# AST-565 Homework Set 9

## Problem 1: Relativistic Hamiltonian under paraxial approximation

Starting from the relativistic Hamiltonian

$$H = (m_b^2 c^4 + c^2 \mathbf{p}^2)^{1/2} + e_b \varphi(\mathbf{x}, t) \,,$$

where $e_b \varphi$ is the electrostatic potential energy of a particle, $\mathbf{p} = \gamma m \mathbf{v}$ is the particle momentum, and $\gamma = (1 + \mathbf{p}^2 \,/\, m_b^2 c^2)^{1/2}$ is the kinematic mass factor. Under the paraxial approximation $p_x^2$ , $p_y^2$ , $(p_z - p_b)^2 \ll p_b^2$ , Taylor expand $\gamma m_b c^2 = [m_b^2 c^4 + c^2 (p_x^2 + p_y^2) + c^2 (p_b + p_z - p_b)^2]^{1/2}$ about $p_z = p_b = \gamma_b m_b \beta_b c$ . Retaining terms to leading order in $p_x^2$ , $p_y^2$ , $(p_z - p_b)^2 \ll p_b^2$ , show that the relativistic Hamiltonian reduces to

$$H = \gamma_b m_b c^2 + V_b (p_z - p_b) + \frac{1}{2\gamma_b m_b} (p_x^2 + p_y^2) + \frac{1}{2\gamma_b^3 m_b} (p_z - p_b)^2 + e_b \varphi(\mathbf{x}, t) \,.$$

Here, $\gamma_b = (1 + p_b^2 \,/\, m_b^2 c^2)^{1/2} = (1 - \beta_b^2)^{-1/2}$ and $V_b = \beta_b c$ are constants.

Problem 2: Smoothing focusing model (Problem 3.14, Davidson and Qin)

(a) Assume $\langle \kappa_q(s) \rangle = 0$ and make use of $(d\ /\ ds)\int_{s_0}^{s} ds\kappa_q(s) = \kappa_q(s)$ and the assumed periodicity of $\delta a(s) = \delta a(s+S)$ to show that

$$\left\langle \kappa_q(s)\delta a(s) \right\rangle = -\left\langle \left( \int_{s_0}^{s} ds\kappa_q(s) \right) \frac{d}{ds}\delta a(s) \right\rangle .$$

(b) Integrate $\dfrac{d^2}{ds^2}\delta a(s) + \kappa_q(s)\overline{a} = 0$ once to show that

$$\frac{d}{ds}\delta a(s) = \frac{d}{ds}\delta a\bigg|_{s=s_0} - \left( \int_{s_0}^{s} ds\kappa_q(s) \right)\overline{a},$$

where $\overline{a}(s)$ is treated as slowly varying over a lattice period $S$.

(c) Make us of the results from (a) and (b) to derive

$$\langle \kappa_q(s)\delta a(s) \rangle = \left\{ \left\langle \left[ \int_{s_0}^{s} ds\kappa_q(s) \right]^2 \right\rangle - \left\langle \int_{s_0}^{s} ds\kappa_q(s) \right\rangle^2 \right\} \overline{a} .$$