{\epsilon_0} \int d^2\mathbf{x}'_\perp \ f_\perp$$

+ boundary conditions on ϕ

Continuous focusing equilibria:

The applied focusing force is constant in s:

$$\mathbf{F}_\perp \propto k_{\beta 0}^2 \mathbf{x}_\perp \qquad k_{\beta 0}^2 = \text{const}$$

No energy pumped into or out of the beam:

$$\Rightarrow \quad H_\perp = \text{const}$$

Therefore any $f_\perp(H_\perp)$ with $f_\perp(H_\perp) \geq 0$ produces a valid stationary $(\partial/\partial s = 0)$ equilibrium.

Concavity arguments show that

$$\frac{\partial f_\perp(H_\perp)}{\partial H_\perp} \leq 0$$

Fowler, Gardner;
extended by: Davidson, PRL **81**, 991 (1998)

is a sufficient condition for equilibrium stability.

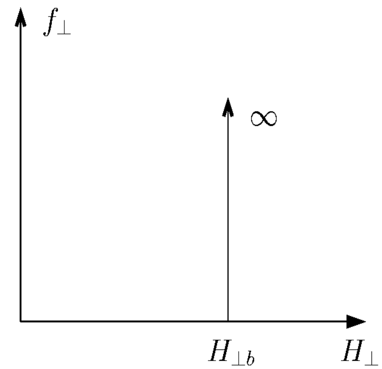
Nonlinear equilibrium equations must be solved for specific $f_\perp(H_\perp)$

- Range of behavior can be understood from a few simple choices

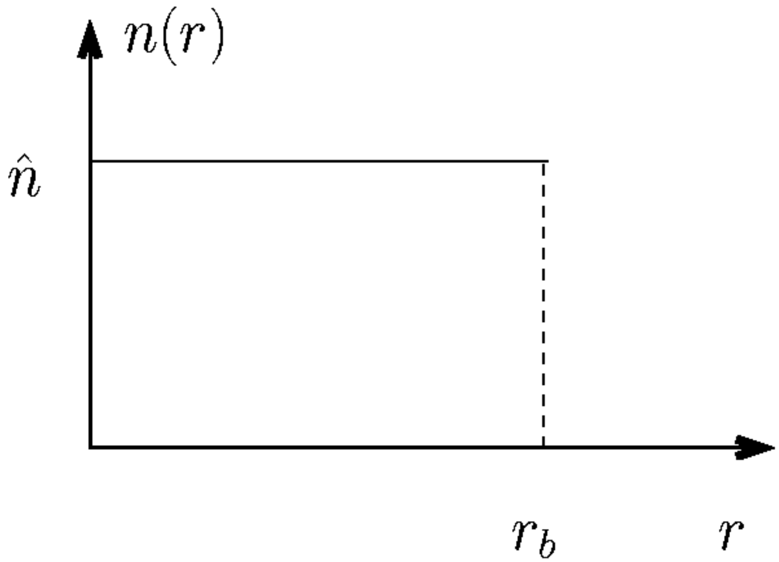
KV (Kapchinkskij-Vladimirskij) Equilibrium [can be unstable]

$$\boxed{f_\perp \propto \delta(H_\perp - H_{\perp b})}$$

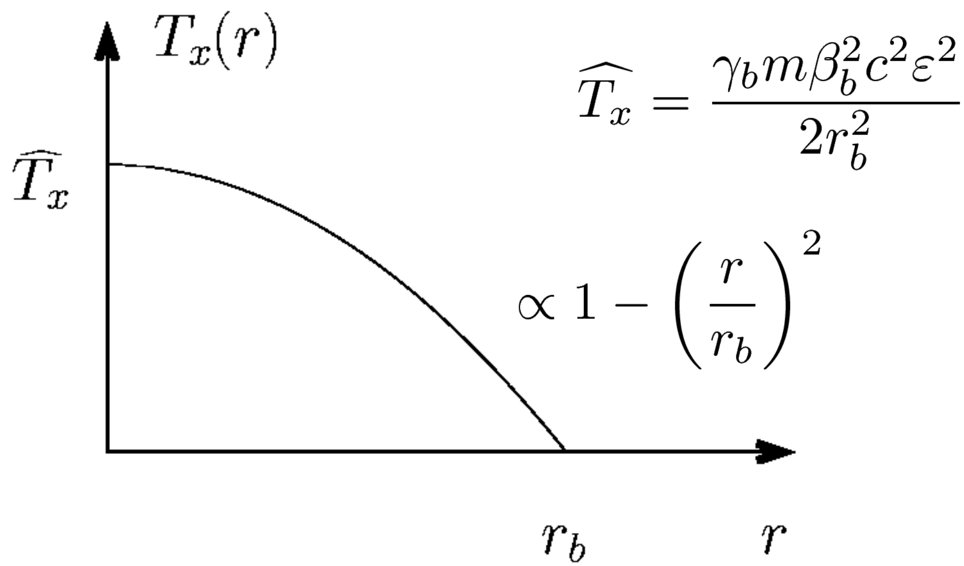
$$H_{\perp b} = \text{const}$$



Density - Uniform



Temperature - Parabolic

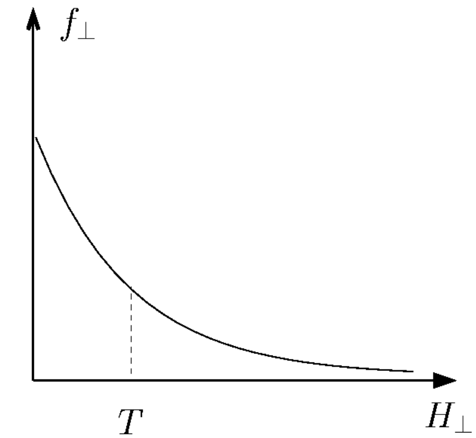


- Analytical solution with all linear orbits with same oscillation frequency
- No change in equilibrium form with varying parameters

Thermal Equilibrium [stable]

$$\boxed{f_\perp \propto \exp(-H_\perp/T)}$$

$$T = \text{const} > 0$$

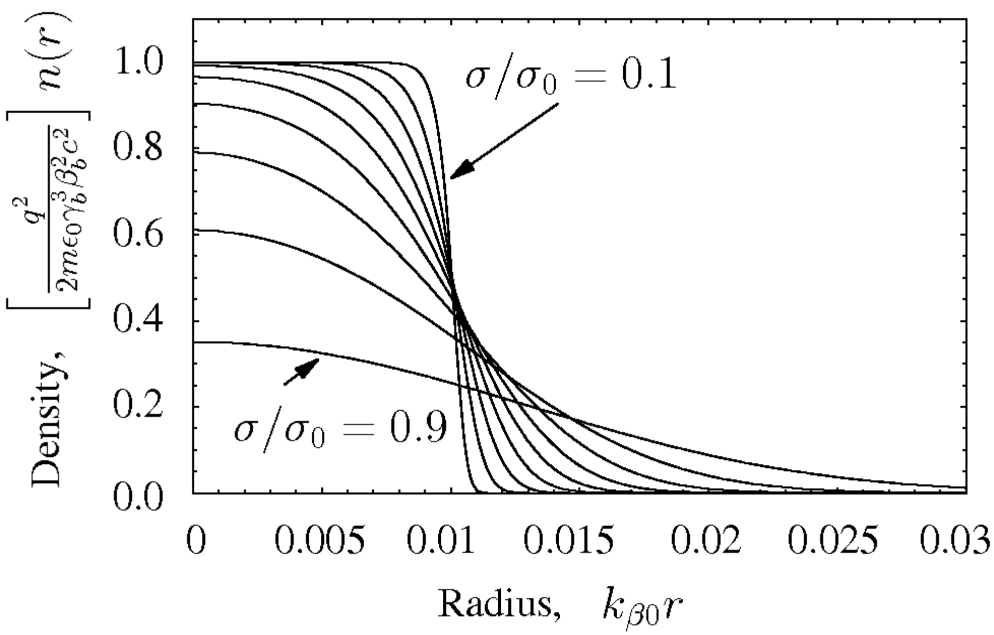


See:

Davidson, *Noneutral Plasmas*, 1990

Davidson and Qin, *Intense Charge Particle Beams*, 2001

Density



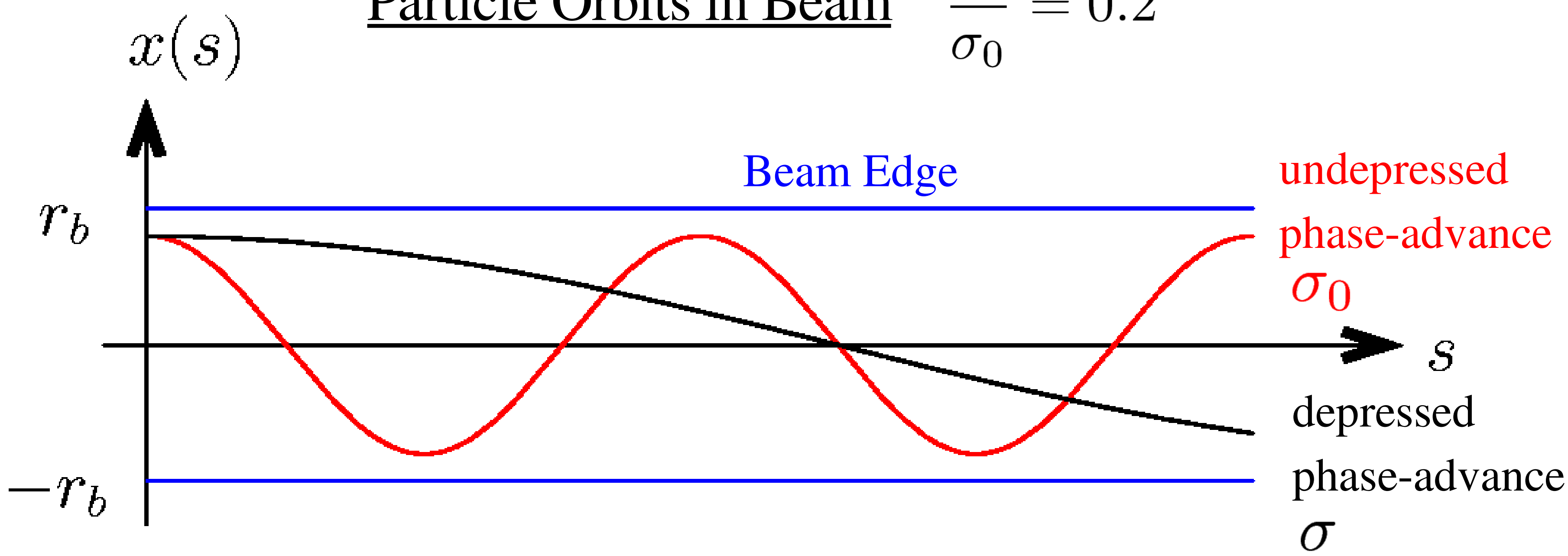
Temperature

$$T_x = \frac{\int d^2x'_\perp \; x'^2 f_\perp}{\int d^2x'_\perp \; f_\perp}$$

$$= \text{const}$$

To measure the relative strength of space-charge to applied focusing forces we employ a convenient, normalized parameter σ/σ_0

<u>Particle Orbits in Beam</u> $\frac{\sigma}{\sigma_0} = 0.2$

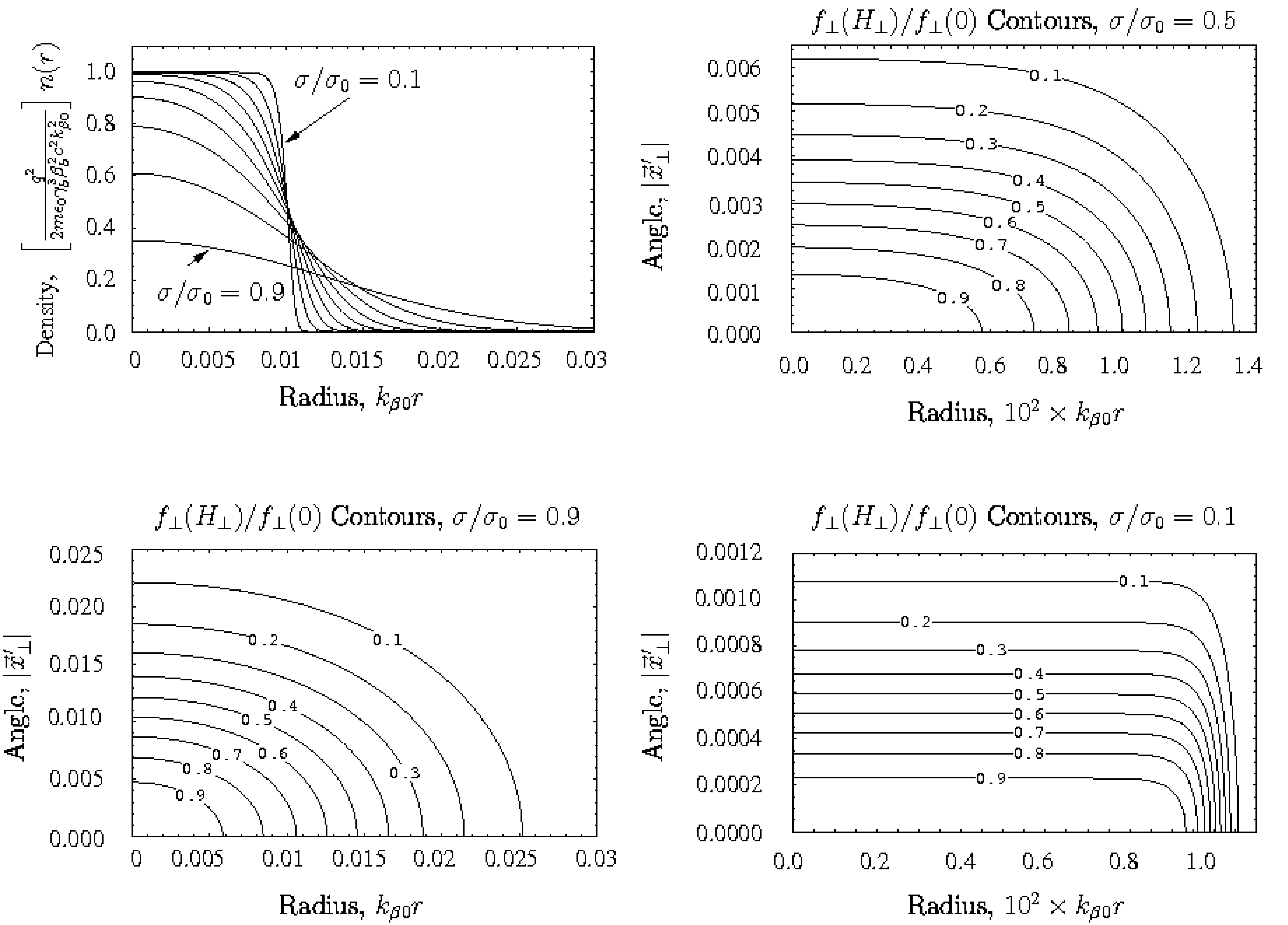


$$0 \leq \frac{\sigma}{\sigma_0} \leq 1$$

Max Space-Charge Intensity

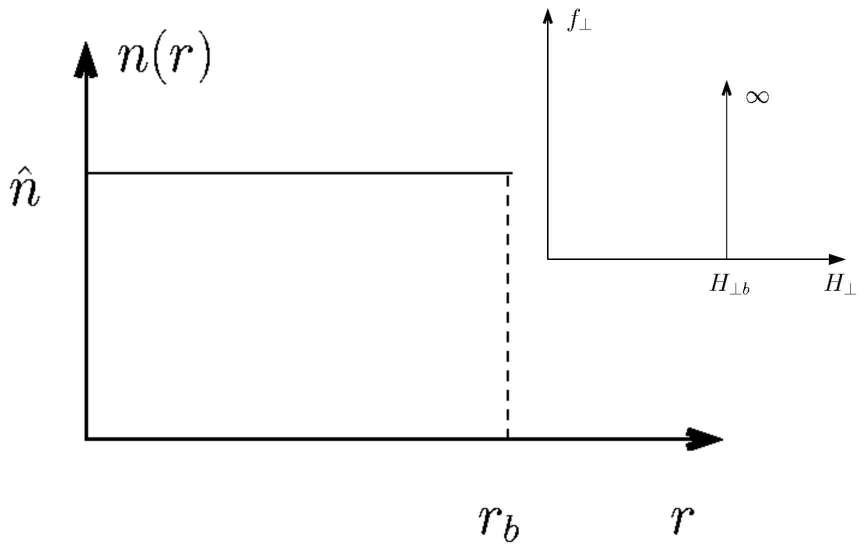
No Space-Charge Intensity

Thermal Equilibrium – Phase-Space Structure

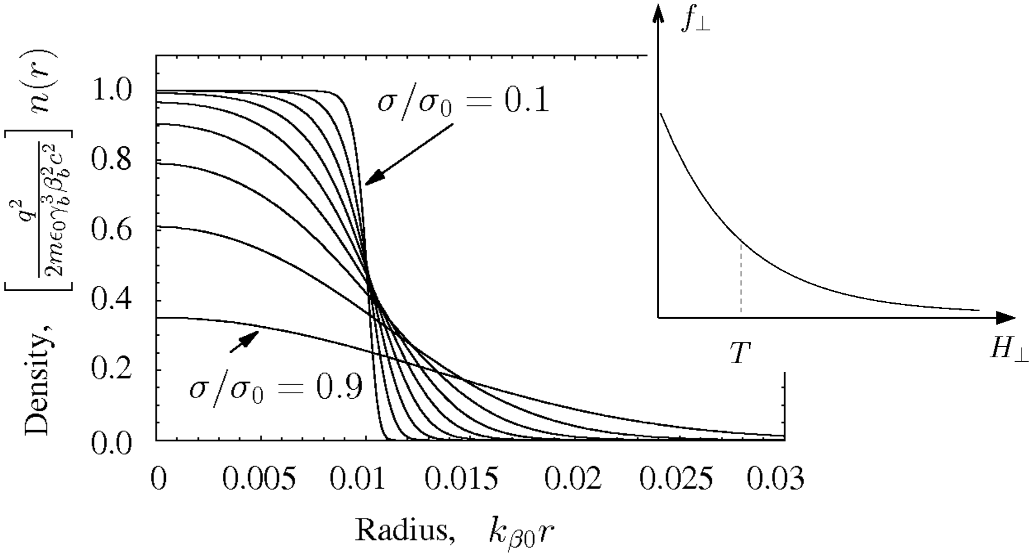


Comparison of density profiles of continuous focusing equilibria

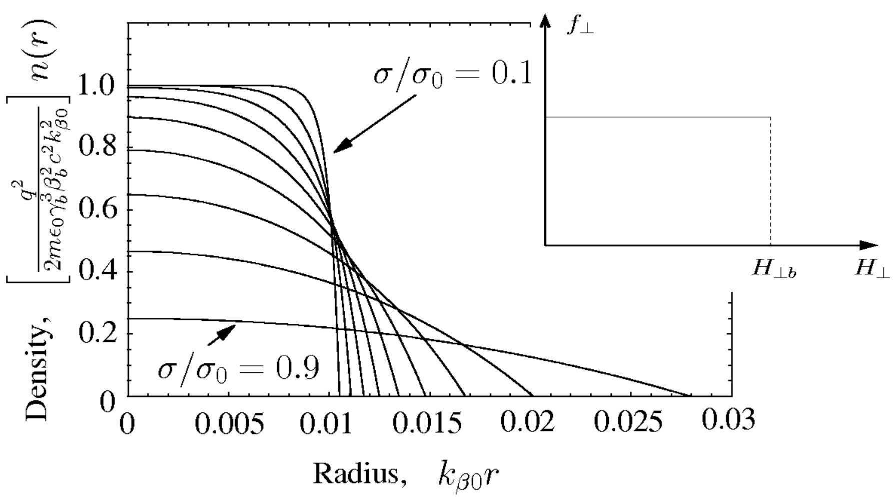
KV: $f_\perp \propto \delta(H_\perp - H_{\perp b})$



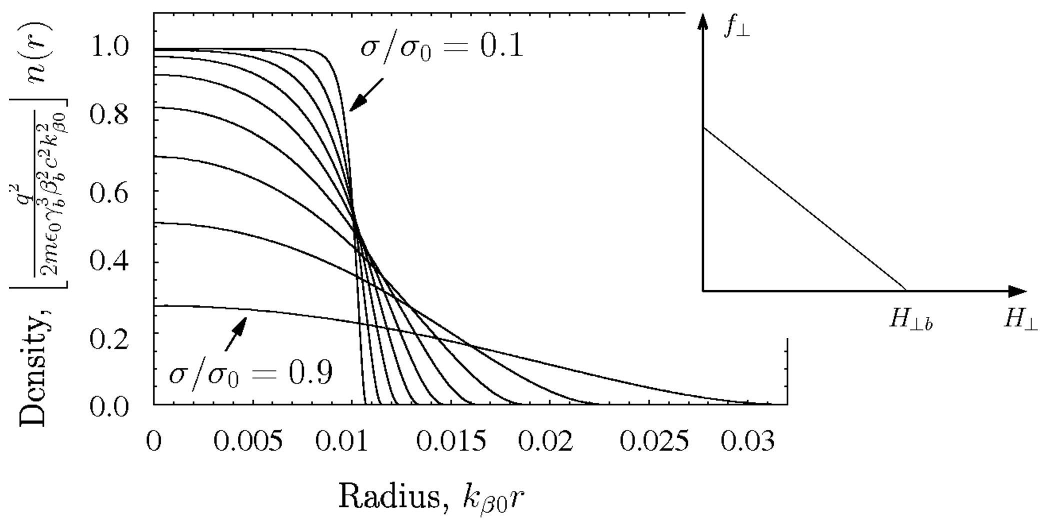
Thermal: $f_\perp \propto \exp(-H_\perp/T)$



Waterbag: $f_\perp \propto \Theta(H_{\perp b} - H_\perp)$



Parabolic: $f_\perp \propto (H_{\perp b} - H_\perp)\Theta(H_{\perp b} - H_\perp)$



Ron Davidson's work in this area (subset!)

Summary Books:

Davidson and Qin, Intense Charge Particle Beams, 2001

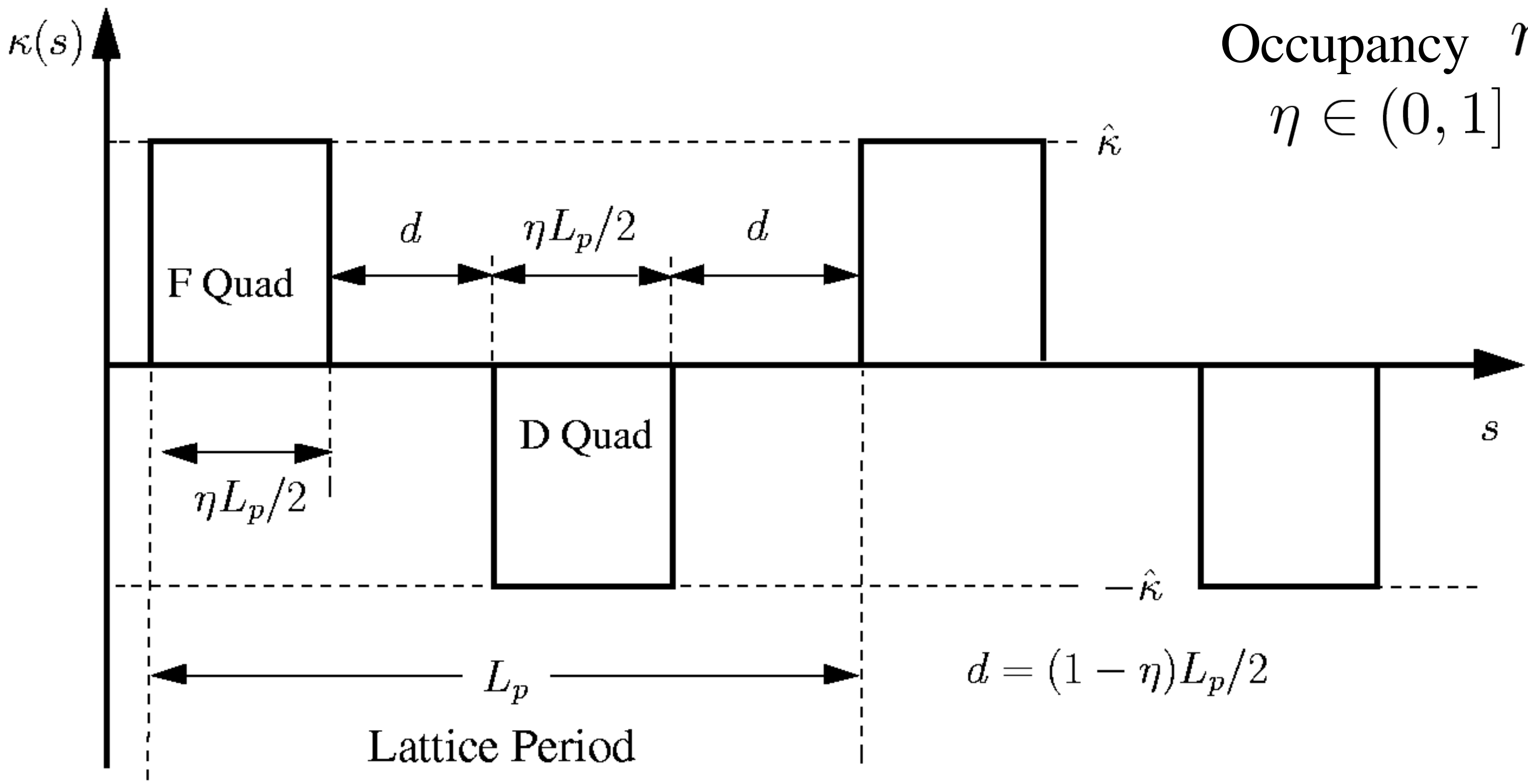
Davidson, *Physics of Noneutral Plasmas*, 1990

Davidson, Theory *of Noneutral Plasmas*, 1974

Papers

Davidson and Krall, PRL **22**, 839 (1969)	Thermal Equilibrium
Davidson and Krall, Phys. Fluids **12**, 1543 (1970)	Thermal Equilibrium
Davidson, J. Plasma Phys. **6**, 229 (1971)	Debye screening
Davidson, Phys. Rev. Lett. **81**, 991 (1998)	Stability Properties
Davidson, Phys. Plasmas **5**, 3459 (1998)	Stability Properties
Lund and Davidson, Phys. Plasmas **5**, 3028 (1998)	KV Interpretation + Stability
Davidson and Qin, PRSTAB **2**, 114401 (1999)	Thermal Equilibrium
Davidson, Qin, and Lund, PRSTAB 6, 024402 (2003)	Thermal Equilibrium

Real beam focusing systems are typically far from continuous
Example: Periodic quadrupole focusing lattice, piecewise constant



Occupancy η

$\eta \in (0, 1]$

Periodic or s-varying focusing pumps energy into and out of the beam:

- $H_\perp \neq \text{const}$
- Can smooth, physically appealing equilibria be constructed?

Hamiltonians:

Continuous focusing $k_{\beta 0}^2 = \text{const}$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3 \beta_b^2 c^2}\phi$$

Quadrupole focusing $\kappa_q(s) \neq \text{const}$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}\kappa_q(s)x^2 - \frac{1}{2}\kappa_q(s)y^2 + \frac{q}{m\gamma_b^3 \beta_b^2 c^2}\phi$$

To measure the strength of the applied focusing, we employ a parameter σ_0 that measures the phase-advance of oscillations of a single-particle in the lattice

$$0 \quad \leq \quad \sigma_0 \quad \leq \quad 180^\circ$$

Zero Applied Focus Strength

Max Applied Focus Strength

- Higher order stability requires for high space-charge intensity $(\sigma/\sigma_0 \ \text{small})$

$$\sigma_0 \lesssim 80^\circ$$

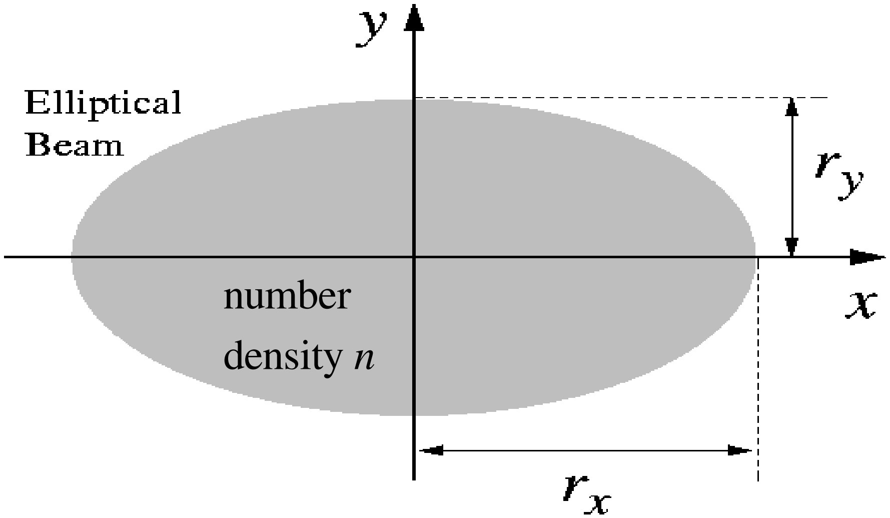
Lund and Chawla, NIMA **561**, 203 (2006)
Lund, Barnard, Bukh, Chawla, and Chilton, NIMA **577**, 173 (2007)

- Achieving good transport harder with increasing σ_0
  - Hard driven system expected to be more problematic

Invariant to apply for periodic systems with strong space-charge

[Kapchinskij and Vladimirskij, Proc. Int. Conf. On High Energy Accel., 1959]

Assume a uniform density elliptical beam in a periodic focusing lattice



Line-Charge:

$$\lambda = qn(s)\pi r_x(s) r_y(s)$$
$$= \text{const}$$

Perveance:

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2 c^2}$$
$$= \text{const}$$

Particle equations of motion within the beam (linear equation for known edge radii):

$$\boxed{\begin{aligned} x''(s) + \left\{\kappa_x(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)}\right\} x(s) = 0 \\ y''(s) + \left\{\kappa_y(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_y(s)}\right\} y(s) = 0 \end{aligned}}$$

Particle orbits have quadratic Courant-Snyder invariants:

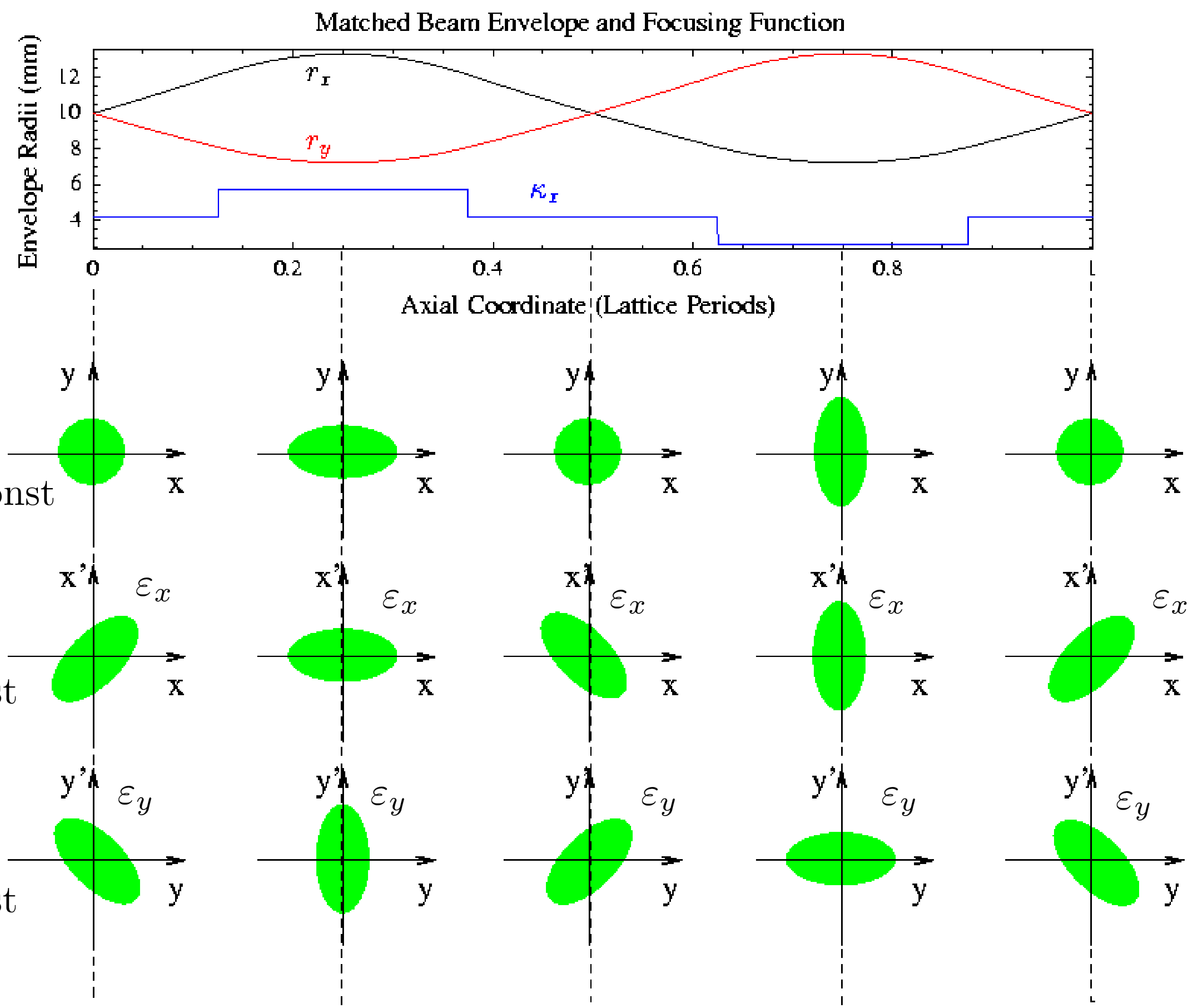
$$\boxed{\begin{aligned} \left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 = C_x = \text{const} \\ \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r_y' y}{\varepsilon_y}\right)^2 = C_y = \text{const} \end{aligned}}$$

Kapchinskij and Vladimirskij constructed a delta-function distribution of a linear combination of these Courant-Snyder invariants that generates the correct uniform density elliptical beam needed for consistency with the assumptions:

$$\boxed{f_\perp = \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta\left[C_x + C_y - 1\right]}$$

- Other forms cannot generate the required uniform density elliptical beam

The ellipticity of the KV equilibrium projections evolve in s. Initial conditions that yield periodic evolution gives the "matched" beam.



Projection

x-y

area: $\pi r_x r_y \neq$ const

x-x'

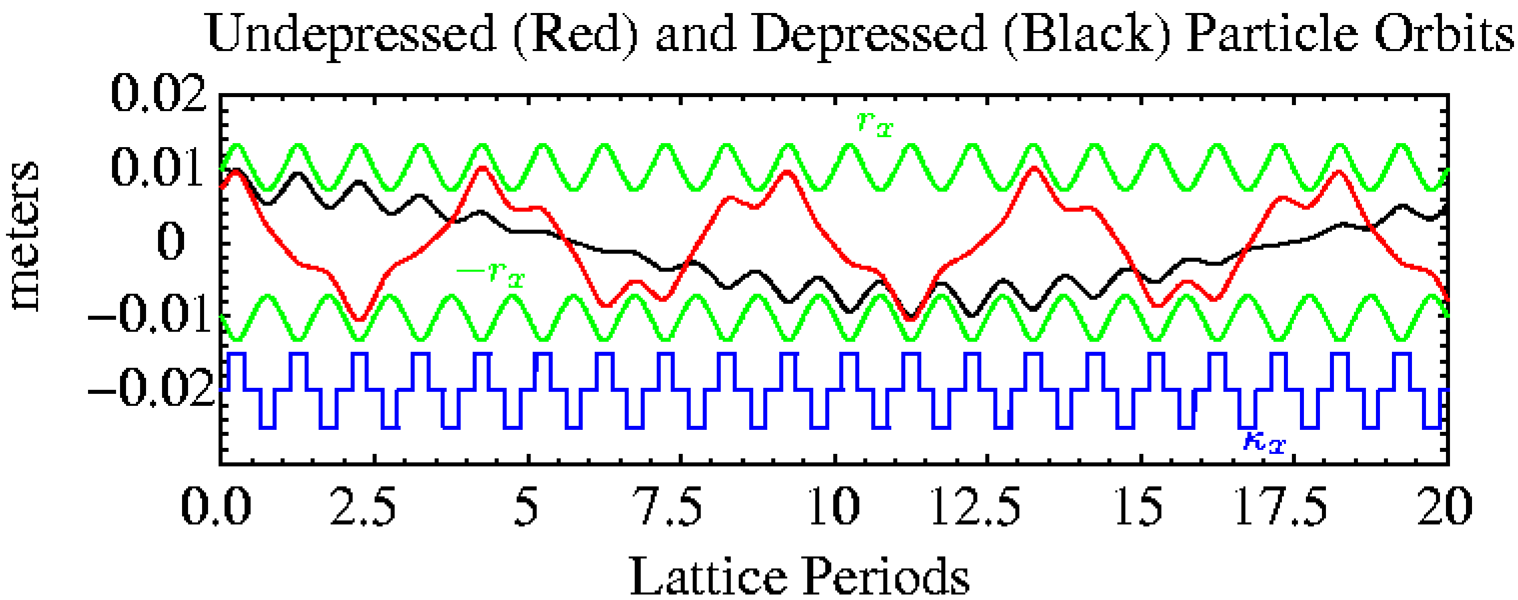
area: $\pi \varepsilon_x =$ const
(CS Invariant)

y-y'

area: $\pi \varepsilon_y =$ const
(CS Invariant)

Particle orbits within a matched KV beam are complicated

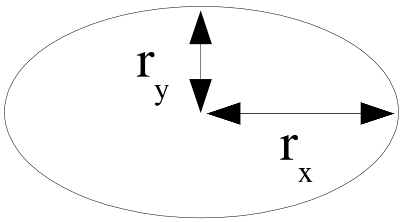
x-plane Particle Orbit



Can a smooth distribution equilibrium be constructed in a periodic focusing lattice with linear forces?

Temperature flutters with lattice periodicity

Elliptical Beam



$$\varepsilon_x^2 \propto T_x r_x^2 \simeq \text{const} \implies T_x \propto \frac{1}{r_x^2}$$

Characteristic Plasma Frequency of Collective Effects

Simple estimate:

$$\sigma_{\text{plasma}} \sim \frac{L_p}{r_b}\sqrt{2Q}$$

$$\text{Typical: } \sigma_{\text{plasma}} \sim 105^\circ/\text{period}$$

Unclear whether such processes conspire to make smooth equilibrium possible

- No exact equilibrium constructed other than singular KV in ~ 50 years

Try applying smooth continuous focusing results and Courant-Snyder invariant forms to generate better classes of distributions to apply in simulations

Construct a simple, approx. equilibrium-like "pseudo-equilibrium" distribution

Assume focusing lattice is given:

$$\kappa_x(s), \quad \kappa_y(s) \qquad \text{specified}$$

Step 1:

For each particle (2D or 3D beam) specify 2nd order moments at axial coordinate *s*

Envelope coordinates/angles:

$$r_x(s) = 2\langle x^2\rangle_\perp^{1/2} \qquad r'_x(s) = 2\langle xx'\rangle_\perp / \langle x^2\rangle_\perp^{1/2}$$

$$r_y(s) = 2\langle y^2\rangle_\perp^{1/2} \qquad r'_y(s) = 2\langle yy'\rangle_\perp / \langle y^2\rangle_\perp^{1/2}$$

Emittance:

$$\varepsilon_x(s) = 4[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2]^{1/2}$$

$$\varepsilon_y(s) = 4[\langle y^2\rangle_\perp \langle y'^2\rangle_\perp - \langle yy'\rangle_\perp^2]^{1/2}$$

Perveance:

$$Q = \frac{q\lambda(s)}{2\pi\epsilon_0 m\gamma_b^3(s)\beta_b^2(s)c^2}$$

Step 2:

Define an rms matched continuous focusing equilibrium beam for each particle:

$$\boxed{\begin{aligned} r_b(s) &= \sqrt{r_x(s) r_y(s)} \\ \varepsilon_b(s) &= \sqrt{\varepsilon_x(s)\varepsilon_y(s)} \\ k_{\beta 0}^2 &= \frac{Q(s)}{r_b^2(s)} + \frac{\varepsilon_b^2(s)}{r_b^4(s)} \end{aligned}}$$

Load a stable continuous focusing distribution $f_\perp(H_\perp)$ with $\partial f_\perp(H_\perp)/\partial H_\perp \leq 0$

- Equations generally highly nonlinear and must be solved numerically
  - Setup to specify with natural variables .... can be difficult

Step 3:

Transform coordinates loaded:

$$\boxed{\begin{aligned} x_f &= \frac{r_x}{r_b} x_i \\ x_f' &= \frac{\varepsilon_x}{\varepsilon_b}\frac{r_b}{r_x} x_i' + \frac{r_x'}{r_b} x_i \end{aligned}}$$

- Preserves linear force Courant-Snyder invariant

Procedure will apply to any s-varying focusing channel

- Focusing channel need not be periodic
- Beam can be initially rms envelope matched *or* mismatched if launched in a periodic transport channel
- Can apply to both 2D transverse and 3D beams

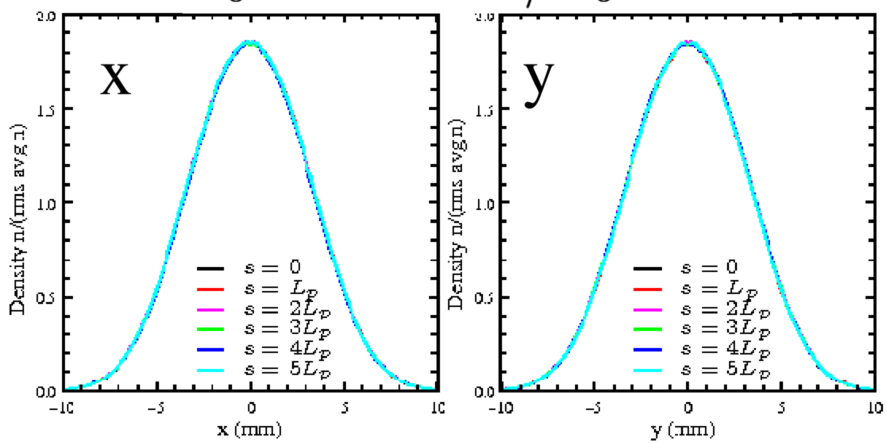
Procedure is approximate

- Transform is for linear fields
- Beam edge expected to manifest an error that will launch waves

Transient evolution of initial pseudo-equilibrium distributions with thermal core form in a FODO quadrupole focusing lattice

- Density profiles along x and y axes
- Snapshots at lattice period intervals over 5 periods

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.9$



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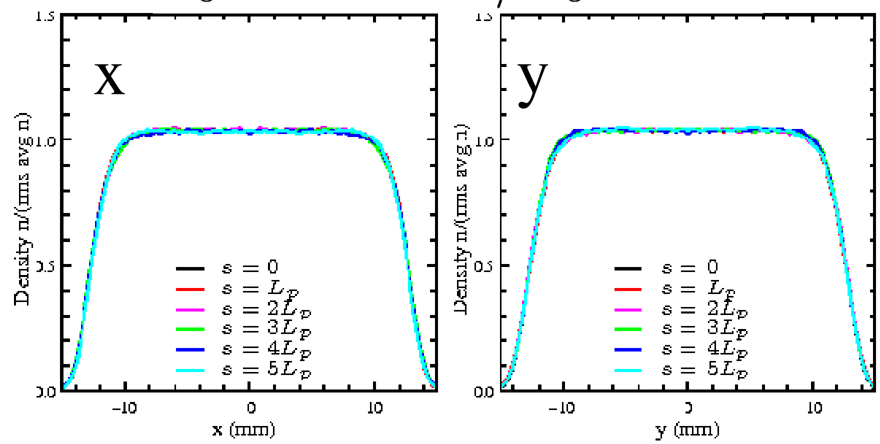
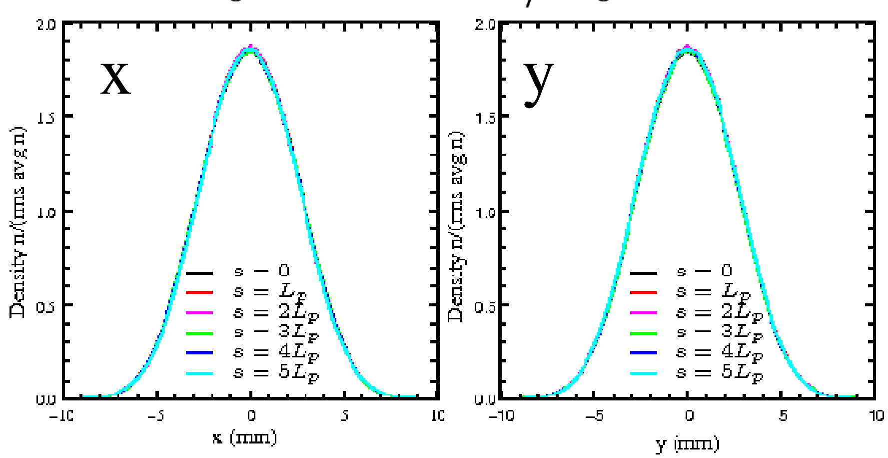


$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.9$



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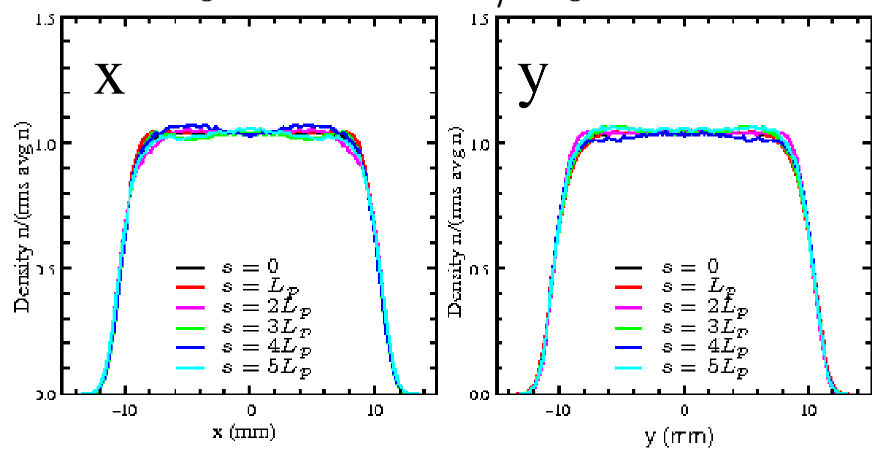
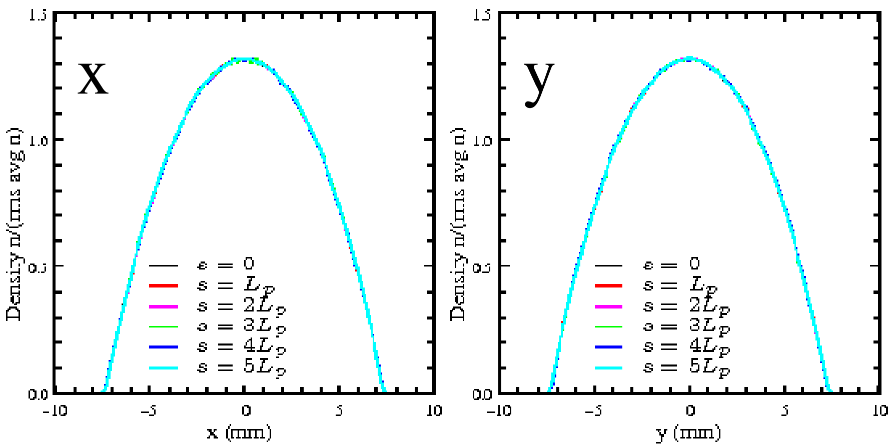


Transient evolution of initial pseudo-equilibrium distributions with waterbag core form in a FODO quadrupole focusing lattice

- Density profiles along x and y axes
- Snapshots at lattice period intervals over 5 periods

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.9$



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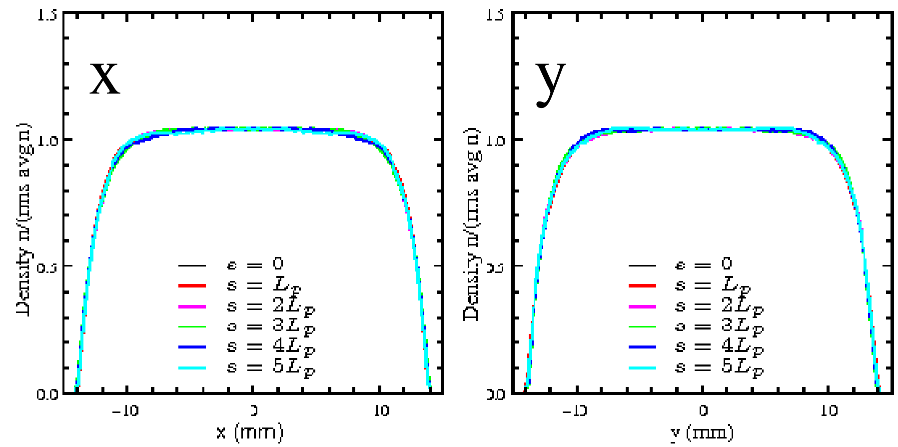

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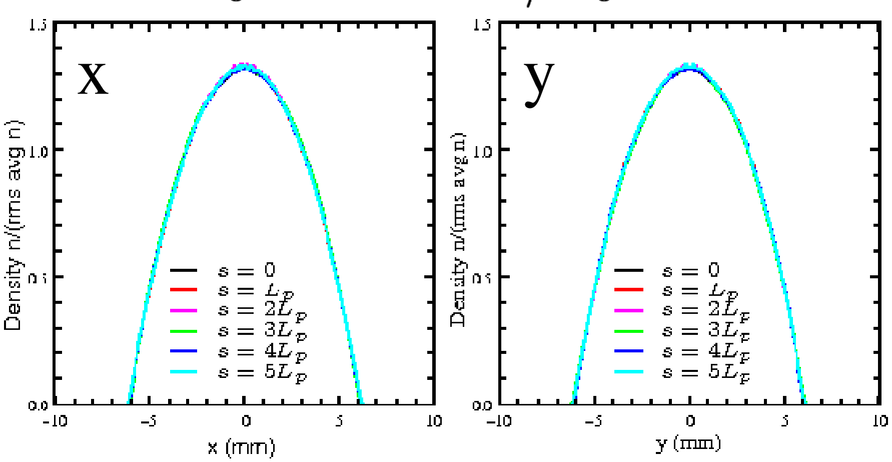

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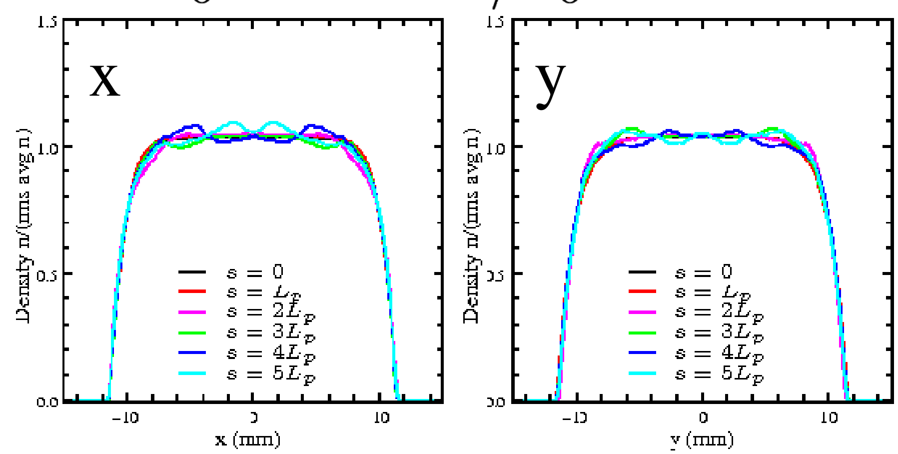


Compare pseudo-equilibrium loads with other accelerator loads

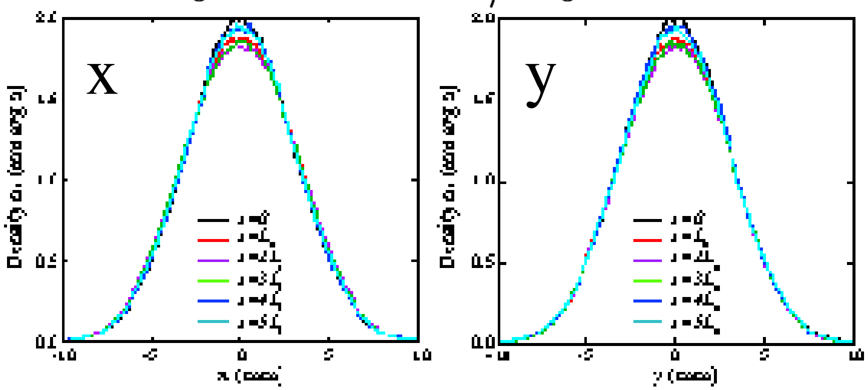
- Comparison distribution from linear-field Courant-Snyder invariants

  Batygin, Nuc. Inst. Meth. A **539**, 455 (2005)

- Thermal/Gaussian forms with weak space-charge

Linear-Field Courant-Snyder:

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.9$



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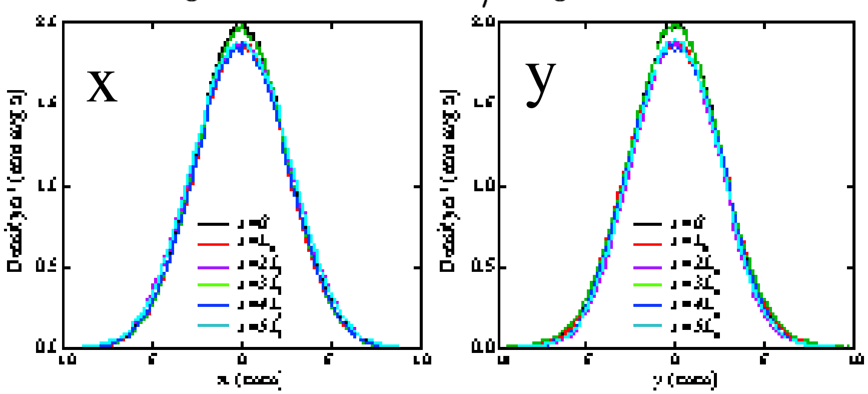
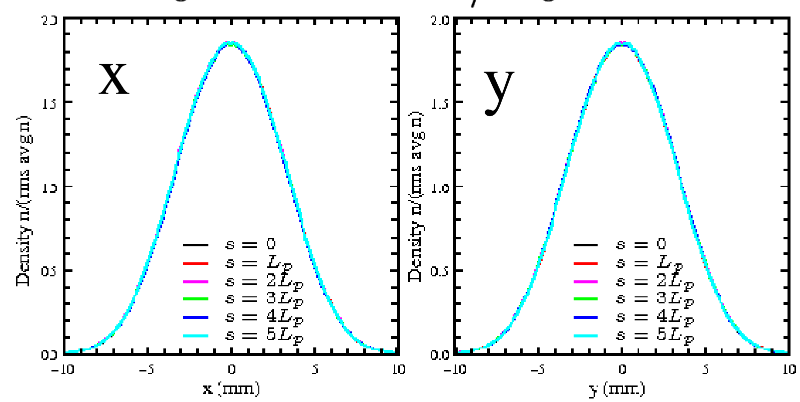


Pseudo-Equilibrium

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.9$



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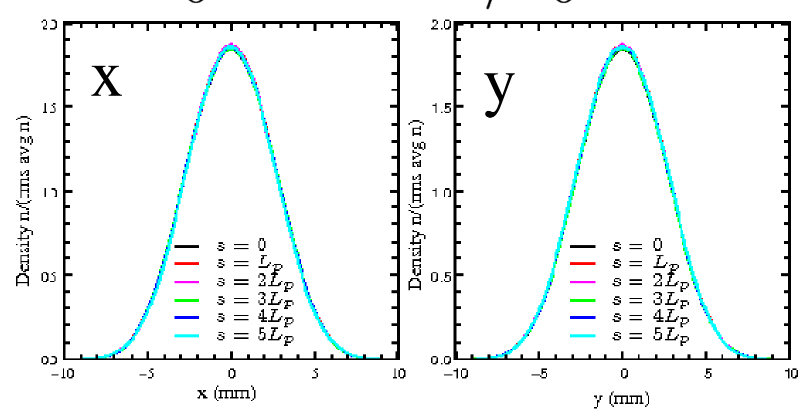

Compare pseudo-equilibrium loads with other accelerator loads

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Linear-Field Courant-Snyder:

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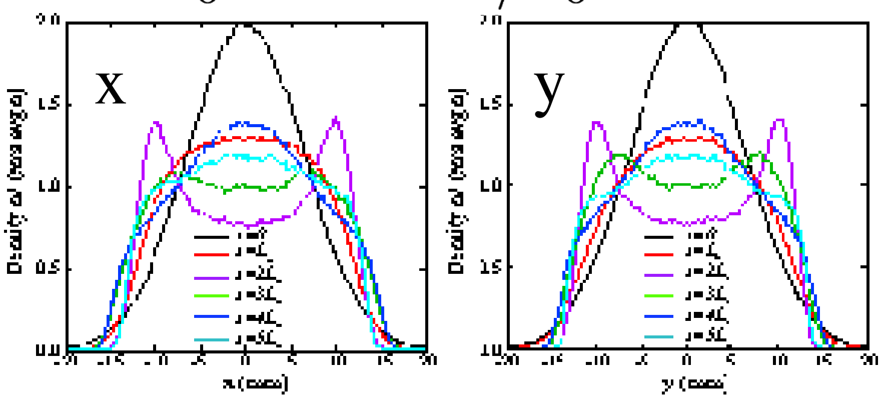

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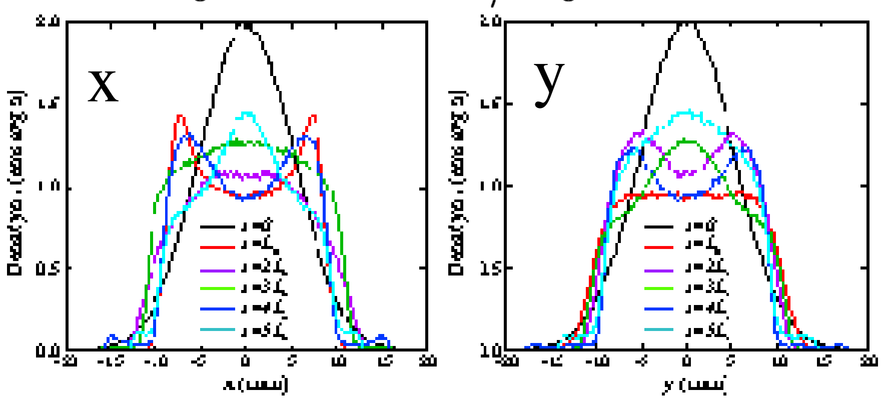
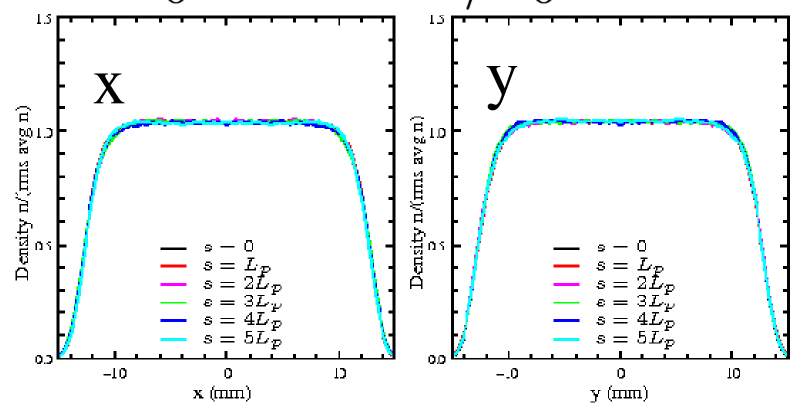
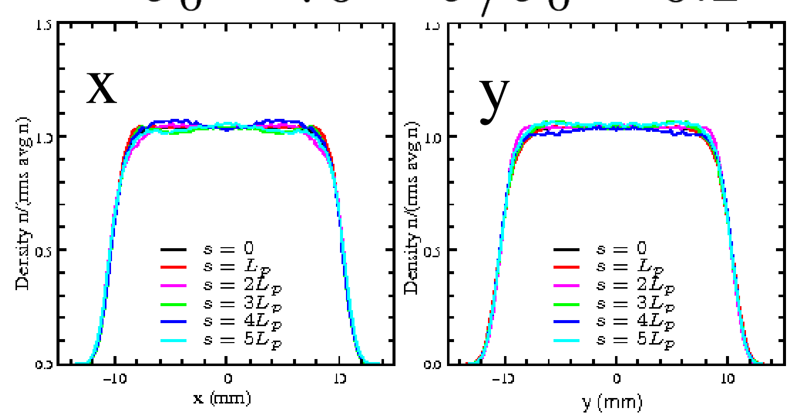


Pseudo-Equilibrium

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.2$



$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.2$



Conclusions

Standard ansatz-type initial distributions employed in codes include:

- KV
- Semi-Gaussian
- Fixed form functions of linear-field Courant-Snyder invariants

can result, when space-charge is strong, in beams that are poorly adapted to linear transport lattices with s-varying focusing forces.

- Large transient waves launched
- Complicates identifying problematic system physics from characteristics of poor initial conditions

A simple, procedure generates "pseudo-equilibrium" distributions with more nearly equilibrium-like properties:

- Transform smooth, stable continuous-focusing distributions
- Procedure formulated in terms of standard accelerator inputs to apply directly

Current Emittance rm Envelope

Review Submitted: Lund, Kikuchi, and Davidson, "Generation of initial Vlasov distributions for simulation of charged particle beams with high space-charge intensity," *Phys. Rev. Special Topics – Accelerators and Beams*, 2007.

Future Directions

Other methods exist to generate improved initial distributions:

- Relaxation methods
  - Pseudo-equilibrium distributions might provide an improved relaxation seed
- Systematic perturbation theories
- Simulating off source in short systems
- Synthesize from experimental measurements

Fundamental Question:

Do smooth equilibrium solutions exist to the Vlasov-Poisson system in periodic (or s-varying) linear focusing channels?

- Singular KV distribution is the only exact solution constructed in ~ 50 years

Ron Davidson and colleagues have employed Hamiltonian averaged perturbation methods that suggest an approach:

Davidson and Qin, *Phys. Rev. Spec. Topics – Accel. Beams* **2**, 074401 (1999)

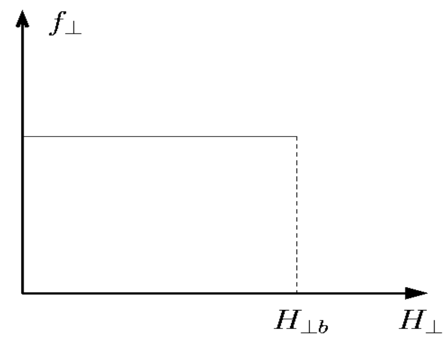
Davidson and Qin, *Physics of Intense Charged Particle Beams* (World Sci., 2001)

Should check results in simulation and apply!

Extra Slides

Waterbag Equilibrium [stable]

$$\boxed{f_\perp \propto \Theta(H_{\perp b} - H_\perp)}$$

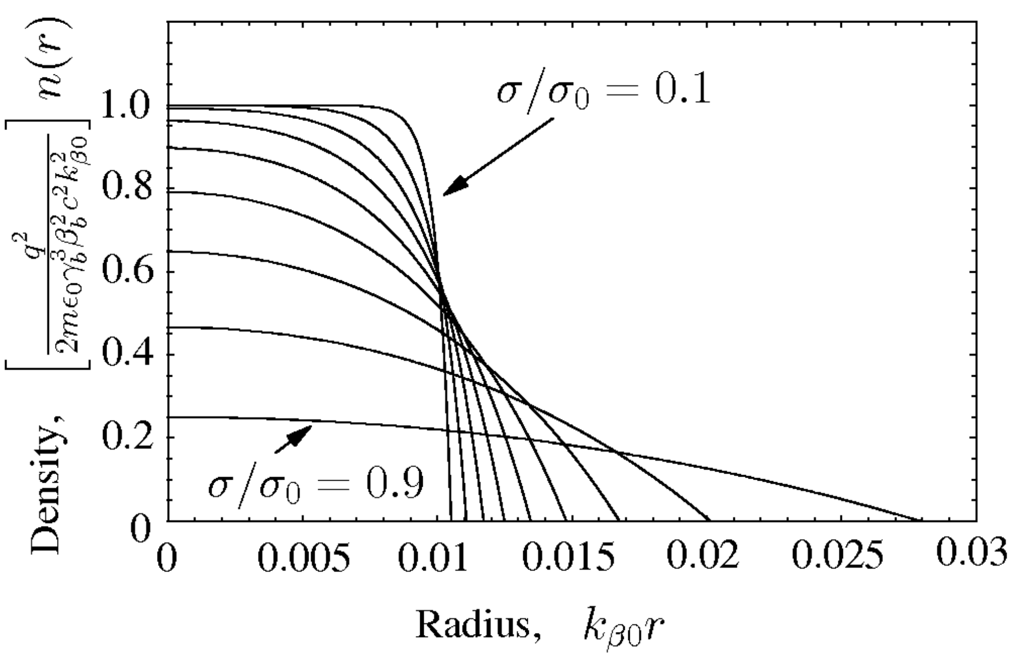


$$\Theta(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x \end{cases}$$

See:

M. Reiser, *Charged Particle Beams*, (1994)

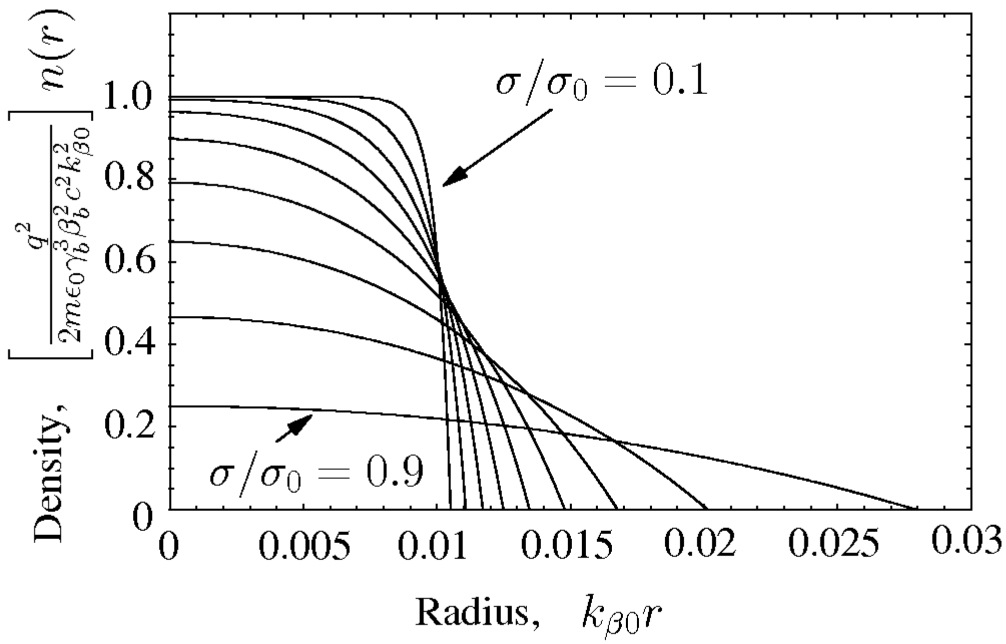
Density



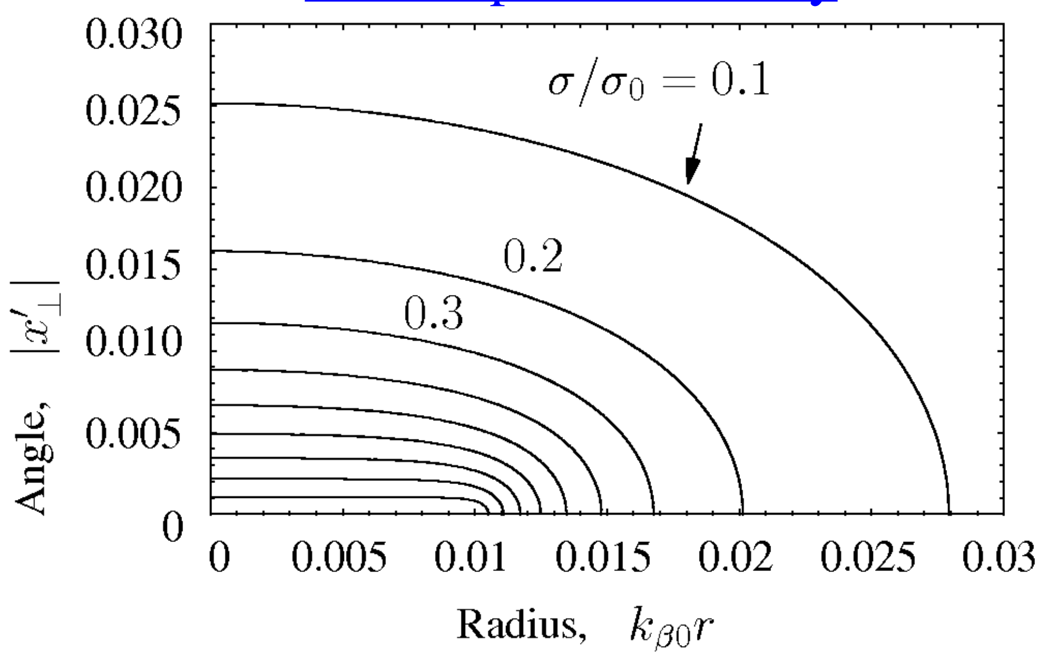
Temperature

$$T_x \propto \text{Density}$$

Density



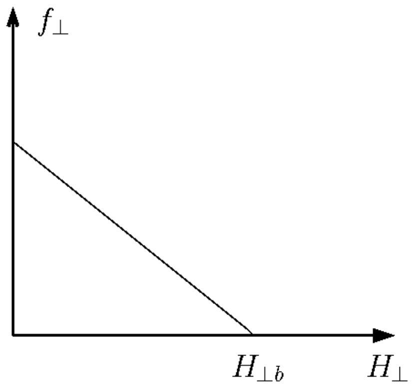
Phase-Space Boundary



Parabolic Equilibrium [stable]

$$\boxed{f_\perp \propto (H_{\perp b} - H_\perp)\Theta(H_{\perp b} - H_\perp)}$$

$$\Theta(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x \end{cases}$$

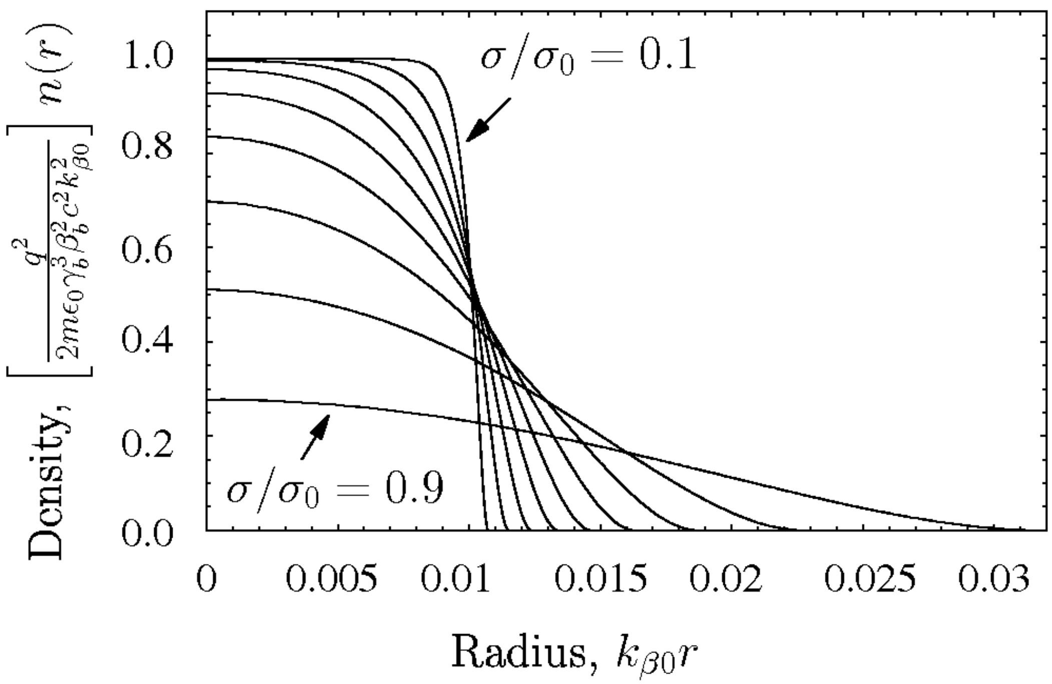


See:

Lund, Kikuchi, Davidson, submitted (2007)

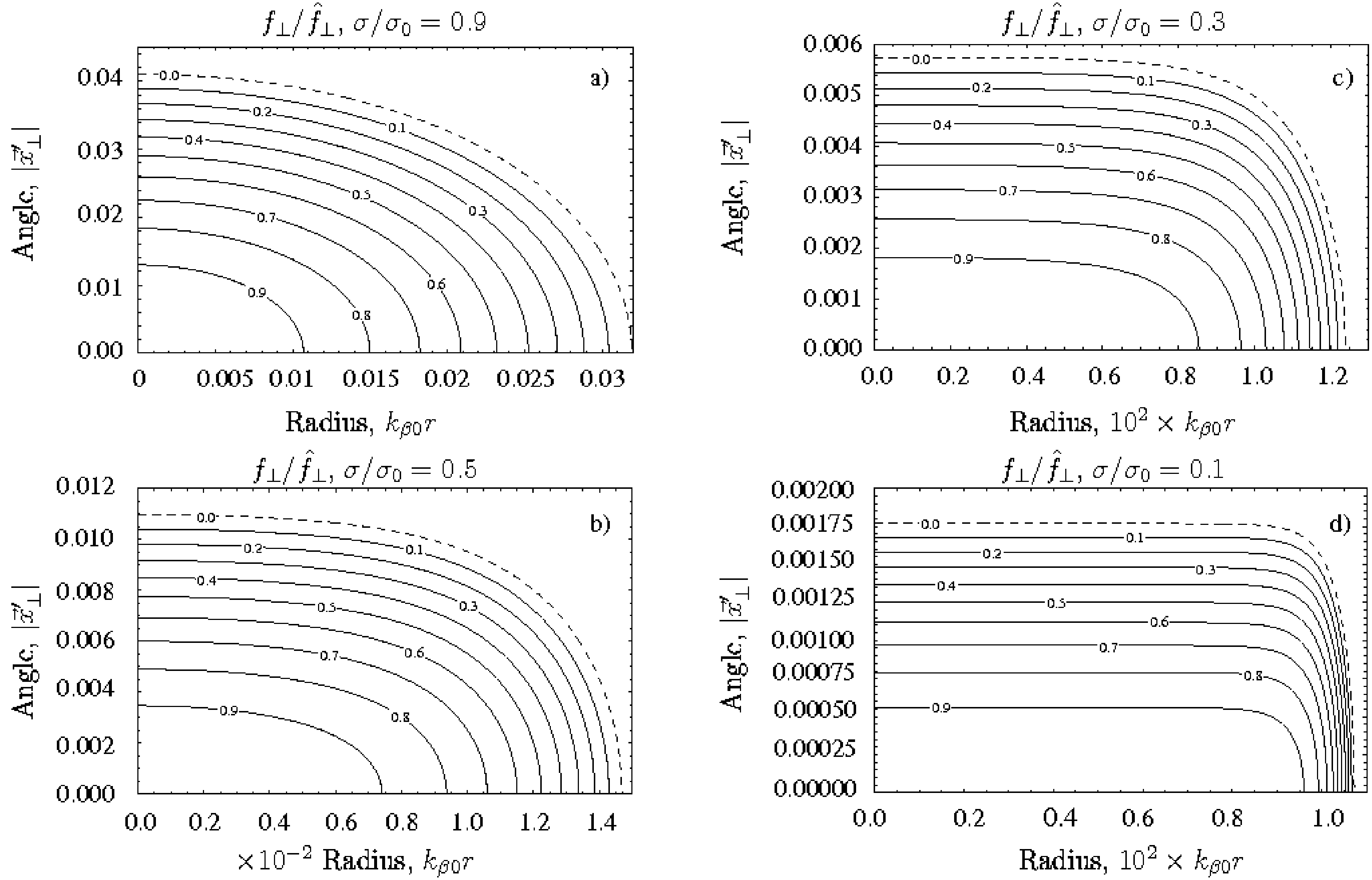
Density

Temperature



$$T_x \propto \sqrt{\text{Density}}$$

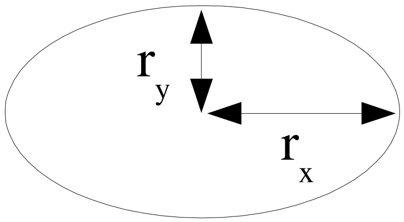
Parabolic Equilibrium – Phase-Space Structure



Large envelope flutter associated with strong focusing can result in a rapid high-order oscillating force imbalance acting on edge particles of the beam

Temperature Flutter

Elliptical rms Equivalent Beam



$$\varepsilon_x^2 \propto T_x r_x^2 \simeq \text{const} \implies T_x \propto \frac{1}{r_x^2}$$

Example Systems	$(r_{max}/r_{min})^2$
AG Trans: $\sigma_0 = 60^o$	~ 2.5
AG Trans: $\sigma_0 = 100^o$	~ 4.9
Matching Section	~ 15 Possible

Characteristic Plasma Frequency of Collective Effects

Continuous Focusing Estimate

$$\sigma_{\text{plasma}} \sim \frac{L_p}{r_b}\sqrt{2Q}$$

Typical: $\sigma_{\text{plasma}} \sim 105^\circ/\text{period}$

- Temperature asymmetry in beam will rapidly fluctuate with lattice periodicity
  - Converging plane => Warmer
  - Diverging plane => Colder
- Collective plasma wave response slower than lattice frequency
  - Beam edge will not be able to adapt rapidly enough
  - Collective waves will be launched from lack of local force balance near the edge

Initial Distributions: Types of Specified Loads

Due to the practical difficulty of always carrying out simulations off the source, two alternative methods are commonly applied:

1) Load an idealized initial distribution
   - Specify at some specific time
   - Based on physically reasonable theory assumptions

2) Load experimentally measured distribution
   - Construct/synthesize a distribution based on experimental measurements

Discussion:

The 2nd option of generating a distribution from experimental measurements, unfortunately, often has practical difficulties:

- Real diagnostics often are far from ideal 6D snapshots of beam phase-space
  - Distribution must be reconstructed from partial data
  - Typically many assumptions must be made in the synthesis process
- Process of measuring the beam can itself change the beam
- It can sometimes be helpful to understand processes and limitations starting from cleaner, more idealized initial beam states

Initial Distributions: Types of Specified Loads Continued (2)

Discussion Continued:

Because of the practical difficulties of loading a distribution based exclusively on experimental measurements, idealized distributions are often loaded:

- Employ distributions based on reasonable, physical ansatzes
- Use limited experimental measures to initialize:
  - Energy, current, rms equivalent beam sizes and emittances
- Simpler initial state can often aid insight:
  - Fewer simultaneous processes can allow one to more clearly understand how limits arise
  - Seed perturbations of relevance when analyzing resonance effects, instabilities, halo, etc.

A significant complication is that there are no known exact smooth equilibrium distribution functions valid for periodic focusing channels:

- Approximate theories valid for low phase advances may exist
  Davidson, Struckmeier, and others

Formulate a simple approximate procedure to load an initial distribution that reflects features one would expect of an equilibrium