Geometry and Fusion Energy

-- Geometric gyrokinetic theory for magnetic thermonuclear fusion plasmas

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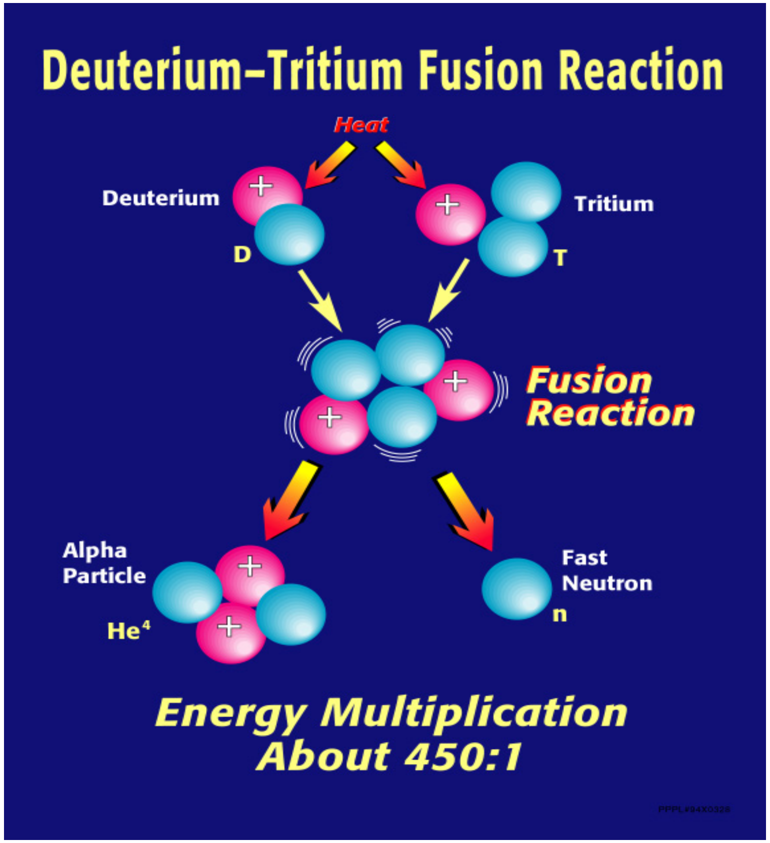
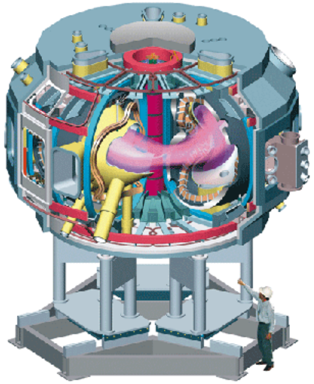
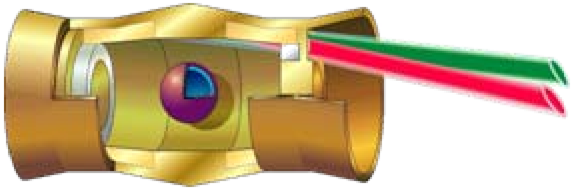
http://www.pppl.gov/~hongqin/Gyrokinetics.php

Thermonuclear fusion brings the ultimate energy source



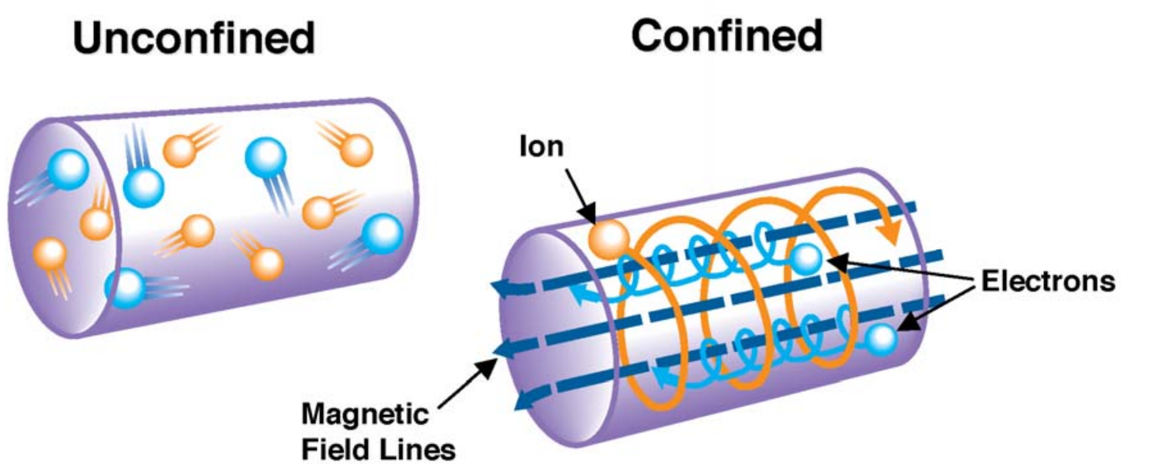


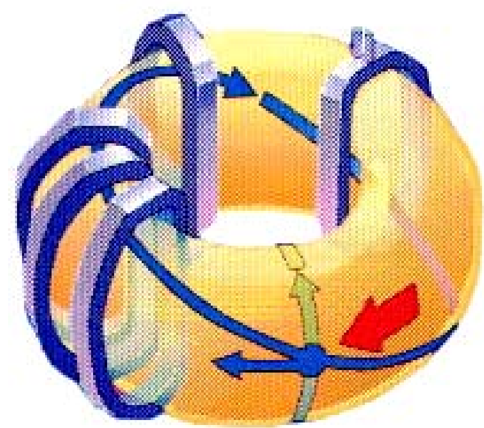
- Magnetic confinement



- Inertia confinement

Confinement by magnetic bottles

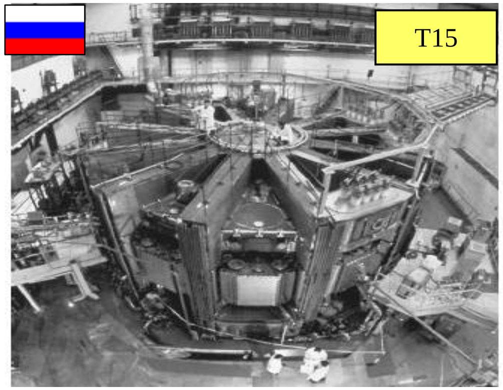


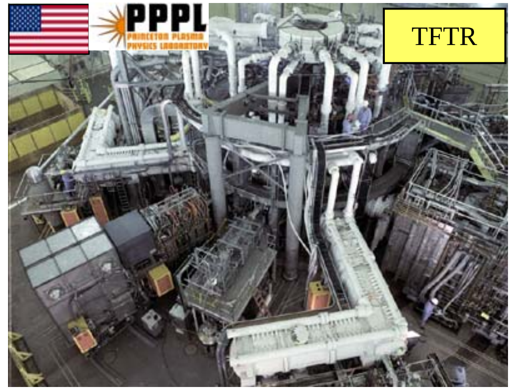


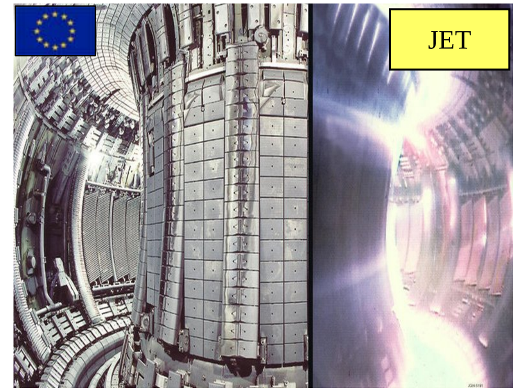
$\boxed{\text{need } s^1 \times s^1}$

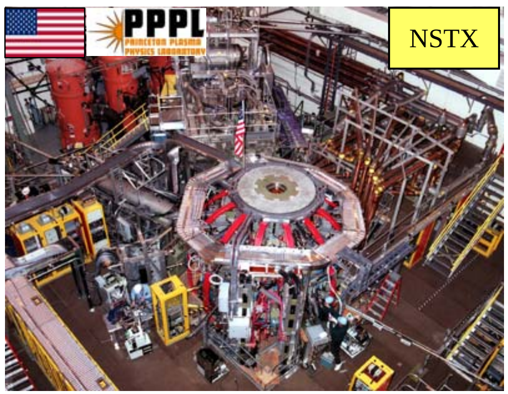
Poincare-Hopf Theorem
Hairy Donut Theorem

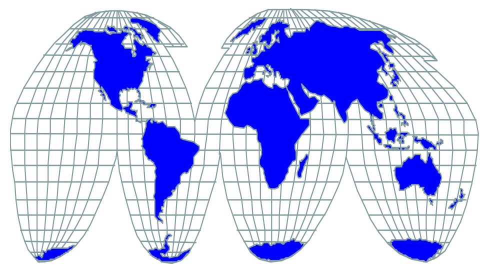
Magnetic fusion devices around the world

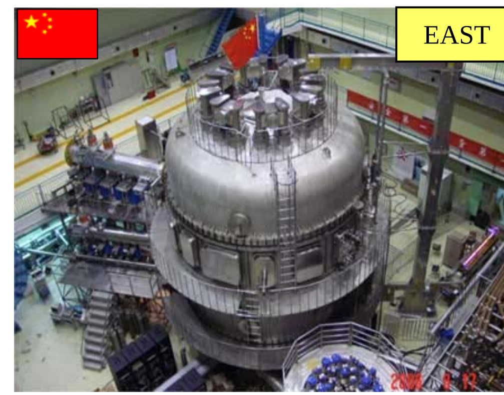


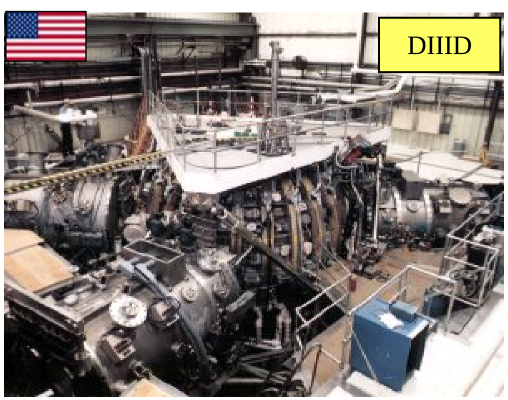


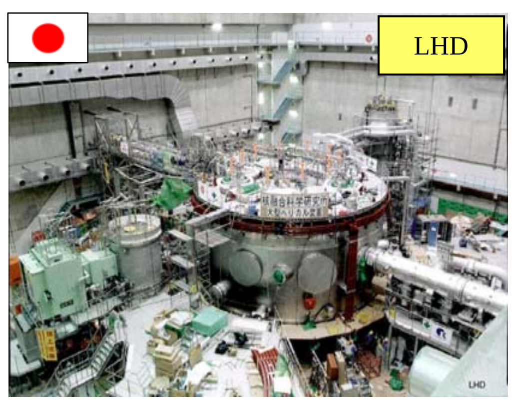


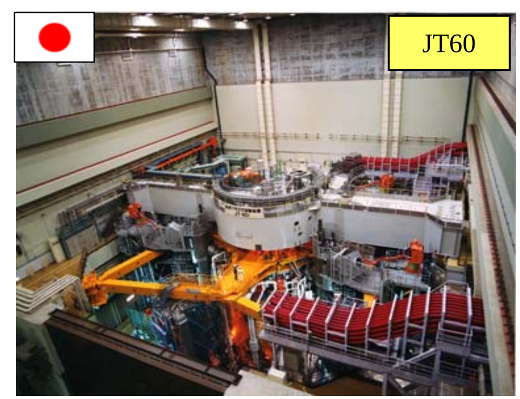




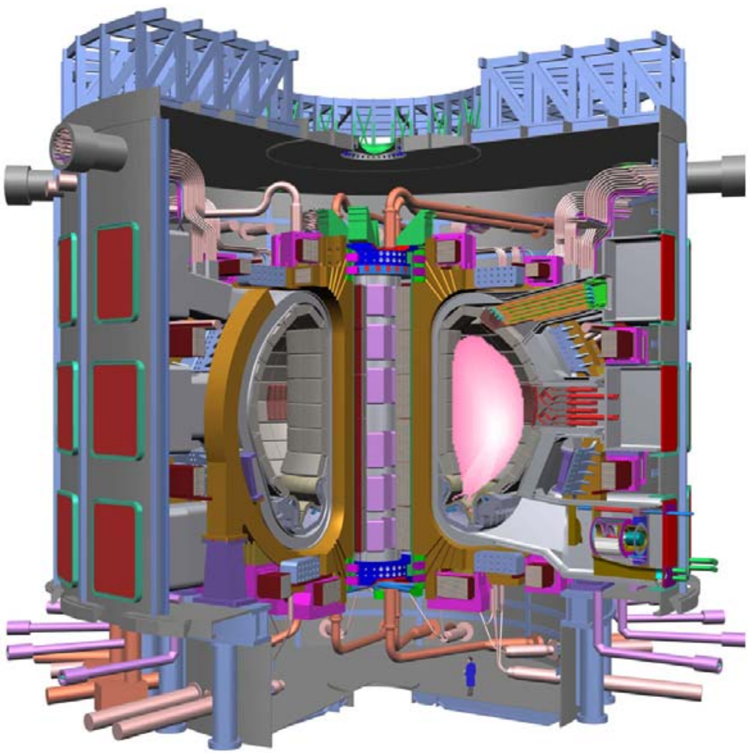


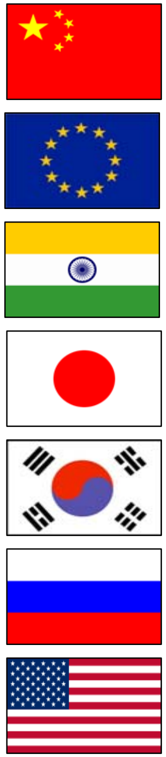






ITER – biggest magnetic bottle





- 500 – 700 MW thermal fusion power
- 400sec – 1 hr pulse length
- 10B USD

Vlasov-Maxwell equations for magnetized fusion plasmas

$$\left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}} + \frac{e_j}{m_j}\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\frac{\partial}{\partial \boldsymbol{p}}\right] f_j(\boldsymbol{x},\boldsymbol{p},t) = 0$$

$$\nabla\times\boldsymbol{B} = \frac{4\pi}{c}\sum_j e_j \int d^3p\,\boldsymbol{v} f_j(\boldsymbol{x},\boldsymbol{p},t) + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$$

$$\nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla\cdot\boldsymbol{E} = 4\pi\sum_j e_j \int d^3p\, f_j(\boldsymbol{x},\boldsymbol{p},t)$$

$$\nabla\cdot\boldsymbol{B} = 0$$

distribution function in phase space $(\boldsymbol{x},\boldsymbol{p})$

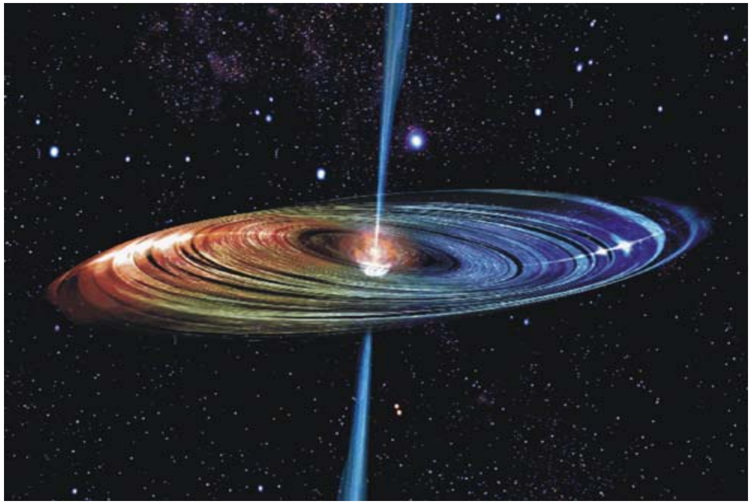
velocity integral

Problems:

1. Not geometric
2. Difficult to compute

Problem 1.

Physics = Geometry → not geometric = not physical



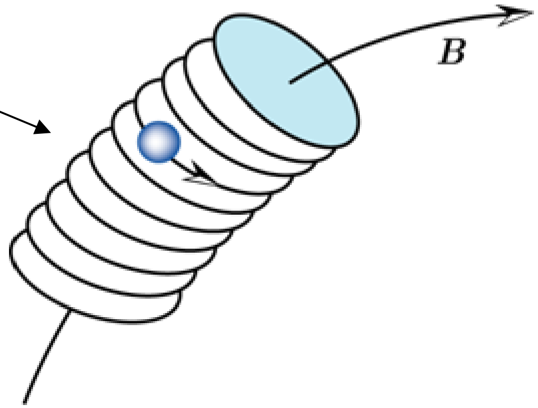
- No metric in phase space.
- Curved spacetime?
- How to make good approximations?

Problem 2.

Fast gyromotion $\rightarrow$ very small time-step

Highly oscillatory

$$\left[\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}} + \frac{e_j}{m_j}\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\frac{\partial}{\partial \boldsymbol{p}}\right] f_j(\boldsymbol{x},\boldsymbol{p},t) = 0$$

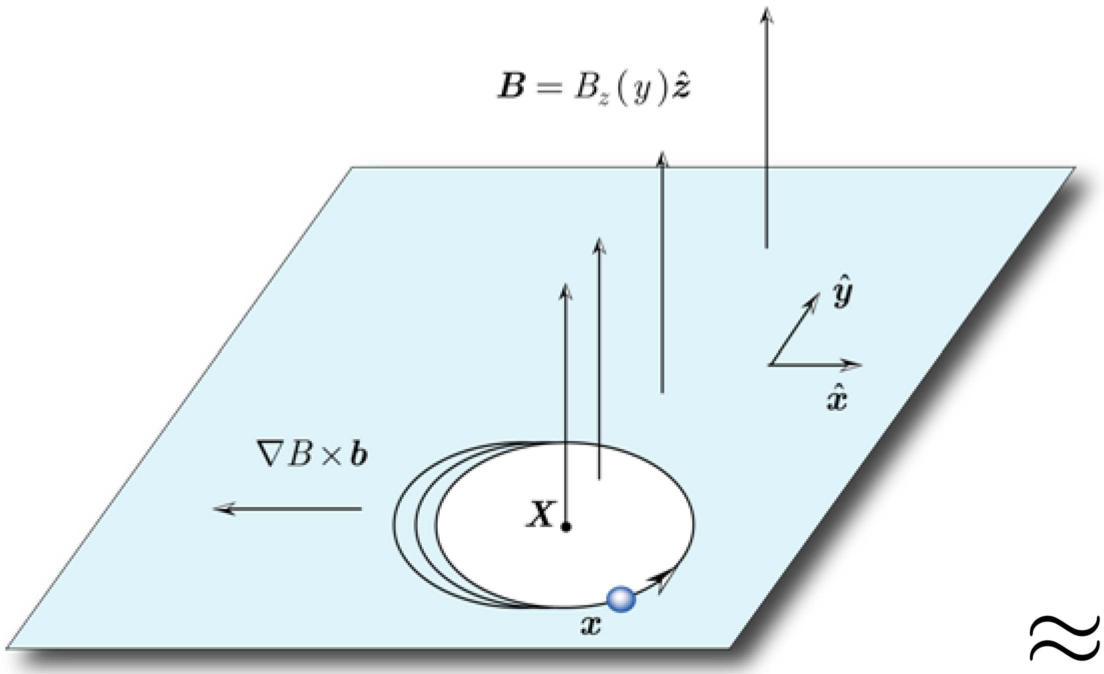


- Possible solution: "average-out" the fast gyromotion.
  - Theoretically appealing
  - Algorithmically efficient
  - Does not really work

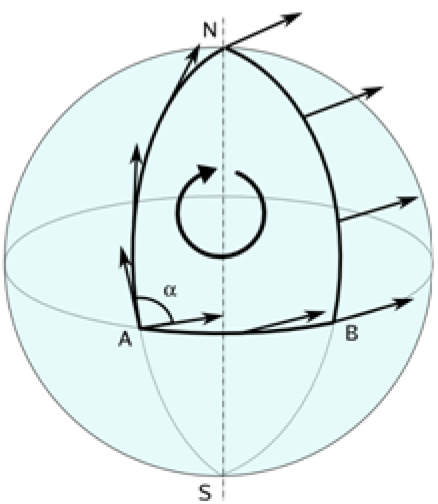
$$\boxed{\frac{1}{2\pi}\int_0^{2\pi} d\theta\,[\text{ Valsov} - \text{Maxwell eqs. }]}$$

$$\left\langle \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}}\right\rangle, \quad \left\langle \frac{\mathbf{v}\times\mathbf{B}}{c}\cdot\frac{\partial f}{\partial \mathbf{p}}\right\rangle ?$$

Drift of gyrocenters in inhomogeneous magnetic field



$\approx$



- ❑ Particles drift away from magnetic field line.
- ❑ It is an (an-)holonomy effect.

Solution to both problems: go geometric.

Better solution to Problem 2:
Decouple the gyromotion by finding the gyro-symmetry

↑

Symmetry is defined by a good coordinate.

↑

Go coordinate independent first
to find a good coordinate.

缘木求鱼

Outline

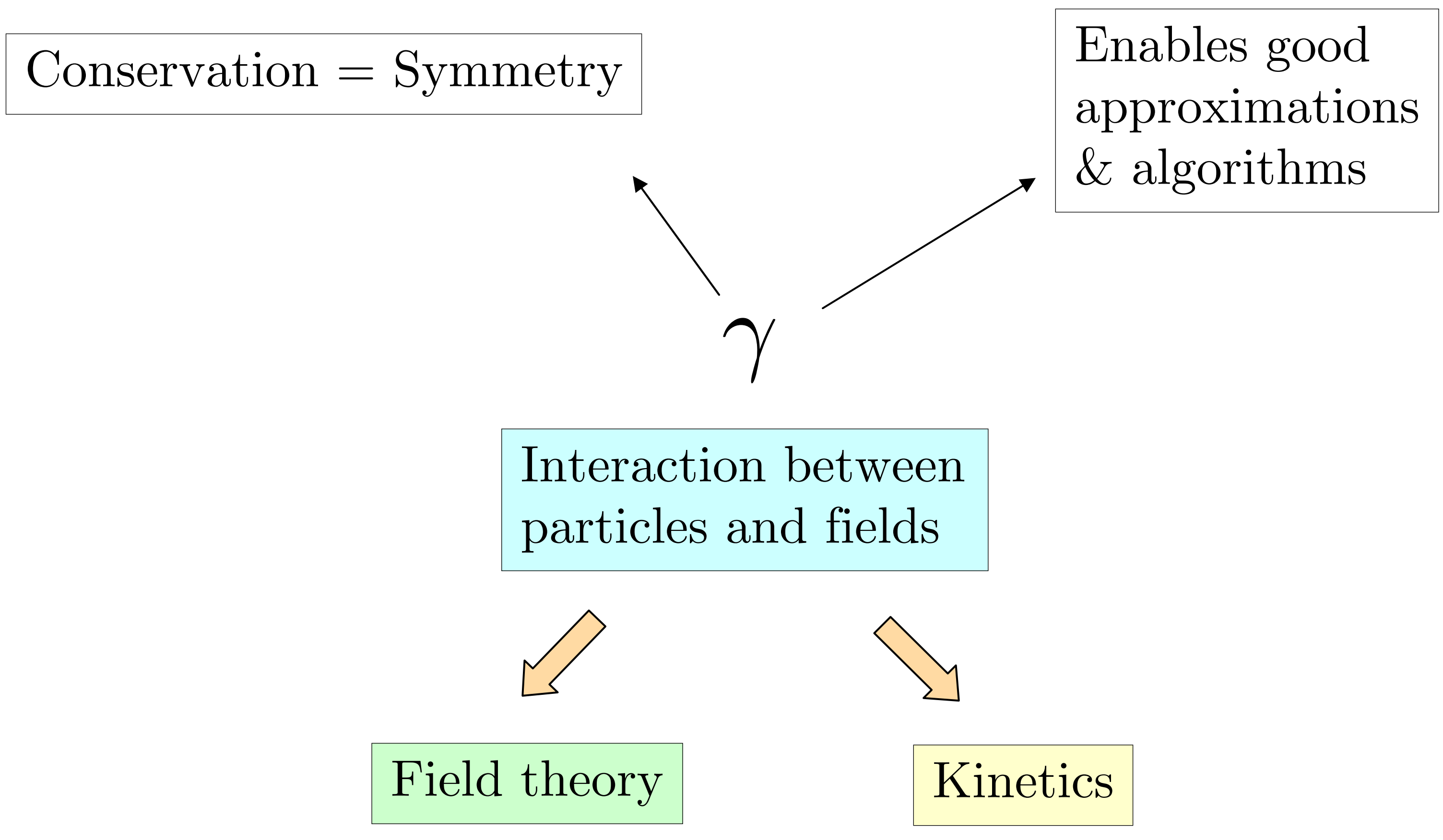
Geometric field theory

Geometrically generalized Vlasov-Maxwell system

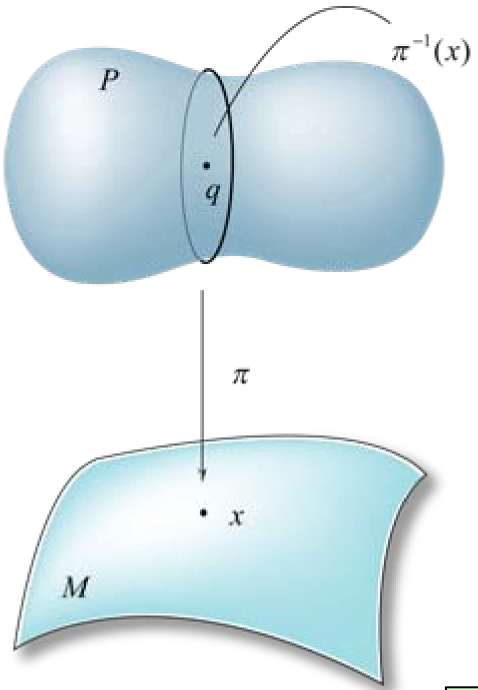
Gyro-symmetry

Geometric gyrokinetic theory

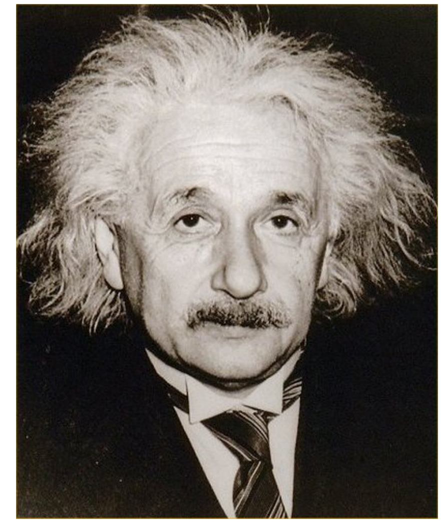
Geometric field theory from Poincare-Cartan-Einstein 1-form γ



Where does γ live? -- phase space



Phase space is a fiber bundle $\pi : P \longrightarrow M$.



Cotagent bundle

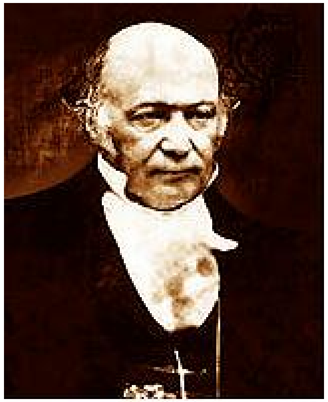
$$P = \{(x,p)\,|\, x \in M,\, p \in T^*_x M,\, g^{-1}(p,p) = -m^2c^2\}$$

7D Spacetime

Inverse of metric

Poincare-Cartan-Einstein 1-form $\gamma \to$ particle dynamics

Hamilton's Eq.



Inner product

Exterior derivative

$$i_\tau d\gamma = 0$$

Worldline

Different for different particles: charged particle, neutron, polymer, virus, ...

Geometrically generalized Vlasov-Maxwell system --- A field theory

Liouville 6-form

$$\Omega \equiv -\frac{1}{6} d\gamma \wedge d\gamma \wedge d\gamma ,$$

$$L_\tau \Omega = i_\tau d\Omega + d(i_\tau \Omega) = 0.$$

Liouville Theorem

Vlasov Eq.

$$L_\tau f = i_\tau df = 0.$$

$$L_\tau (f\Omega) = (L_\tau f)\Omega + (L_\tau \Omega) f = 0.$$

Conservative form

Geometrically generalized Vlasov-Maxwell system --- A field theory

$$S = \int_x L,$$

$$L(x) = -\frac{1}{2} dA \wedge *dA + 4\pi \int_{\pi^{-1}(x)} f\Omega \wedge \gamma$$

Fiber integral

Variational derivative

$$\frac{\delta S}{\delta A} = E(L) = 0$$

Euler derivative

$$d * dA = 4\pi * j,$$

$$j^{\alpha}(x) = \int_{x'} \int_{\pi^{-1}(x')} f\Omega \wedge \frac{\delta \gamma(x')}{\delta A_{\alpha}(x)}, (\alpha = 0,1,2,3).$$

Second order field theory → 4-diamagnetic current

$$\boxed{\begin{aligned} j^{\alpha}(x) = &\int_{x'} \delta\left(x - x'\right) \int_{\pi^{-1}(x')} f\Omega \wedge \frac{\partial \gamma(x')}{\partial A_{\alpha}(x)} \\ &- \frac{\partial}{\partial x^{\beta}} \left[\int_{x'} \delta\left(x - x'\right) \int_{\pi^{-1}(x')} f\Omega \wedge \frac{\partial \gamma(x')}{\partial A_{\alpha,\beta}(x)} \right], \ (\alpha = 0,1,2,3). \end{aligned}}$$

4-diamagnetic current

$$\frac{\partial A_{\alpha}}{\partial x^{\beta}}$$

Valid for any γ. Exact conservation properties.

Allow physics models, approximations build into γ.

Example: 1-form for charge particles with Lorentz force

$$\gamma = A + p = (\boldsymbol{A} + \boldsymbol{v}) \cdot d\boldsymbol{x} - \left[\frac{v^2}{2} + \phi \right] dt \, ,$$

$A \equiv (-\phi, \boldsymbol{A})$, four vector potential, 1-form

$p \equiv (-v^2/2, \boldsymbol{p})$, 1-form momentum

Action

$$L(x) = -\frac{1}{2} dA \wedge *dA + 4\pi \int_{\pi^{-1}(x)} f\Omega \wedge \gamma$$

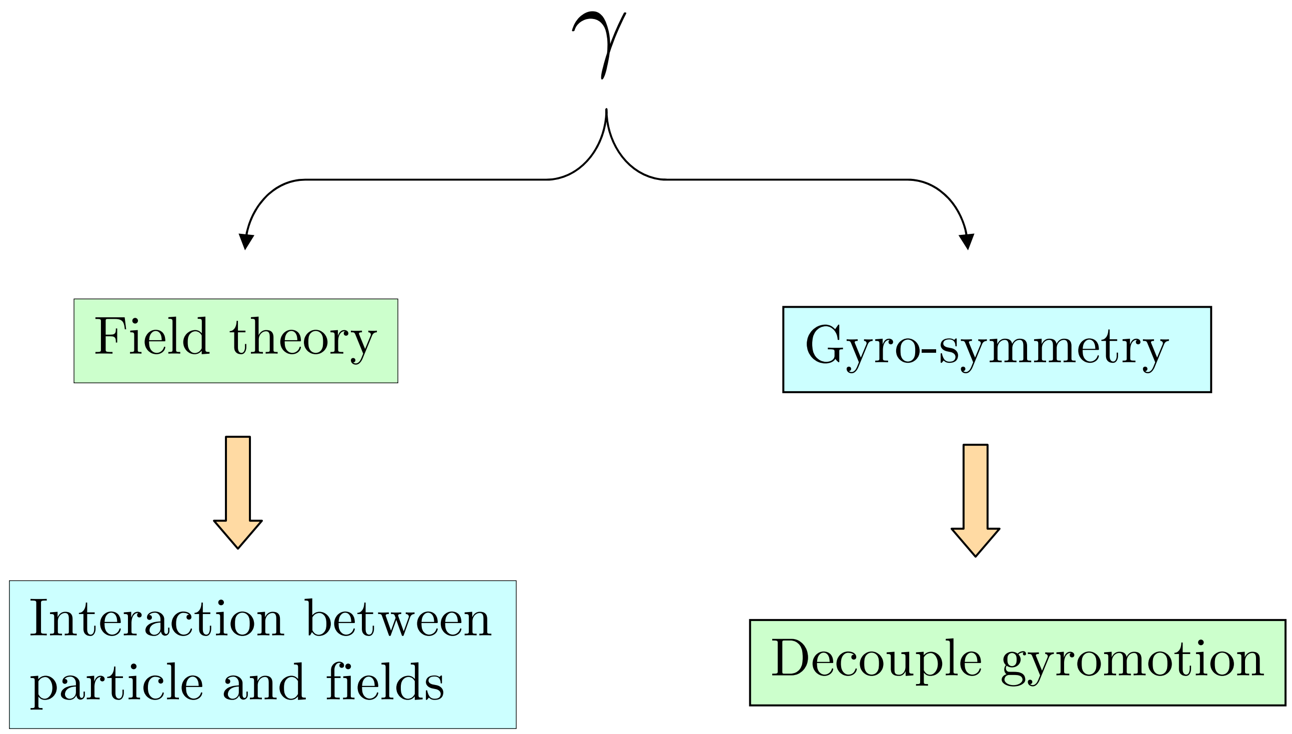
Vlasov

$$df(\tau) = 0, \; i_\tau d\gamma = 0$$

Maxwell

$$d * dA = 4\pi \int_{\pi^{-1}(x)} f\Omega$$

Geometric gyrokinetic theory = field theory + gyro-symmetry



What is gyrosymmetry?

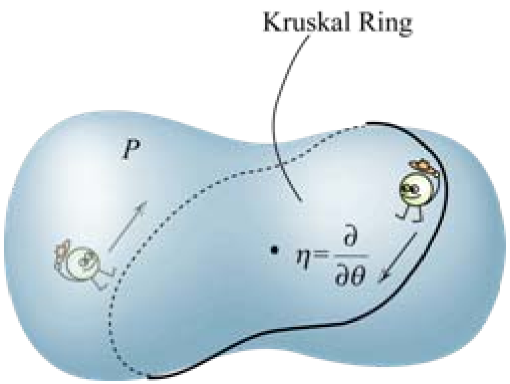
Noether's Theorem

$$L_\eta \gamma = dS$$

$$L_\eta \gamma = d\left(i_\eta \gamma\right) + i_\eta d\gamma = dS$$

$$d(\gamma \cdot \eta) \cdot \tau = dS \cdot \tau$$

$\gamma \cdot \eta - S$ is conversed.

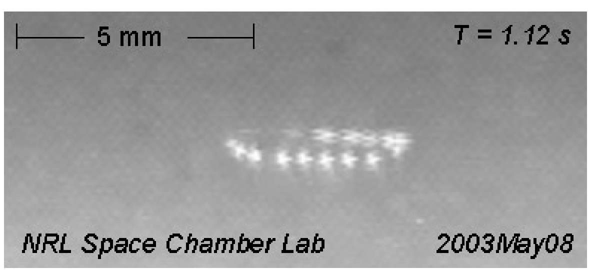


$$\eta = v_x \left(\frac{1}{B} \frac{\partial}{\partial x} + \frac{\partial}{\partial v_y} \right) + v_y \left(\frac{1}{B} \frac{\partial}{\partial y} - \frac{\partial}{\partial v_x} \right)$$

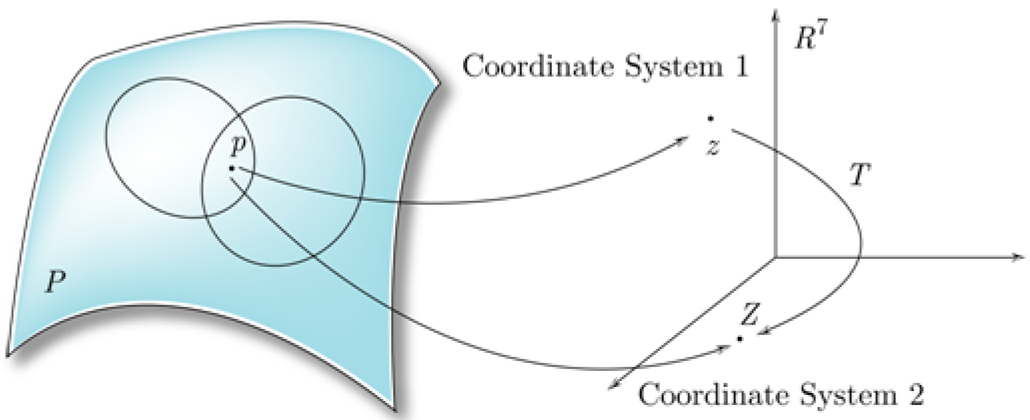
$$\mu = \frac{v_x^2 + v_y^2}{2B}$$

$$\eta = \frac{\partial}{\partial \theta}$$

Gyrophase coordinate



Dynamics under Lie group of coordinate transformation



Continuous Lie group,

$$g : z \mapsto Z = g(z,\varepsilon)$$

Vector field (Lie algbra),

$$G = dg/d\varepsilon\,\big|_{\varepsilon=0}\ ,$$

$$-G = dg^{-1}/d\varepsilon\,\big|_{\varepsilon=0}\ .$$

γ in Z

Pullback

Taylor expansion

$$\Gamma(Z) = g^{-1*}\gamma(z) = \gamma\left[g^{-1}(Z)\right] = \gamma(Z) - L_{G(Z)}\gamma(Z) + O(\varepsilon^2)$$

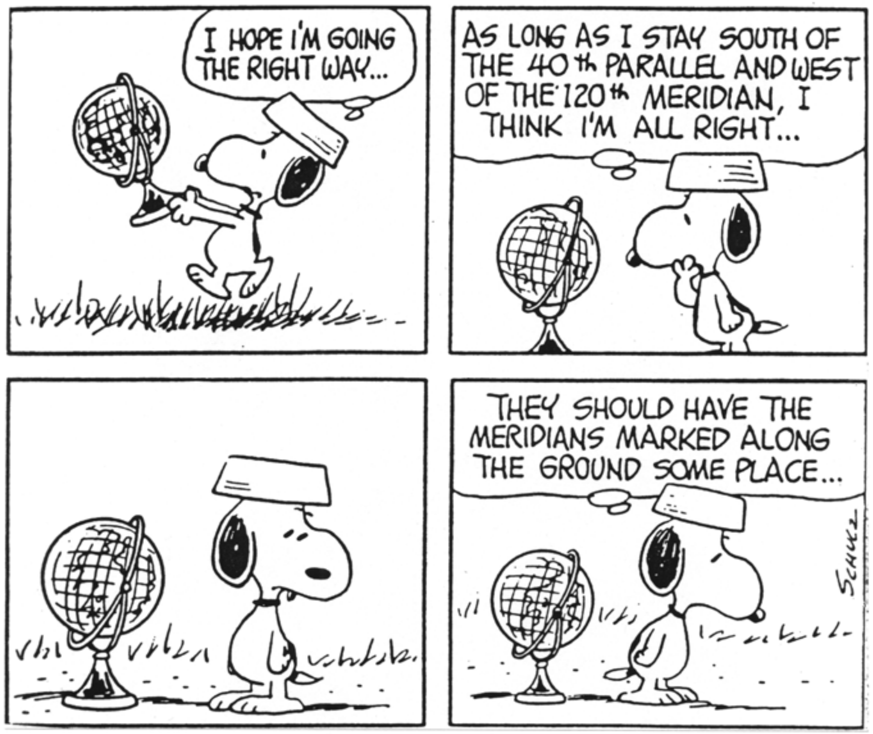
$$= \gamma(Z) - i_{G(Z)}d\gamma(Z) - d\left[\gamma\cdot G(Z)\right] + O(\varepsilon^2)\,.$$

Cartan's formula

Insignificant

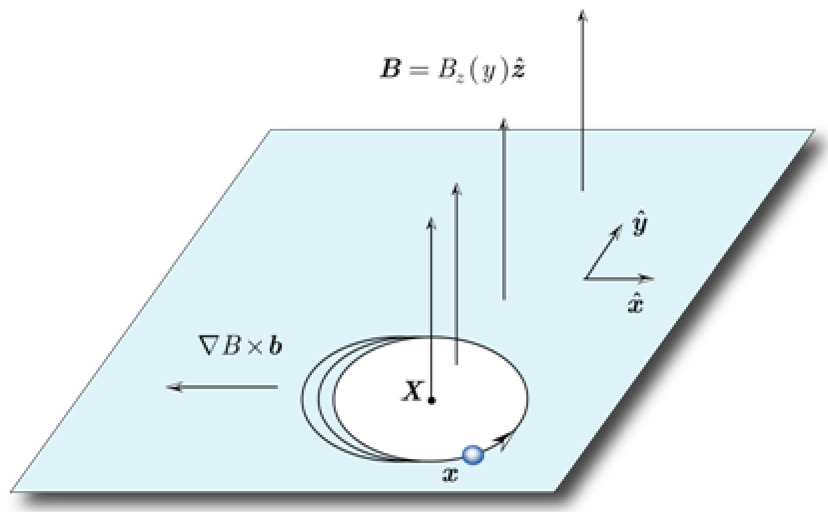
No need for Poisson bracket

Perturbation techniques — quest of good coordinates



Peanuts by Charles Schulz. Reprint permitted by UFS, Inc.

γ in the gyrocenter coordinate



$$\gamma = A + p = (\boldsymbol{A} + \boldsymbol{v}) \cdot d\boldsymbol{x} - \frac{v^2}{2} dt \,,$$

$$\boldsymbol{x} \equiv \boldsymbol{X} + \rho \,,$$
$$\boldsymbol{v} \equiv u\boldsymbol{b} + w\boldsymbol{c}$$
$$\rho = \frac{\boldsymbol{v}}{B} \times \boldsymbol{b}, \;\; \boldsymbol{b} \equiv \frac{\boldsymbol{B}}{B} \,,$$

$$\gamma = (\, A + u\boldsymbol{b}\,) \cdot d\boldsymbol{X} + \frac{w^2}{2B} d\theta - \left(\frac{u^2 + w^2}{2} \right) dt \,,$$

gyrocenter coordinate

Gyrocenter dynamics

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}$$

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$$

$$\boxed{i_\tau d\gamma = 0}$$

∇B drift

$$\frac{d\boldsymbol{X}}{dt} = \frac{\boldsymbol{B}^\dagger}{B_\parallel^\dagger}(u + \frac{\mu}{2}\boldsymbol{b} \cdot \boldsymbol{\nabla} \times \boldsymbol{b}) - \frac{\boldsymbol{b} \times \boldsymbol{E}^\dagger}{B_\parallel^\dagger}$$

$$\frac{du}{dt} = \frac{\boldsymbol{B}^\dagger \cdot \boldsymbol{E}^\dagger}{B_\parallel^\dagger}$$

$$\frac{d\theta}{dt} = B + \boldsymbol{R} \cdot \frac{d\boldsymbol{X}}{dt} - R + \frac{u}{2}\boldsymbol{b} \cdot \boldsymbol{\nabla} \times \boldsymbol{b}$$

$$\frac{d\mu}{dt} = 0$$

$$\boldsymbol{B}^\dagger \equiv \boldsymbol{\nabla} \times (\boldsymbol{A} + u\boldsymbol{b}), \; B_\parallel^\dagger = \boldsymbol{B}^\dagger \cdot \boldsymbol{b}$$

$$\boldsymbol{E}^\dagger \equiv \mu \boldsymbol{\nabla} B - u\frac{\partial \boldsymbol{b}}{\partial t}$$

Gyrokinetic equations

$$\boxed{L_{\tau} F = i_{\tau} dF = 0.}$$

$$\frac{dZ_j}{dt}\frac{\partial F}{\partial Z_j} = 0\,,\left(0 \le j \le 6\right).$$

$$F = \langle F \rangle.$$

$$\frac{\partial}{\partial \theta}\left(\frac{dZ}{dt}\right) = 0\,,$$

$$\frac{\partial \langle F \rangle}{\partial t} + \frac{d\mathbf{X}}{dt}\cdot \nabla_{\mathbf{x}} \langle F \rangle + \frac{du}{dt}\frac{\partial \langle F \rangle}{\partial u} = 0\,,$$

Gyrokinetic field theory

$$\Omega \equiv -\frac{1}{6} d\gamma \wedge d\gamma \wedge d\gamma ,$$

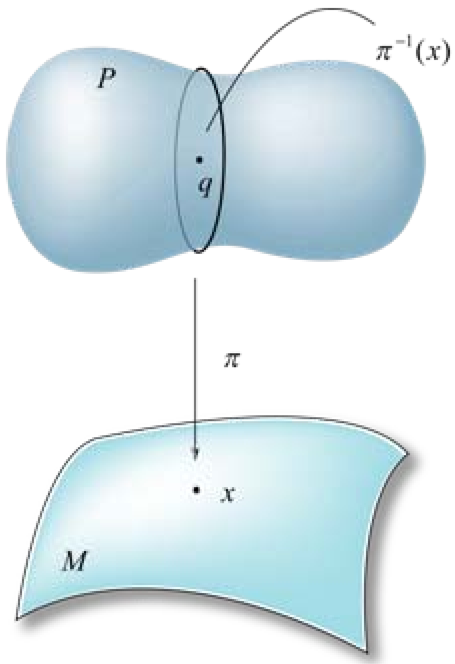
$$\begin{aligned}
\Omega = {} & B_{\|}^{\dagger} dX^1 \wedge dX^2 \wedge dX^3 \wedge du \wedge d\mu \wedge d\theta \\
& + \left[A^{\dagger}_{j,t} b_i - A^{\dagger}_{i,j} u - b_j H_{,i} \right] dt \wedge dX^j \wedge dX^i \wedge du \wedge d\mu \wedge d\theta \\
& + \left[A^{\dagger}_{i,j} H_{,l} + A^{\dagger}_{i,j} A^{\dagger}_{l,t} \right] dX^j \wedge dX^i \wedge dt \wedge d\mu \wedge d\theta \\
& - A^{\dagger}_{i,j} b_l H_{,\mu} dX^j \wedge dX^i \wedge dt \wedge du \wedge d\mu .
\end{aligned}$$

$\boxed{\gamma}$

$$d * dA = 4\pi * j ,$$

$$j^{\alpha}(x) = \int_{x'} \int_{\pi^{-1}(x')} F\Omega \wedge \frac{\delta \gamma(x')}{\delta A_{\alpha}(x)} , (\alpha = 0,1,2,3).$$

Conclusion



$+ \quad \gamma$

Geometric field theory

↓

Geometrically generalized Vlasov-Maxwell system

↓ Gyro-symmetry

Geometric gyrokinetic theory