# Symmetries and Invariants of the Oscillator Equation and Envelope Equation with Time-Dependent Frequency

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Oscillator with t-dependent frequency

$$\ddot{u} + \kappa(t)u = 0. \qquad (1)$$

$$I = \frac{u^2}{w^2} + \left(\dot{w}u - w\dot{u}\right)^2 \text{ is an invariant.}$$

$$\ddot{w} + \kappa(t)w - \frac{1}{w^3} = 0.$$

Courant-Snyder invariant (1958)

Envelope equation. Birkhoff (1908, concept)

Kulsrud (1957) had two eqs. equivalent to the envelope eq.

Takayama (1983) summarized different methods for deriving the invariant.

Noether's Theorem (1918) links invariants and symmetries

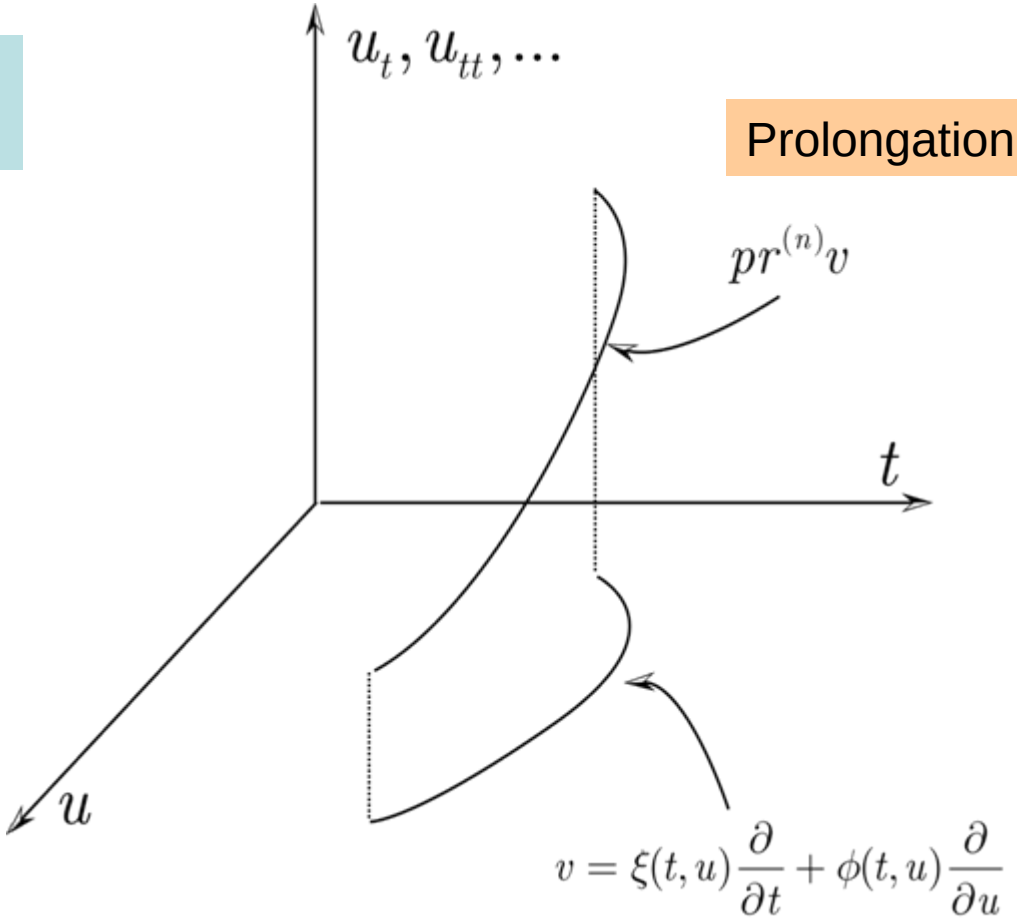
Symmetry in time ➔ energy conservation

Symmetry in configuration ➔ momentum conservation

What is the symmetry for the Courant-Snyder invariant?

## What is a symmetry for a differential equation?

Symmetry is a vector field.
S. Lie (1890s)



## What is symmetry?

$$v = \xi(t,u)\frac{\partial}{\partial t} + \phi(t,u)\frac{\partial}{\partial u}$$

Prolongation

$$\boxed{pr^{(2)}v = \xi\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} + \phi^{u}\frac{\partial}{\partial \dot{u}} + \phi^{uu}\frac{\partial}{\partial \ddot{u}}}$$

$$\phi^{u} \equiv \phi_t + (\phi_u - \xi_t)\dot{u} - \xi_u \dot{u}^2,$$

$$\begin{aligned}\phi^{uu} \equiv & -3\xi_u \dot{u}\ddot{u} + (\phi_u - 2\xi_t)\ddot{u} - \xi_{uu}\dot{u}^3 \\ & +(\phi_{uu} - 2\xi_{tu})\dot{u}^2 + (2\phi_{ut} - \xi_{tt})\dot{u} + \phi_{tt}\ .\end{aligned}$$

Determining eq.

$$\boxed{pr^{(2)}v\left[\ddot{u} + \kappa(t)u\right] = \phi^{uu} + \kappa\phi + \xi\dot{\kappa}u = 0}$$

## Symmetry for Eq. (1)

$$\boxed{\begin{aligned}\xi &= a(t)u + b(t)\\ \phi &= \dot{a}(t)u^2 + c(t)u + d(t)\end{aligned}}$$

$$\ddot{a} + \kappa a = 0,\ (2\mathrm{D})$$

$$\dddot{b} + 4\kappa\dot{b} + 2\dot{\kappa}b = 0,\ (3\mathrm{D})$$

$$\dot{c} - \frac{\ddot{b}}{2} = 0,\ (1\mathrm{D})$$

$$\ddot{d} + \kappa d = 0.\ (2\mathrm{D})$$

$\beta$ function

Lie sub-algebras

$$g_a \equiv \left\{ au\frac{\partial}{\partial t} + \dot{a}u^2\frac{\partial}{\partial u} \middle| \ddot{a} + \kappa a = 0 \right\}$$

$$g_b \equiv \left\{ b\frac{\partial}{\partial t} + \frac{\dot{b}}{2}u\frac{\partial}{\partial u} \middle| \dddot{b} + 4\kappa\dot{b} + 2\dot{\kappa}b = 0 \right\}$$

$$g_c \equiv \left\{ c_0\frac{\partial}{\partial u} \middle| c_0 \in R \right\}$$

$$g_d \equiv \left\{ d\frac{\partial}{\partial u} \middle| \ddot{d} + \kappa d = 0 \right\}$$

## Lie algebra and sub-algebras of the symmetry group

(Qin & Davidson 2005)

?! (2D)

Wronskian (2D)

$$g = g_a \oplus g_b \oplus g_c \oplus g_d$$

Courant-Snyder symmetry (3D)

Scaling (1D)

## Noether's Theorem (1918) links invariants and symmetries

Infinitesimal divergence symmetry:

$$pr^{(2)}v(L) + L\frac{d\xi}{dt} = \frac{dB(t,u)}{dt}$$

$B(t,u)$--- arbitrary function.
$L$ --- Lagrangian.

Then $pr^{(2)}v(L) = \dfrac{dA}{dt} + \xi\dfrac{dL}{dt}$ for some function $A(t,u)$,
and $I = B - A - L\xi$ is an invariant.

## Sub-algebras of the infinitesimal divergence symmetry

Infinitesimal divergence symmetry ➔ symmetry

$$g = g_b \oplus g_d$$

Courant-Snyder symmetry (3D)

Wronskian (2D)

## Courant-Snyder invariant and symmetry

$$\dddot{b} + 4\kappa\dot{b} + 2\dot{\kappa}b = 0,$$

$\beta$ function

$$b = 2w^2,$$

$$g_b : v = 2w^2 \frac{\partial}{\partial t} + \dot{w}wu\frac{\partial}{\partial u},$$

$$\ddot{w} + \kappa(t)w - \frac{\varepsilon}{w^3} = 0.$$

$$I_{cs} = \frac{u^2}{w^2} + \left(\dot{w}u - w\dot{u}\right)^2 \text{ is an invariant.}$$

Envelope equation.
Birkhoff (1908, concept)

Envelope eq. is the determining eq. for the Courant-Snyder symmetry.

## Symmetry for the envelope equation

- Courant-Snyder symmetry: $g_b$
- The determining equation is another envelope equation.

**Theorem 1.** For an arbitrary function $\kappa(t)$ and $w_1$, $w_2$ satisfying

$$\ddot{w}_1 + \kappa w_1 = \frac{\varepsilon_1}{w_1^3},$$

$$\ddot{w}_2 + \kappa w_2 = \frac{\varepsilon_2}{w_2^3},$$

where $\varepsilon_1$ and $\varepsilon_2$ are real constants, the quantity

$$I = \varepsilon_1 \left( \frac{w_2}{w_1} \right)^2 + \varepsilon_2 \left( \frac{w_1}{w_2} \right)^2 + \left( w_2 \dot{w}_1 - \dot{w}_2 w_1 \right)^2$$

is an invariant.

Symmetry $\Rightarrow$ better algorithm for matched $\beta$ function.

$$w_g = w_p \left( \frac{I - \sqrt{I^2 - 4}\sin\left[-2(\psi + C)\right]}{2} \right)^{1/2}$$

$$\psi = \int_0^t \frac{1}{w_p^2} dt.$$

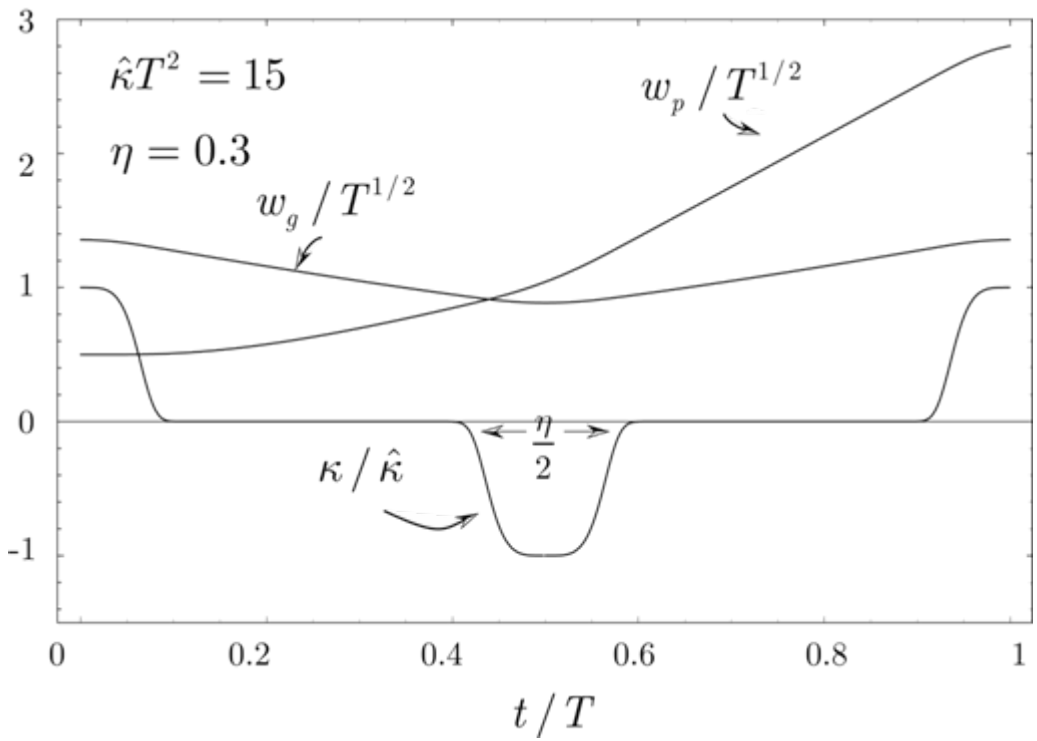
Select $I$ and $C$ such that

$w_g(0) = w_g(T)$ and $\dot{w}_g(0) = \dot{w}_g(T)$.

Only one numerical integration for the envelope equation.

Conventional method needs many.

## An example


$\hat{\kappa}T^2 = 15$
$\eta = 0.3$
$w_p / T^{1/2}$
$w_g / T^{1/2}$
$\kappa / \hat{\kappa}$
$\frac{\eta}{2}$
3
2
1
0
-1
0
0.2
0.4
0.6
0.8
1
$t / T$

## Conclusions

Symmetry ➔ Courant-Snyder invariant.

Symmetry ➔ better algorithm for the envelope equation.

Symmetry ➔ other applications at **http://nonneutral.pppl.gov.**