<u>Vlasov-Maxwell Theory for Collective Dynamics of High Intensity Beams</u> (Davidson & Qin, Chaps. 4, 8, 10)

- Nonlinear Vlasov-Maxwell equations
- Collective eigenmode and instabilities
    - Surface mode
    - Body mode
    - Stability theorem
    - Two stream instability
- Nonlinear delta-f particle simulation method

Paraxial Approximation

$$p_x^2,\ p_y^2,\ (p_z - p_b)^2 \ll p_b^2,\ a,b \ll S$$
$$p_b = \gamma_b m_b \beta_b c$$
$$\gamma_b = (1 + p_b^2 / m_b^2 c^2)^{1/2}$$

Electrostatic and magnetostatic approximations for self-fields

$$\mathbf{E}^s = -\nabla\phi^s,\ \ \mathbf{B}^s = \nabla \times A_z^s \mathbf{e}_z$$
$$e_b(\mathbf{E}^s + \frac{1}{c}\mathbf{v} \times \mathbf{B}^s) = e_b(-\frac{1}{\gamma_b^2}\nabla_\perp \phi - \mathbf{e}_z \frac{\partial \phi}{\partial z})$$

Smooth focusing model

$$\mathbf{F}_{foc} = \gamma_b m_b \omega_{\beta\perp}^2 (x\mathbf{e}_x + y\mathbf{e}_y)$$

Long coating beams (single species)

Vlasov

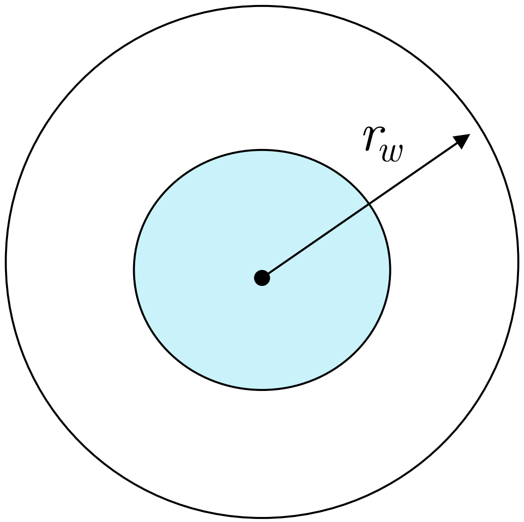
$$\boxed{\frac{\partial f_b}{\partial t} + \mathbf{v} \cdot \frac{\partial f_b}{\partial \mathbf{x}} + [-\gamma_b m_b \omega_{\beta\perp}^2 (x\mathbf{e}_x + y\mathbf{e}_y) - e_b(\frac{1}{\gamma_b^2} \nabla_\perp \phi + \mathbf{e}_z \frac{\partial \phi}{\partial z})] \cdot \frac{\partial f_b}{\partial \mathbf{p}} = 0}$$

Maxwell

$$\boxed{\nabla^2 \phi = -4\pi e_b \int d^3 p f_b}$$

B.C.

$$\boxed{\phi(r = r_w, \theta, z, t) = const.}$$



Equilibrium (1D, long costing beam)

$$f_b^0 = f_b^0(r, \mathbf{p}) = f_b^0(H_\perp, p_z)$$
$$\phi^0 = \phi^0(r)$$

$$H_\perp = \frac{1}{2\gamma_b m_b}(p_x^2 + p_y^2) + \frac{1}{2}\gamma_b m_b \omega_{\beta\perp}^2 r^2 + \frac{e_b}{\gamma_b^2}\phi^0(r)$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\phi^0 = - 4\pi e_b \int d^3p f_b^0$$

Linear eigenmode analysis

$$f_b(\mathbf{x},\mathbf{p},t) = f_b^0(r,\mathbf{p}) + \delta f_b(\mathbf{x},\mathbf{p},t)$$
$$\phi(\mathbf{x},t) = \phi^0(r) + \delta\phi(\mathbf{x},t)$$

along unperturbed orbit

$$\left(\frac{d}{dt}\right)_0 \delta f_b = \{\frac{\partial}{\partial t} + \mathbf{v}_\perp \cdot \frac{\partial}{\partial \mathbf{x}_\perp} + v_z \frac{\partial}{\partial z} - (\gamma_b m_b \omega_{\beta\perp}^2 \mathbf{x}_\perp + \frac{e_b}{\gamma_b^2 r}\frac{\partial \phi^0}{\partial r}\mathbf{x}_\perp) \cdot \frac{\partial}{\partial \mathbf{p}_\perp}\}\delta f_b$$
$$= e_b(\frac{1}{\gamma_b^2}\nabla_\perp \delta\phi \cdot \frac{\partial f_b^0}{\partial \mathbf{p}_\perp} + \frac{\partial \delta\phi}{\partial z}\frac{\partial f_b^0}{\partial p_z})$$

$$\nabla_\perp^2 \delta\phi + \frac{\partial^2}{\partial z^2}\delta\phi = -4\pi e_b \int d^3 p \delta f_b$$

$$\frac{\partial f_b^0}{\partial \mathbf{p}_\perp} = \frac{\mathbf{p}_\perp}{\gamma_b m_b}\frac{\partial f_b^0}{\partial H_\perp}$$

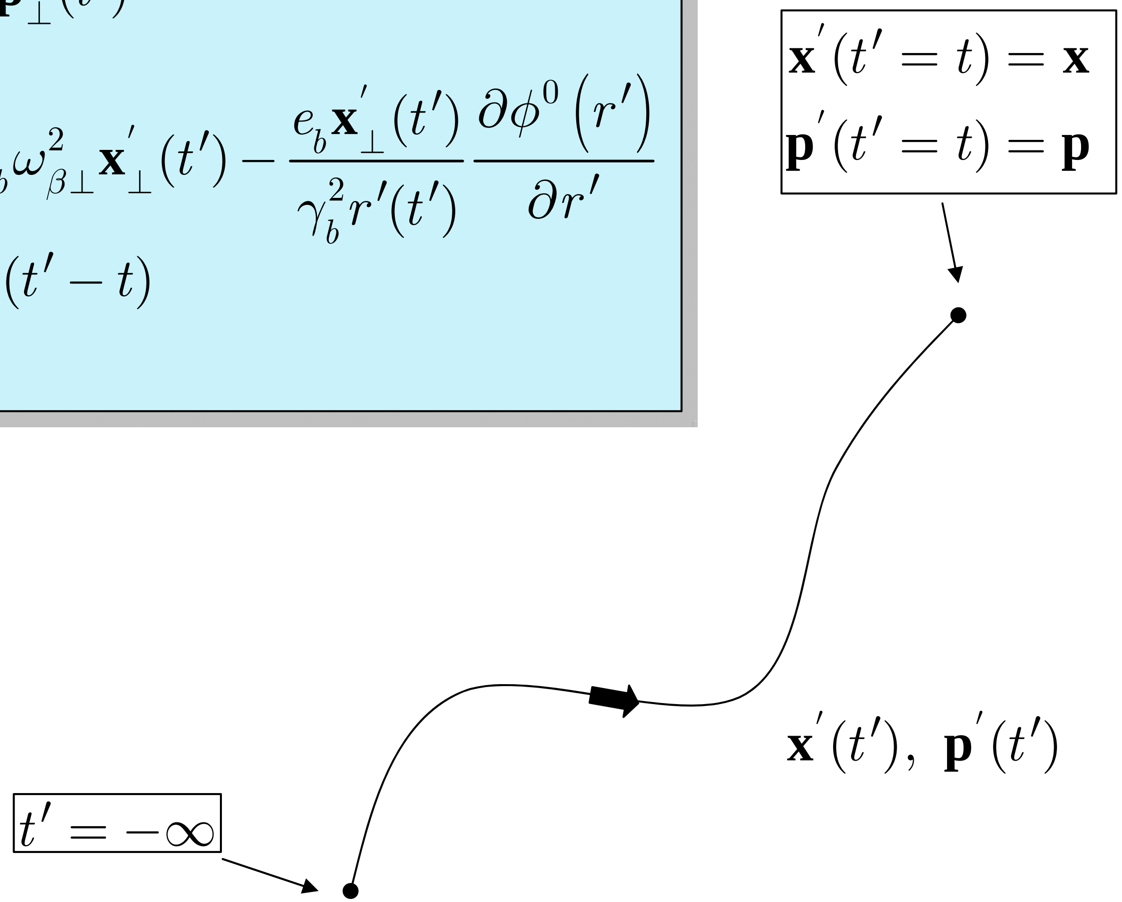
<u>Unperturbed orbit</u>

$$\frac{d}{dt'}\mathbf{x}'_\perp(t') = \frac{1}{\gamma_b m_b}\mathbf{p}'_\perp(t')$$

$$\frac{d}{dt'}\mathbf{p}'_\perp(t') = -\gamma_b m_b \omega^2_{\beta\perp}\mathbf{x}'_\perp(t') - \frac{e_b \mathbf{x}'_\perp(t')}{\gamma_b^2 r'(t')}\frac{\partial\phi^0\left(r'\right)}{\partial r'}$$

$$z'(t') = z + v_z(t'-t)$$

$$p'_z(t') = p_z$$



Integration along unperturbed orbit

$$\boxed{\delta f_b(t' = -\infty) = 0}$$

$$\delta f_b(\mathbf{x},\mathbf{p},t) = \frac{e_b}{\gamma_b^2}\frac{\partial f_b^0}{\partial H_\perp}\int_{-\infty}^{t} dt' \frac{\mathbf{p}_\perp^{'}}{\gamma_b m_b}\cdot\nabla_\perp' \delta\phi(\mathbf{x}^{'},t') + e_b\frac{\partial f_b^0}{\partial p_z}\int_{-\infty}^{t} dt' \frac{\partial}{\partial z'}\delta\phi(\mathbf{x}^{'},t')$$

$$\boxed{\delta\phi(\mathbf{x},t) - \int_{-\infty}^{t} dt'(\frac{\partial}{\partial t'} + v_z'\frac{\partial}{\partial z'})\delta\phi(\mathbf{x}^{'},t')}$$

$$\boxed{\frac{d}{dt'}\delta\phi(\mathbf{x}^{'},t') = (\frac{\partial}{\partial t'} + \frac{\mathbf{p}_\perp^{'}}{\gamma_b m_b}\cdot\frac{\partial}{\partial \mathbf{x}_\perp^{'}} + v_z'\frac{\partial}{\partial z'})\delta\phi(\mathbf{x}_\perp^{'},z',t')}$$

$$\delta f_b(\mathbf{x},\mathbf{p},t) = \sum_{n=-\infty}^{\infty} \delta f_b(\mathbf{x}_\perp,\mathbf{p},k_z,\omega)\exp(ik_z z - i\omega t)$$

$$\delta\phi(\mathbf{x},t) = \sum_{n=-\infty}^{\infty} \delta\phi(\mathbf{x}_\perp,k_z,\omega)\exp(ik_z z - i\omega t)$$

$$\exp\{ik_z z'(t') - i\omega t'\} = \exp(ik_z z - i\omega t)\exp\{-i(\omega - k_z v_z)(t'-t)\}\ .$$

$$\begin{aligned}
\delta f_b(\mathbf{x}_\perp,\mathbf{p},k_z,\omega) &= \frac{e_b}{\gamma_b^2}\frac{\partial f_b^0}{\partial H_\perp}[\delta\phi(\mathbf{x}_\perp,k_z,\omega) \\
&+ i(\omega - k_z v_z)\int_{-\infty}^{t} dt'\delta\phi[\mathbf{x}_\perp'(t'),k_z,\omega]\exp\{-i(\omega-k_z v_z)(t'-t)\}] \\
&+ e_b\frac{\partial f_b^0}{\partial p_z} ik_z\int_{-\infty}^{t} dt'\delta\phi[\mathbf{x}_\perp'(t'),k_z,\omega]\exp\{-i(\omega-k_z v_z)(t'-t)\} \\
&= \frac{e_b}{\gamma_b^2}\frac{\partial f_b^0}{\partial H_\perp}[\delta\phi(\mathbf{x}_\perp) + (\omega-k_z v_z)I(\mathbf{x}_\perp,\mathbf{p})] + e_b\frac{\partial f_b^0}{\partial p_z}k_z I(\mathbf{x}_\perp,\mathbf{p})
\end{aligned}$$

$$\boxed{I(\mathbf{x}_\perp,\mathbf{p},k_z,\omega) = i\int_{-\infty}^{0} d\tau\delta\phi[\mathbf{x}_\perp'(\tau),k_z,\omega]\exp\{-i(\omega-k_z v_z)\tau\}\ .}$$

Eigenvalue Eq.

$$\nabla_{\perp}^{2}\delta\phi(\mathbf{x}_{\perp}) - k_{z}^{2}\delta\phi(\mathbf{x}_{\perp}) = -\frac{4\pi e_{b}^{2}}{\gamma_{b}^{2}}\int d^{3}p\,\frac{\partial f_{b}^{0}}{\partial H_{\perp}}[\delta\phi(\mathbf{x}_{\perp}) + (\omega - k_{z}v_{z})I(\mathbf{x}_{\perp},\mathbf{p})]$$
$$-4\pi e_{b}^{2}k_{z}\int d^{3}p\,\frac{\partial f_{b}^{0}}{\partial p_{z}}I(\mathbf{x}_{\perp},\mathbf{p})$$

$$I(\mathbf{x}_{\perp},\mathbf{p},k_{z},\omega) = i\int_{-\infty}^{0} d\tau\,\delta\phi[\mathbf{x}_{\perp}^{'}(\tau),k_{z},\omega]\exp\{-i(\omega - k_{z}v_{z})\tau\}\ .$$

Kapchinskij-Vladimirskij beam equilibrium

$$f_b^0(r,\mathbf{p}) = F_b(H_\perp)G_b(p_z) \qquad \int_{-\infty}^{\infty} dp_z G(p_z) = 1$$

$$F_b(H_\perp) = \frac{\hat{n}_b}{2\pi\gamma_b m_b}\delta(H_\perp - \hat{T}_{\perp b}) = \frac{\hat{n}_b}{2\pi\gamma_b m_b}\delta\left[\frac{p_\perp^2}{2\gamma_b m_b} - \hat{T}_{\perp b}(1-\frac{r^2}{r_b^2})\right]$$

$$\boxed{\begin{aligned} n_b^0(r) &= \begin{cases} \hat{n}_b = const., & 0 \le r < r_b \\ 0\,, & r_b < r \le r_w \end{cases} \\ \varphi_0(r) &= -e_b \pi \hat{n}_b r^2 \end{aligned}} \longrightarrow \boxed{\begin{aligned} \frac{1}{2}\gamma_b m_b \nu^2 r_b^2 &= \hat{T}_{\perp b} \\ \nu^2 &= \omega_{\beta\perp}^2 - \frac{1}{2\gamma_b^2}\omega_{pb}^2 \end{aligned}}$$

Unperturbed orbit:

$$\mathbf{x}_\perp'(\tau) = \mathbf{x}_\perp \cos(\nu\tau) + \frac{1}{\gamma_b m_b \nu}\mathbf{p}_\perp \sin(\nu\tau)$$

$$\mathbf{p}_\perp'(\tau) = \mathbf{p}_\perp \cos(\nu\tau) - \gamma_b m_b \nu \mathbf{x}_\perp \sin(\nu\tau)$$

$\boxed{\text{Consider the mode with } k_z = 0}$

$$\delta\phi(\mathbf{x}_\perp) = \sum_{\ell=-\infty}^{\infty} \delta\phi^\ell(r)\exp(i\ell\theta)$$

$$I(\mathbf{x}_\perp, \mathbf{p}_\perp) = i\int_{-\infty}^{0} d\tau\delta\phi[\mathbf{x}'_\perp(\tau)]\exp(-i\omega\tau)$$

$$\boxed{\nabla_\perp^2\delta\phi(\mathbf{x}_\perp) = -\frac{4\pi e_b^2}{\gamma_b^2}\int d^2p\,\frac{\partial F_b}{\partial H_\perp}[\delta\phi(\mathbf{x}_\perp) + \omega I(\mathbf{x}_\perp, \mathbf{p}_\perp)]}$$

$$\boxed{\int d^2p = \int_0^{2\pi} d\varphi \int_0^{\infty} dp_\perp p_\perp}$$

gyrophase

Do the gyrophase integral first

$$I^{\ell}(r,p_{\perp})=i\int_0^{2\pi}\frac{d\varphi}{2\pi}\int_{-\infty}^{0}d\tau\delta\phi^{\ell}[r'(\tau)]\exp\{i\ell[\theta'(\tau)-\theta]-i\omega\tau\}$$

surface

$$\boxed{\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\delta\phi^{\ell}(r)-\frac{\ell^2}{r^2}\delta\phi^{\ell}(r)=&\frac{\hat{\omega}_{pb}^2 r_b}{v_{Tb}^2\gamma_b^2}\{\delta\phi^{\ell}(r)+\omega[I^{\ell}]_{p_{\perp}=0}\}\delta(r-r_b)\\&+\frac{\omega_{pb}^2(r)}{\gamma_b^2}\omega[\frac{\gamma_b^2 m_b^2}{p_{\perp}}\frac{\partial I^{l}}{\partial p_{\perp}}]_{p_{\perp}^2=\hat{p}_{\perp b}^2(1-r^2/r_b^2)}\end{aligned}}$$

body

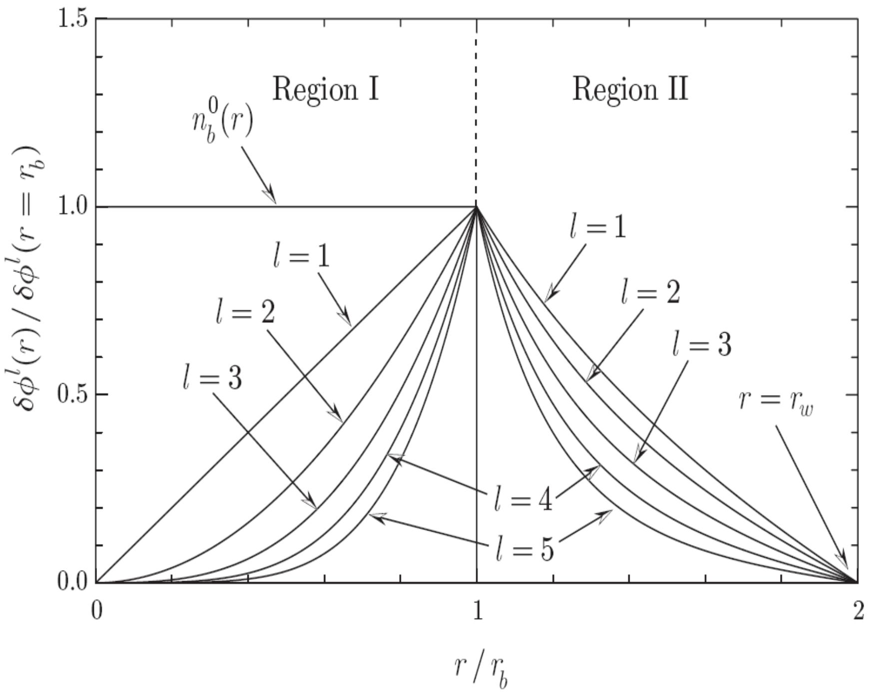
$$\hat{p}_{\perp b}^2\equiv 2\gamma_b m_b\hat{T}_{b\perp}=(\gamma_b m_b v_{Tb})^2$$

$$v_{Tb}^2\equiv 2\hat{T}_{\perp b}/\gamma_b m_b$$

Surface mode

Assume $\dfrac{\partial I^{l}}{\partial p_{\perp}} = 0$

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\delta\phi^{\ell}(r) - \frac{\ell^2}{r^2}\delta\phi^{\ell}(r) = \frac{\hat{\omega}_{pb}^2 r_b}{v_{Tb}^2\gamma_b^2}\{\delta\phi^{\ell}(r) + \omega[I^{\ell}]_{p_{\perp}=0}\}\delta(r-r_b)}$$



$$\delta\phi_I^{\ell}(r) = \hat{\phi}^{\ell} r^{\ell}\ , \quad 0 \le r < r_b$$

$$\delta\phi_{II}^{\ell}(r) = \hat{\phi}^{\ell} r^{\ell}\,\frac{(1 - r_w^{2\ell} / r^{2\ell})}{(1 - r_w^{2\ell} / r_b^{2\ell})}, \quad r_b < r \le r_w$$

$$\boxed{r'^{\ell}\exp(i\ell\theta') = (x' + iy')^{\ell}}$$

$$I^{\ell} = i\hat{\phi}^{\ell}\int_{-\infty}^{0} d\tau \exp(-i\ell\theta - i\omega\tau)\int_{0}^{2\pi}\frac{d\varphi}{2\pi}[x'(\tau) + iy'(\tau)]^{\ell}$$

$$= i\frac{\delta\phi^{\ell}(r)}{2^{\ell}}\int_{-\infty}^{0} d\tau \exp(-i\omega\tau)[\exp(i\nu\tau) + \exp(-i\nu\tau)]^{\ell}$$

$$\longrightarrow \boxed{\frac{\partial I^{l}}{\partial p_{\perp}} = 0}$$

$$\boxed{x'(\tau) + iy'(\tau) = r\exp(i\theta)\cos(\nu\tau) + \frac{p_{\perp}}{\gamma_b m_b \nu}\exp(i\varphi)\sin(\nu\tau)}$$

$$\mathbf{x}'_{\perp}(\tau) = \mathbf{x}_{\perp}\cos(\nu\tau) + \frac{1}{\gamma_b m_b \nu}\mathbf{p}_{\perp}\sin(\nu\tau)$$

$$\mathbf{p}'_{\perp}(\tau) = \mathbf{p}_{\perp}\cos(\nu\tau) - \gamma_b m_b \nu \mathbf{x}_{\perp}\sin(\nu\tau)$$

$$\delta\phi^{\ell}(r) + \omega I^{\ell} = \Gamma_b^{\ell}(\omega)\delta\phi^{\ell}(r)$$

$$\mathrm{Im}\,\omega > 0$$

$$\Gamma_b^{\ell}(\omega) = -\frac{1}{2^{\ell}}\sum_{m=0}^{\ell}\frac{\ell!}{m!(\ell-m)!}\frac{(\ell-2m)\nu}{[\omega-(\ell-2m)\nu]}$$

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\delta\phi^{\ell}(r) - \frac{\ell^2}{r^2}\delta\phi^{\ell}(r) = \frac{\hat{\omega}_{pb}^2}{\gamma_b^2\nu^2 r_b}\Gamma_b^{\ell}(\omega)\delta\phi^{\ell}(r)\delta(r-r_b)$$

$$\delta\phi_I^{\ell}(r) = \hat{\phi}^{\ell} r^{\ell}\ , \quad 0 \le r < r_b$$

$$\delta\phi_{II}^{\ell}(r) = \hat{\phi}^{\ell} r^{\ell}\,\frac{(1 - r_w^{2\ell}\,/\,r^{2\ell})}{(1 - r_w^{2\ell}\,/\,r_b^{2\ell})}, \quad r_b < r \le r_w$$

$$[r\frac{\partial}{\partial r}\delta\phi_{II}^{\ell}]_{r=r_b(1+\varepsilon)} - [r\frac{\partial}{\partial r}\delta\phi_{I}^{\ell}]_{r=r_b(1-\varepsilon)} = \frac{\hat{\omega}_{pb}^2}{\gamma_b^2\nu^2}\Gamma_b^{\ell}(\omega)\delta\phi^{\ell}(r_b)$$

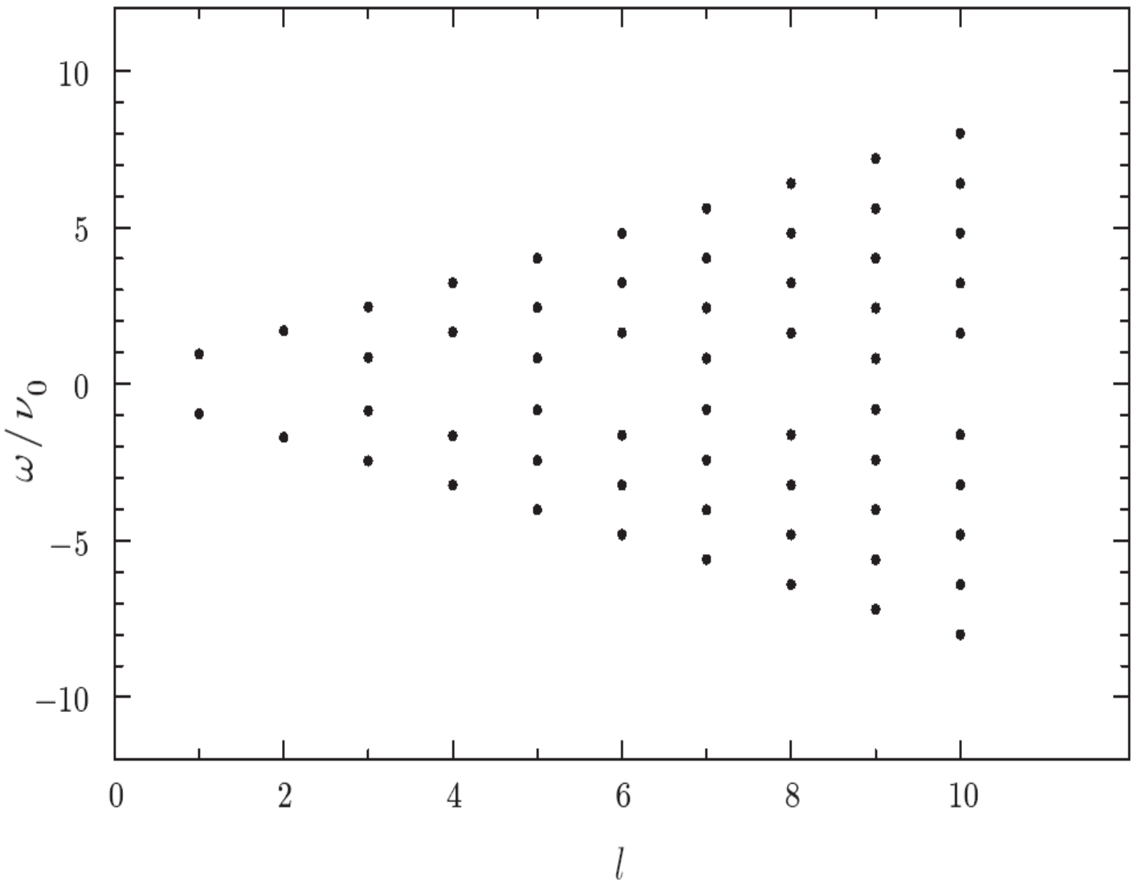
Dispersion relation:

$$\boxed{\begin{aligned}0 = D_\ell(\omega) &= 1 + \frac{\hat{\omega}_{pb}^2}{2\ell\gamma_b^2\nu^2}[1-(\frac{r_b}{r_w})^{2\ell}]\Gamma_b^\ell(\omega)\\ &= 1 - \frac{\hat{\omega}_{pb}^2}{2^{\ell+1}\ell\gamma_b^2\nu^2}[1-(\frac{r_b}{r_w})^{2\ell}]\sum_{m=0}^{\ell}\frac{\ell!}{m!(\ell-m)!}\frac{(\ell-2m)\nu}{[\omega-(\ell-2m)\nu]}\end{aligned}}$$

For every ℓ, there are $\ell+1$ mode with $\mathrm{Im}(\omega)=0$.

$\ell = 1:$ diapole mode

$$\omega^2 = \nu^2 + \frac{\hat{\omega}_{pb}^2}{2\gamma_b^2}(1-\frac{r_b^2}{r_w^2})$$



10
5
0
−5
−10
ω / ν₀
0
2
4
6
8
10
l

$\boxed{k_z \neq 0}$

$$G_b(p_z) = \delta(p_z - \gamma_b m_b V_b) \qquad k_z^2 r_b^2 \ll 1$$

$$0 = D_\ell(\omega - k_z V_b) = 1 + \frac{\hat{\omega}_{pb}^2}{2\ell\gamma_b^2\nu^2}[1 - (\frac{r_b}{r_w})^{2\ell}]\Gamma_b^\ell(\omega - k_z V_b)$$

Doppler shifted

$\ell = 1:$ diapole mode

$$(\omega - k_z V_b)^2 = \nu^2 + \frac{\hat{\omega}_{pb}^2}{2\gamma_b^2}(1 - \frac{r_b^2}{r_w^2})$$

Body mode $\quad \partial / \partial\theta = 0 \quad \partial / \partial z = 0$

$$\delta\phi_n(r) = \begin{cases} \hat{\phi}_n \sum_{j=0}^{n} a_j(\omega)(r / r_b)^{2j}, & 0 \le r < r_b, \\ \hat{\phi}_n \sum_{j=0}^{n} a_j(\omega) \frac{\ell n(r/r_w)}{\ell n(r_b/r_w)}, & r_b < r \le r_w, \end{cases}$$

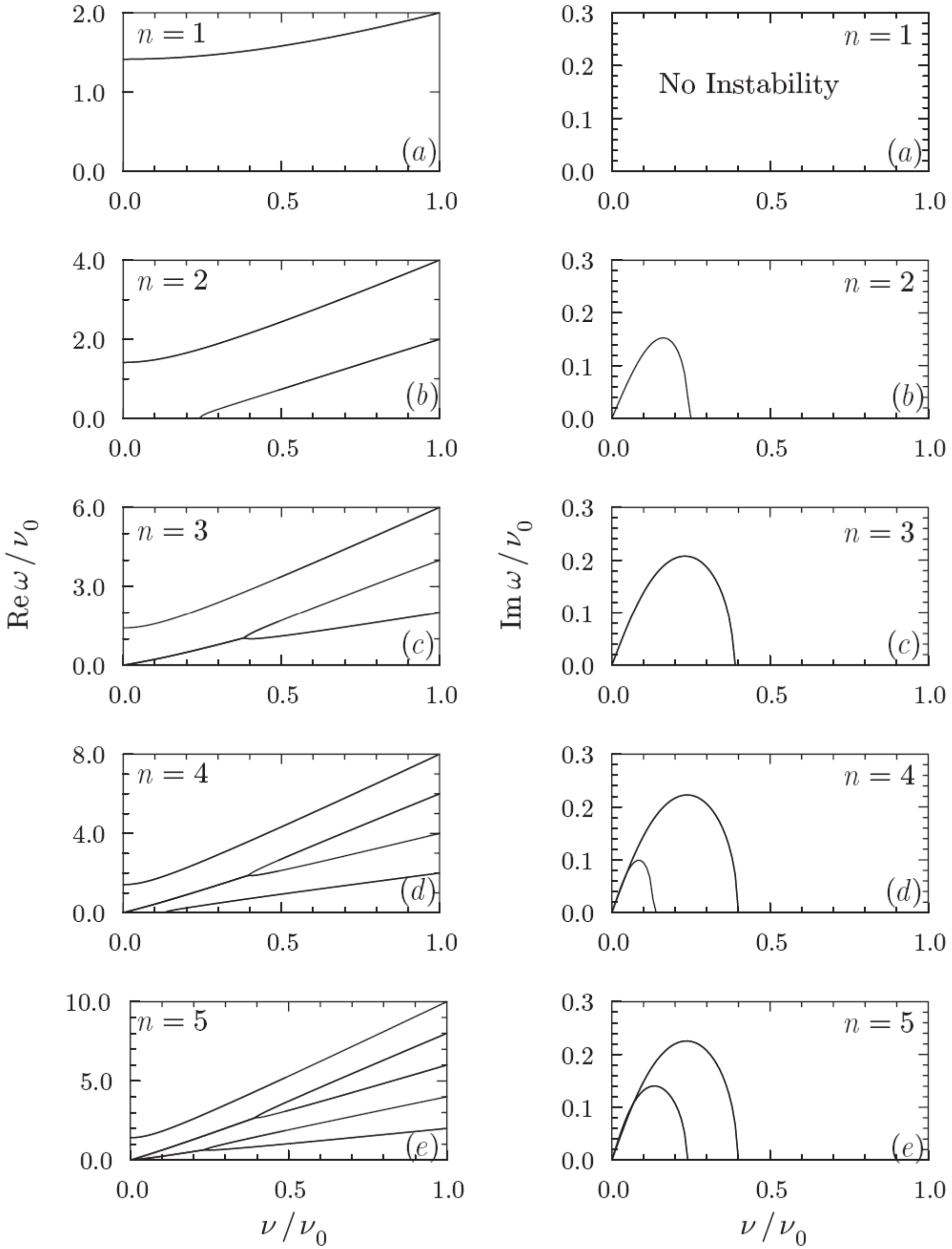
$$\boxed{\delta\phi_n(r) = \begin{cases} \frac{1}{2}\hat{\phi}_n [P_{n-1}(1-2\frac{r^2}{r_b^2}) + P_n(1-2\frac{r^2}{r_b^2})], & 0 \le r < r_b, \\ 0, & r_b < r \le r_w. \end{cases}}$$

$$D_n(\omega) = 2n + \frac{1-(\nu/\nu_0)^2}{(\nu/\nu_0)^2}\left[B_{n-1}\left(\frac{\omega/\nu_0}{\nu/\nu_0}\right) - B_n\left(\frac{\omega/\nu_0}{\nu/\nu_0}\right)\right] = 0$$

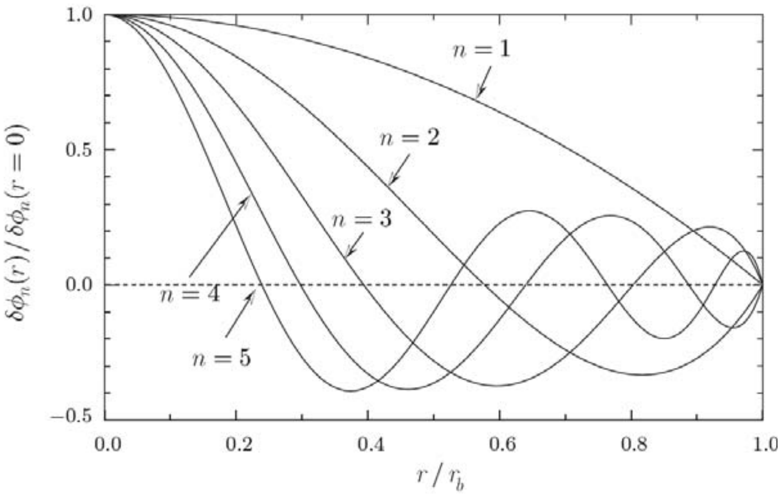
$$B_n(\alpha) \equiv \begin{cases} 1\,,\ n=0\,, \\ \dfrac{[(\alpha/2)^2-0^2][(\alpha/2)^2-2^2]}{[(\alpha/2)^2-1^2][(\alpha/2)^2-3^2]}\cdots\dfrac{[(\alpha/2)^2-(n-1)^2]}{[(\alpha/2)^2-n^2]}\,, \\ \text{for } n=1,3,5\cdots, \\ \dfrac{[(\alpha/2)^2-1^2][(\alpha/2)^2-3^2]}{[(\alpha/2)^2-2^2][(\alpha/2)^2-4^2]}\cdots\dfrac{[(\alpha/2)^2-(n-1)^2]}{[(\alpha/2)^2-n^2]}\,, \\ \text{for } n=2,4,6\cdots \end{cases}$$

$$\nu/\nu_0 \equiv (1-\hat{\omega}_{pb}^2/2\gamma_b^2\omega_{\beta\perp}^2)^{1/2}$$

$$\nu_0 = \omega_{\beta\perp}$$



KV beam is unstable



Nonlinear stability theorem

- Global conservation constrains for the nonlinear Vlasov-Maxwell equations.
- Determine Class of beam distributions that are stable at high intensity.

A sufficient condition for linear and nonlinear stability is:

$$\frac{\partial f\left(H\right)}{\partial H} \leq 0$$

Self-field

where $$H \equiv \frac{1}{2}\mathbf{p}^2 + \psi\left(\mathbf{x}\right) + e_b\phi\left(\mathbf{x}\right)$$

Focusing field

$$\psi\left(\mathbf{x}\right) = \frac{1}{2}m_b\gamma_b\left(\omega_{\beta\perp}^2 r^2 + \omega_z^2 z^2\right)$$

- R. C. Davidson, Physical Review Letters **81**, 991 (1998).
- *Physics of Intense Charged Particle Beams in High Energy Accelerators* (World Scientific, 2001), R. C. Davidson and H. Qin, Chapter 4;

Collective instabilities in intense charged particle beams

One-Component Beams

- Harris and Weibel instability driven by temperature anisotropy

$$T_\perp \gg T_\| \ .$$

- Resistive wall instability

Propagation Through Background speices

- Two-stream instability
- Ion-electron (Electron cloud) instability

Propagation Through Background Plasma

- Resistive hose instability
- Multispecies Weibel instability
- Multispecies two-stream instability

Ion-electron two-stream instability

$$\boxed{\begin{aligned} f_b^0(r,\mathbf{p}) &= \frac{\hat{n}_b}{2\pi\gamma_b m_b}\delta(H_{\perp b}-\hat{T}_{\perp b})G_b(p_z) \\ f_e^0(r,\mathbf{p}) &= \frac{\hat{n}_e}{2\pi m_e}\delta(H_{\perp e}-\hat{T}_{\perp e})G_e(p_z) \end{aligned}}$$

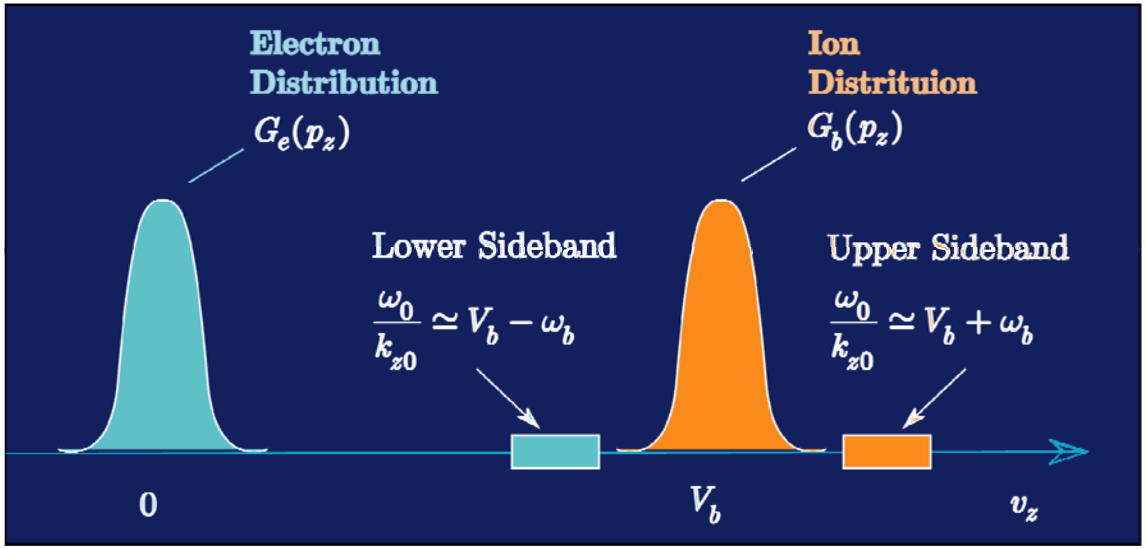
$$G_b(p_z) = \frac{\Delta_b}{\pi[(p_z-\gamma_b m_b V_b)^2+\Delta_b^2]}$$

$$G_e(p_z) = \frac{\Delta_e}{\pi(p_z^2+\Delta_e^2)}$$

$$H_{\perp b} = \frac{1}{2\gamma_b m_b}\mathbf{p}_\perp^2 + \frac{1}{2}\gamma_b m_b \omega_{\beta\perp}^2 r^2 + e_b\psi^0(r)$$

$$H_{\perp e} = \frac{1}{2m_e}\mathbf{p}_\perp^2 - e\varphi^0(r)$$

$$V_b = \int_{-\infty}^{\infty} dp_z v_z G_b(p_z)$$



$$
\boxed{\begin{aligned}&[\frac{2}{1-(r_b/r_w)^{2\ell}}+\frac{\hat{\omega}_{pb}^2}{\ell\gamma_b^2\nu_b^2}\Gamma_b^\ell(\omega)][\frac{2}{1-(r_b/r_w)^{2\ell}}+\frac{\hat{\omega}_{pe}^2}{\ell\nu_e^2}\Gamma_e^\ell(\omega)]\\&=\frac{\hat{\omega}_{pe}^2}{\ell\nu_e^2}\cdot\frac{\hat{\omega}_{pb}^2}{\ell\nu_b^2}\Gamma_e^\ell(\omega)\Gamma_b^\ell(\omega)\end{aligned}}
$$

$$
\Big\downarrow \; l=1
$$

$$
\boxed{[(\omega-k_zV_b+i\,|\,k_z\,|\,v_{T\|b})^2-\omega_b^2][(\omega+i\,|\,k_z\,|\,v_{T\|e})^2-\omega_e^2]=\omega_f^4}
$$

$$
\omega_e^2\equiv\nu_e^2+\frac{1}{2}\hat{\omega}_{pe}^2(1-\frac{r_b^2}{r_w^2})=\frac{1}{2}\frac{\gamma_b m_b}{Z_b m_e}\hat{\omega}_{pb}^2(1-f\frac{r_b^2}{r_w^2}) \qquad f=\hat{n}_e/Z_b\hat{n}_b
$$

$$
\omega_b^2\equiv\nu_b^2+\frac{\hat{\omega}_{pb}^2}{2\gamma_b^2}(1-\frac{r_b^2}{r_w^2})=\omega_{\beta\perp}^2+\frac{1}{2}\hat{\omega}_{pb}^2(f-\frac{1}{\gamma_b^2}\frac{r_b^2}{r_w^2}) \qquad \omega_f^4\equiv\frac{1}{4}f(1-\frac{r_b^2}{r_w^2})^2\frac{\gamma_b m_b}{Z_b m_e}\hat{\omega}_{pb}^4
$$

beam

electron

coupling

$$[(\omega - k_z V_b)^2 - \omega_b^2][\omega^2 - \omega_e^2] = \omega_f^4$$

Strong interaction

$$\omega_0 - k_{z0} V_b = -\omega_b$$
$$\omega_0 = +\omega_e$$

$$\longrightarrow \quad \frac{\omega_0}{k_{z0}} - V_b \simeq -\frac{\omega_b}{\omega_e + \omega_b} V_b$$

$$\omega = \omega_0 + \delta\omega, \;\; k_z = k_{z0} + \delta k_z$$
$$\delta\omega(\delta\omega - V_b \delta k_z) = -\frac{\omega_f^4}{4\omega_e \omega_b} \equiv -\Gamma_0^2$$

$$\longrightarrow$$

$$\mathrm{Re}\,\delta\omega = \frac{1}{2} V_b \delta k_z$$
$$\mathrm{Im}\,\delta\omega = \Gamma_0 [1 - (\frac{V_b \delta k_z}{2\Gamma_0})^2]^{1/2}$$

Nonlinear Vlasov-Maxwell equations for high intensity beams

Collective dynamics described by the Vlasov equation

$$\left\{\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} - \left[m_b\left(\omega_{\beta b}^2\mathbf{x}_\perp + \omega_z^2 z\mathbf{e}_z\right) + e_b(\nabla\phi - v_z c\nabla_\perp A_z)\right]\cdot\frac{\partial}{\partial \mathbf{p}}\right\} f(\mathbf{x},\mathbf{p},t) = 0$$

Smooth focusing field:

$$\mathbf{F}_{foc} = -m_b\omega_{\beta b}^2\mathbf{x}_\perp - m_b\omega_z^2 z\,\mathbf{e}_z$$

distribution function in phase space

Self-electric and self-magnetic fields <u>self-consistently</u> determined from Maxwell's equations.

$$\nabla^2\phi = -4\pi e_b\int d^3p f(\mathbf{x},\mathbf{p},t),$$

$$\nabla^2 A_z = -4\pi c e_b\int d^3p v_z f(\mathbf{x},\mathbf{p},t).$$

$f(\mathbf{x},\mathbf{p},t)$ and self-field (ϕ, A_z) nonlinearly coupled

Particle in cell simulation method

$$f_j = \frac{N_j}{N_{sj}} \sum_{i=1}^{N_{sj}} \delta(\mathbf{x} - \mathbf{x}_{ji}) \delta(\mathbf{p} - \mathbf{p}_{ji})$$

$$\frac{d\mathbf{x}_{ji}}{dt} = (\gamma_j m_j)^{-1} \mathbf{p}_{ji}$$

$$\frac{d\mathbf{p}_{ji}}{dt} = -\gamma_j m_j \omega_{\beta\perp j}^2 \mathbf{x}_{\perp ji} - e_j \left(\nabla\varphi - \frac{1}{c} \mathbf{v}_{ji} \times (\nabla A_z \times \hat{\mathbf{e}}_z)\right)$$

Particle shape

$$\nabla^2 \phi = -4\pi \sum_j e_j n_j \,, \qquad n_j = \int d^3\mathbf{p} f_j(\mathbf{x}, \mathbf{p}, t) = \frac{N_j}{N_{sj}} \sum_{i=1}^{N_{sj}} S(\mathbf{x} - \mathbf{x}_{ji})$$

$$\nabla^2 A_z = -\frac{4\pi}{c} \sum_j J_{zj} \,, \qquad J_{zj} = e_j \int d^3\mathbf{p}\, v_{zj} f_j(\mathbf{x}, \mathbf{p}, t) = \frac{e_j N_j}{N_{sj}} \sum_{i=1}^{N_{sj}} v_{zji} S(\mathbf{x} - \mathbf{x}_{ji})$$

δf particle simulation method reduces noise

$$\begin{aligned} f &= f_0 + \delta f, \\ \phi &= \phi_0 + \delta\phi, \\ A_z &= A_{z0} + \delta A_z. \end{aligned}$$

(f_0, ϕ_0, A_{z0}) – equilibrium

$(\delta f, \delta\phi, \delta A_z)$ – perturbation

Fully nonlinear

$$\frac{dw_i}{dt} = -(1 - w_i)\frac{1}{f_0}\frac{\partial f_0}{\partial \mathbf{p}} \cdot \delta\left(\frac{d\mathbf{p}_i}{dt}\right),$$

$$\delta\left(\frac{d\mathbf{p}_i}{dt}\right) \equiv -e_b\left(\nabla\delta\phi - \frac{v_{zi}}{c}\nabla_\perp \delta A_z\right),$$

Weight function $w \equiv \dfrac{\delta f}{f}$

dynamically determines δf

Statistical noise **significantly**

reduced by a factor of $\dfrac{\delta f}{f}$

δf particle simulation method reduces noise

$$\delta f_j = \frac{N_j}{N_{sj}} \sum_{i=1}^{N_{sj}} w_{ji} \delta(\mathbf{x} - \mathbf{x}_{ji}) \delta(\mathbf{p} - \mathbf{p}_{ji})$$

$$\frac{dw_{ji}}{dt} = -(1 - w_{ji}) \frac{1}{f_{j0}} \frac{\partial f_{j0}}{\partial \mathbf{p}} \cdot \delta\left(\frac{d\mathbf{p}_i}{dt}\right)$$

$$\frac{d\mathbf{x}_{ji}}{dt} = (\gamma_j m_j)^{-1} \mathbf{p}_{ji}$$

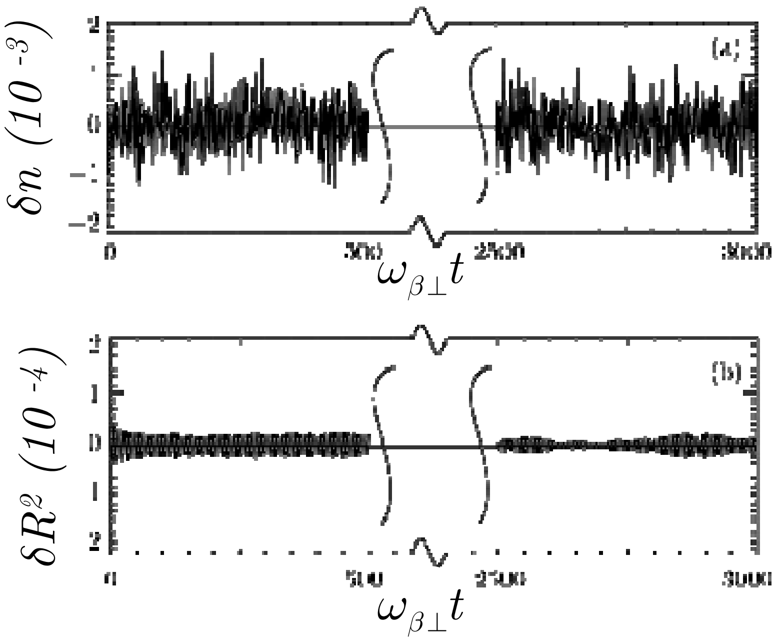
$$\frac{d\mathbf{p}_{ji}}{dt} = -\gamma_j m_j \omega_{\beta\perp j}^2 \mathbf{x}_{\perp ji} - e_j (\nabla \varphi - \frac{1}{c} \mathbf{v}_{ji} \times (\nabla A_z \times \hat{\mathbf{e}}_z))$$

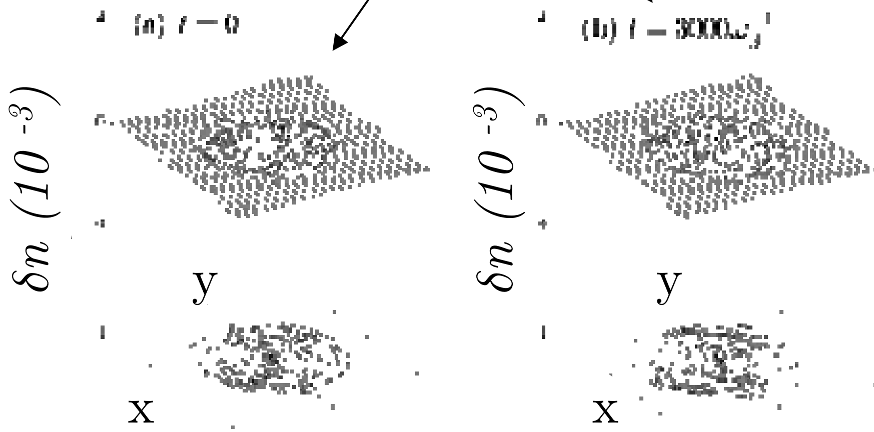
$$\nabla^2 \delta\phi = -4\pi \sum_j e_j \delta n_j \qquad \delta n_j = \int d^3\mathbf{p} \delta f_j(\mathbf{x}, \mathbf{p}, t) = \frac{N_j}{N_{sj}} \sum_{i=1}^{N_{sj}} w_{ji} S(\mathbf{x} - \mathbf{x}_{ji})$$

$$\nabla^2 \delta A_z = -\frac{4\pi}{c} \sum_j \delta J_{zj} \qquad \delta J_{zj} = e_j \int d^3\mathbf{p}\, v_{zj} \delta f_j(\mathbf{x}, \mathbf{p}, t) = \frac{e_j N_j}{N_{sj}} \sum_{i=1}^{N_{sj}} v_{zji} w_{ji} S(\mathbf{x} - \mathbf{x}_{ji})$$

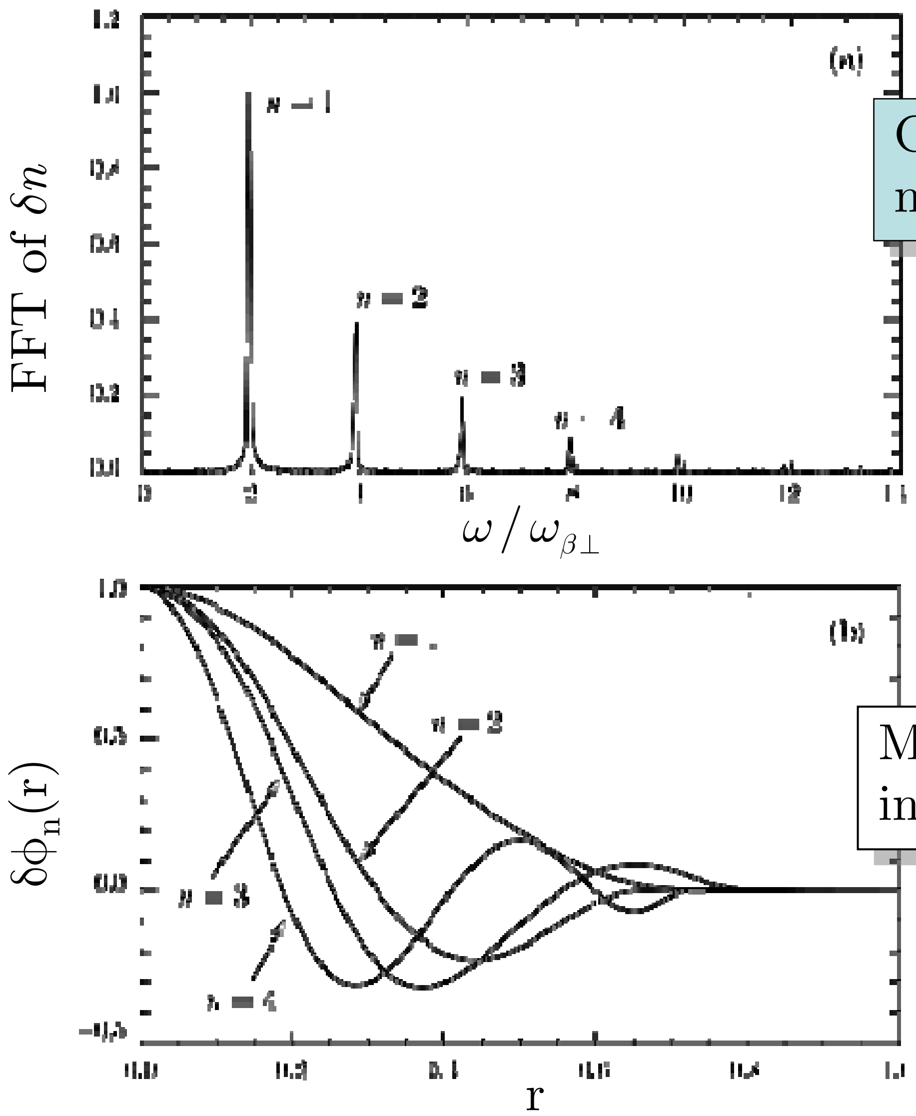
Long-time nonlinear perturbations of a thermal equilibrium beam

contain
perturbation
spectrum





Collective interior mode excitation



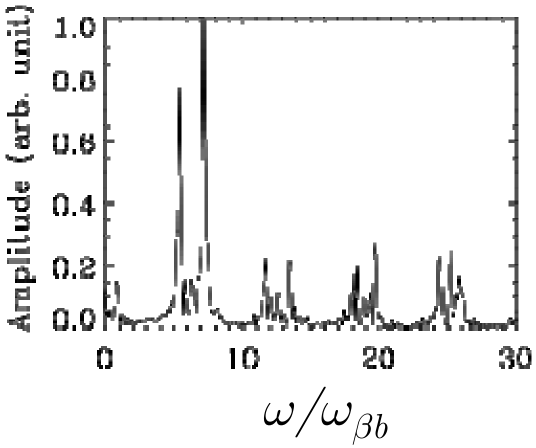
Collective dynamics manifest by eigenmodes

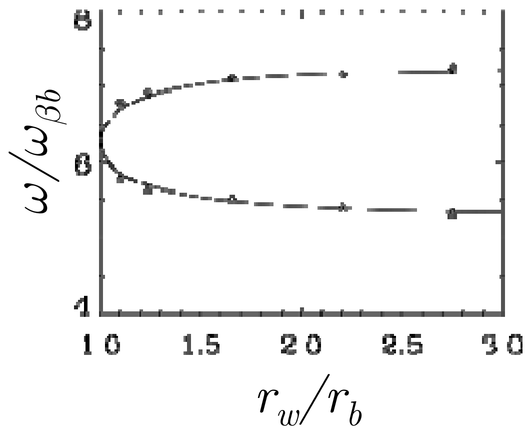
Mode structure confined inside the beam

<u>Collective surface mode excitation</u>

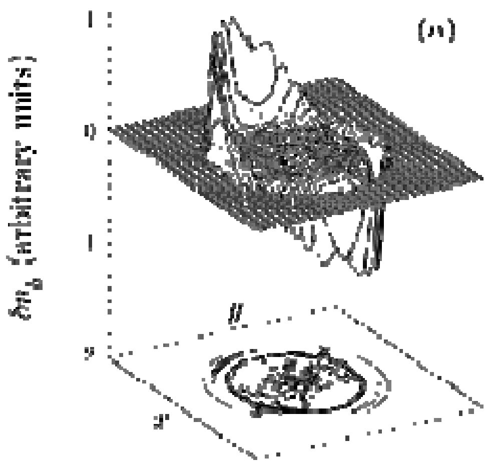
Dispersion relation:

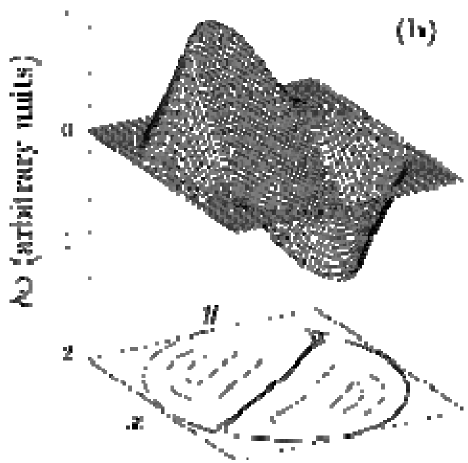
$$\omega = k_z V_b \pm \frac{\hat{\omega}_{pb}}{\sqrt{2}\gamma_b}\sqrt{1 - \frac{r_b^2}{r_w^2}}$$



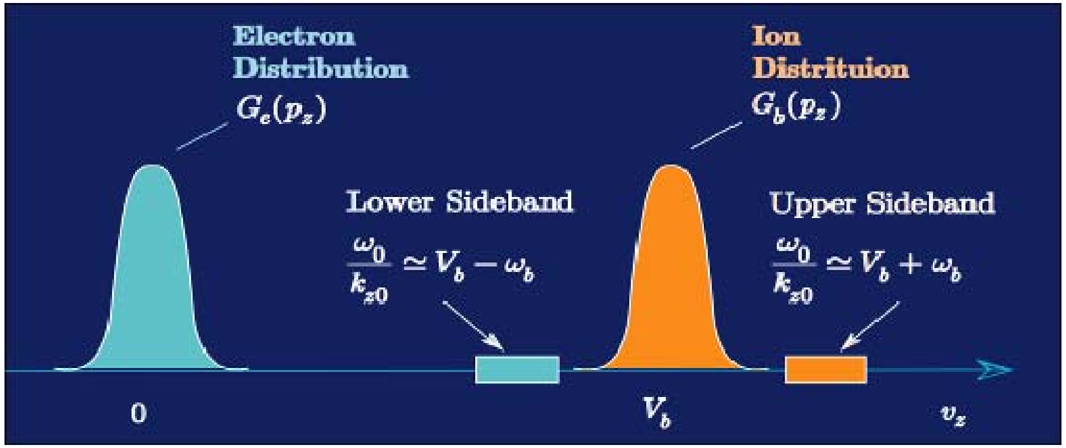


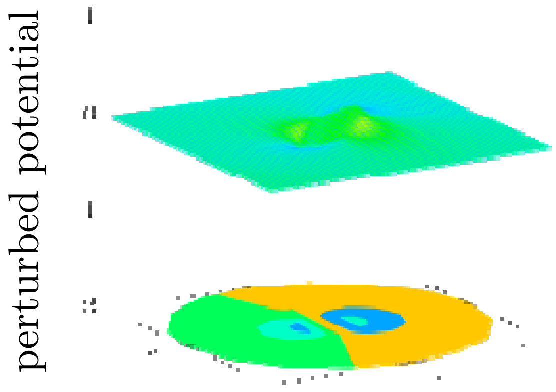
Dipole mode structure





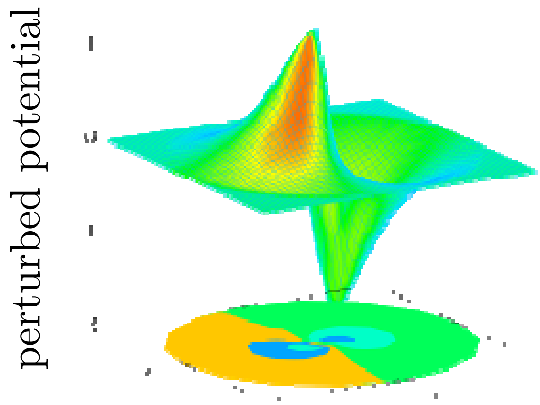
Electron-ion two-stream instability





$t = 0$

- ❑ Surface mode destabilized by background species.
- ❑ Observed in high intensity proton beams (PRS).
- ❑ Could be a show stopper for high intensity accelerators, e.g. SNS.
- ❑ Transverse geometry and dynamics are important.
- ❑ Damping mechanisms (Landau damping) are important.



$t = 200\,/\,\omega_{\beta b}$

Instability properties predicted by BEST simulations

- **Agrees well with experiment observations** (Proton Storage Ring)
  - Mode structure
  - Growth rate
  - Real oscillation frequency
- Late time nonlinear growth observed.
- There are two-phases to the instability.