Geometric Gyrokinetic Theory for Edge Plasmas

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http://www.pppl.gov/~hongqin/Gyrokinetics.php

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Motivations

Geometrically generalized Vlasov-Maxwell system ← Geometric field theory

↓

Geometic gyrokinetic theory ← Gyro-symmetry

↓

Gyrokinetic equations for edge ← Allow for dynamic background electromagnetic field

Classical gyrokinetics: average out gyrophase

- Magnetized plasmas → fast gyromotion.
- "Average-out" the fast gyromotion
  - Theoretically appealing
  - Algorithmically efficient

Highly oscillatory

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}+e_j\left(\mathbf{E}+\frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial}{\partial \mathbf{p}}\right]f_j(\mathbf{x},\mathbf{p},t)=0\ ,$$

$$\nabla\times\mathbf{B}=\frac{4\pi}{c}\sum_j e_j\int d^3p\,\mathbf{v}f_j(\mathbf{x},\mathbf{p},t)+\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}\ ,$$

$$\nabla\cdot\mathbf{E}=4\pi\sum_j e_j\int d^3p\,f_j(\mathbf{x},\mathbf{p},t)\ ,$$

$$\nabla\cdot\mathbf{B}=0.$$

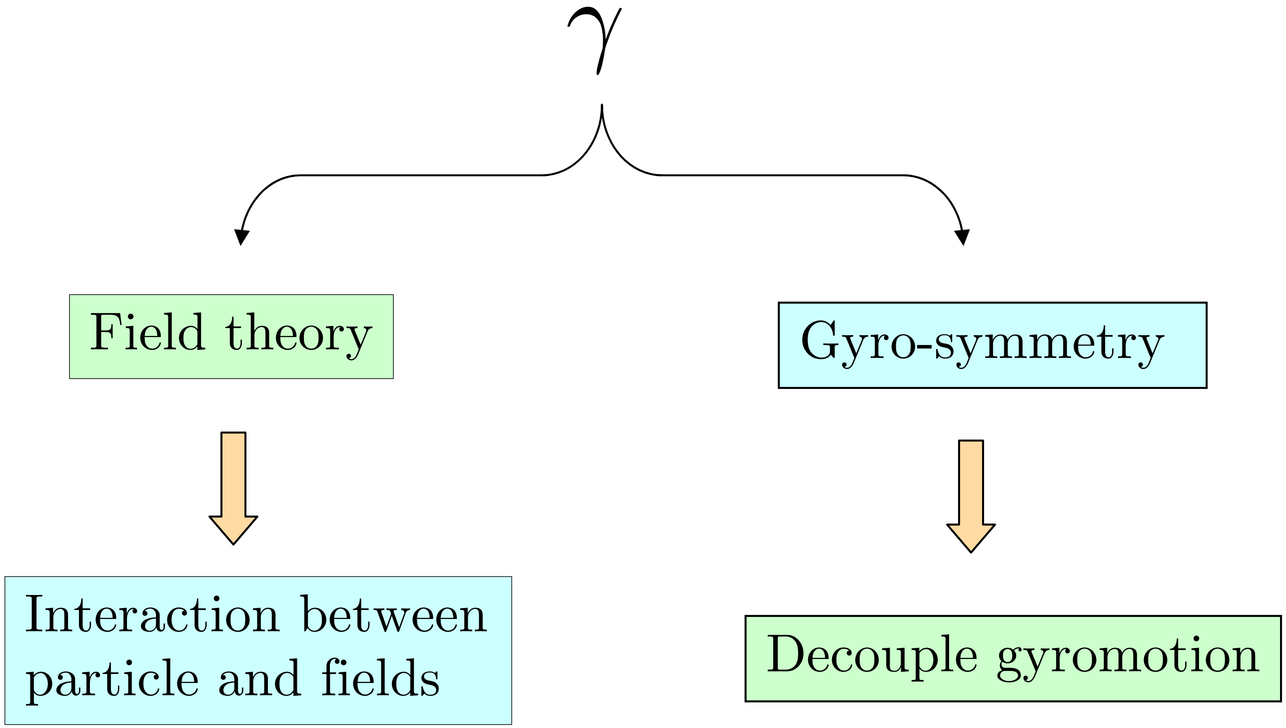
$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\,[\text{ Valsov} - \text{Maxwell eqs. }]$$

Does not work

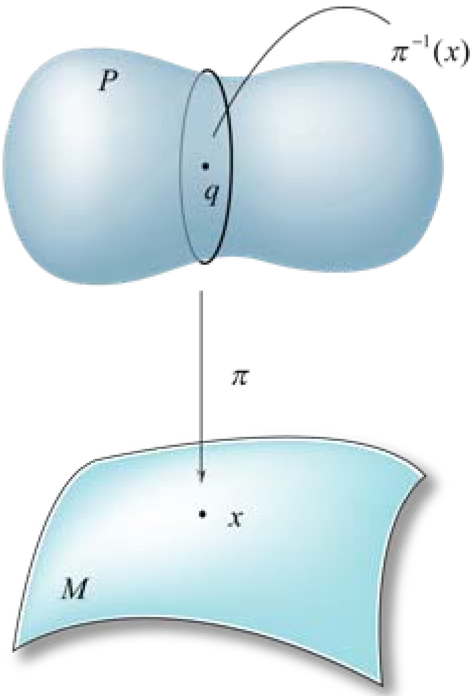
$$\left\langle \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}}\right\rangle,\ \left\langle \frac{\mathbf{v}\times\mathbf{B}}{c}\cdot\frac{\partial f}{\partial \mathbf{p}}\right\rangle ?$$

θ dependent

Modern gyrokinetics = field theory + gyro-symmetry



What is the phase space?



Phase space is a fiber bundle $\pi : P \longrightarrow M$.

7D

Cotagent bundle

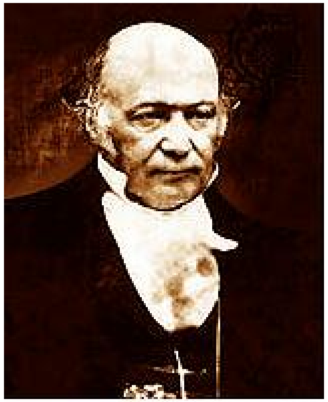
$$P = \{(x,p)\,|\,x \in M,\, p \in T_x^*M,\, g^{-1}(p,p) = -m^2c^2\}$$

Spacetime

Inverse of metric

Poincare-Cartan-Einstein 1-form $\gamma \rightarrow$ particle dynamics

Hamilton's Eq.



Inner product

Exterior derivative

$$i_T d\gamma = 0$$

Worldline

Different for different particles: charged particle, neutron, polymer, virus, ...

Geometrically generalized Vlasov-Maxwell system --- A field theory

Liouville 6-form

$$\Omega \equiv -\frac{1}{3!m^3} d\gamma \wedge d\gamma \wedge d\gamma ,$$

$$L_\tau \Omega = i_\tau d\Omega + d\left(i_\tau \Omega\right) = 0.$$

Liouville Theorem

Vlasov Eq.

$$L_\tau f = i_\tau df = 0.$$

$$L_\tau \left(f\Omega\right) = \left(L_\tau f\right)\Omega + \left(L_\tau \Omega\right) f = 0.$$

Conservative form

Geometrically generalized Vlasov-Maxwell system --- A field theory

$$S = \int_x L,$$

$$L(x) = -\frac{1}{2} dA \wedge *dA + 4\pi \int_{\pi^{-1}(x)} f\Omega \wedge \gamma$$

Variational derivative

Fiber integral

$$\frac{\delta S}{\delta A} = E(L) = 0$$

Euler derivative

$$d * dA = 4\pi * j,$$

$$j^{\alpha}(x) = \int_{x'} \int_{\pi^{-1}(x')} f\Omega \wedge \frac{\delta\gamma(x')}{\delta A_{\alpha}(x)}, (\alpha = 0,1,2,3).$$

Second order field theory → 4-diamagnetic current

$$j^{\alpha}(x) = \int_{x'} \delta\left(x - x'\right) \int_{\pi^{-1}(x')} f\Omega \wedge \frac{\partial \gamma(x')}{\partial A_{\alpha}(x)}$$

$$-\frac{\partial}{\partial x^{\beta}}\left[\int_{x'} \delta\left(x - x'\right) \int_{\pi^{-1}(x')} f\Omega \wedge \frac{\partial \gamma(x')}{\partial A_{\alpha,\beta}(x)}\right], (\alpha = 0,1,2,3).$$

4-diamagnetic current

$$\frac{\partial A_{\alpha}}{\partial x^{\beta}}$$

Valid for any γ. Exact conservation properties.

Allow physics models, approximations build into γ.

Example: 1-form for charge particles with Lorentz force

$\boxed{\times dt}$

$$L = \mathbf{A}\cdot\mathbf{v} + \frac{1}{2}v^2 - \phi = (\mathbf{A}+\mathbf{v})\cdot\mathbf{v} - \left(\frac{1}{2}v^2 + \phi\right)$$

$$\gamma = A + p = (\mathbf{A}+\mathbf{v})\cdot d\mathbf{x} - \left[\frac{v^2}{2} + \phi\right]dt\ ,$$

$A \equiv (-\phi, \mathbf{A})$, four vector potential, 1-form

$p \equiv (-v^2/2, \mathbf{p})$, 1-form momentum

Action

$$L(x) = -\frac{1}{2}dA \wedge *dA + 4\pi\int_{\pi^{-1}(x)} f\Omega \wedge \gamma$$

Vlasov

$$df(\tau) = 0,\ i_\tau d\gamma = 0$$

Maxwell

$$d * dA = 4\pi\int_{\pi^{-1}(x)} f\Omega$$

What is symmetry?

- Coordinate dependent version:

$$\frac{\partial L}{\partial \theta} = 0\,, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0\,.$$

Problem, what is θ?

- Geometric version:

S. Lie (1890s)

Symmetry is group

$$L_\eta \gamma = dS$$

Lie derivative

Symmetry vector field

Advantage: general, stronger, enables techniques to find θ.

Symmetry is invaraint

Noether's Theorem (1918)

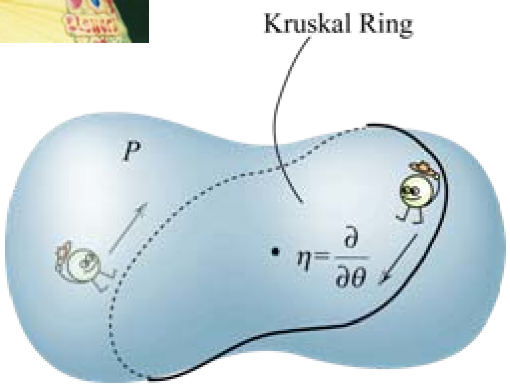
Cartan's formula

$$L_\eta \gamma = d\left(i_\eta \gamma\right) + i_\eta d\gamma = dS$$

$$d(\gamma \cdot \eta) \cdot \tau = dS \cdot \tau$$

$\gamma \cdot \eta - S$ is conversed.

What is gyrosymmetry?



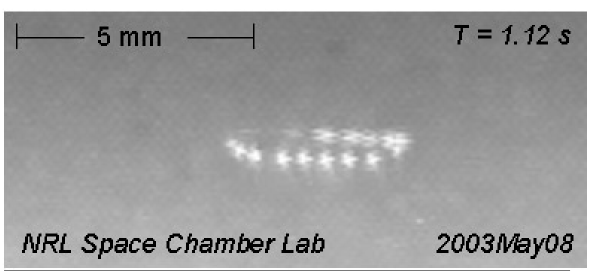
$$\eta = v_x\left(\frac{1}{B}\frac{\partial}{\partial x} + \frac{\partial}{\partial v_y}\right) + v_y\left(\frac{1}{B}\frac{\partial}{\partial y} - \frac{\partial}{\partial v_x}\right)$$

$$\eta = \frac{\partial}{\partial\theta}$$

Noether's Theorem

$$\mu = \frac{v_x^2 + v_y^2}{2B}$$

Gyrophase coordinate



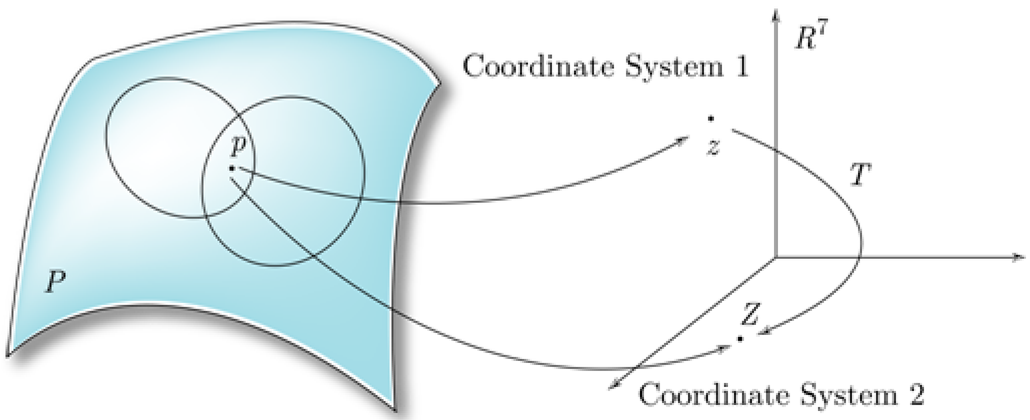
Amatucci, Pop 11, 2097 (2004).

Coordinate perturbation to find η.

Q: How does the dynamics transform?

A: It transforms as an 1-form!

Dynamics under Lie group of coordinate transformation



Continuous Lie group,

$$g : z \mapsto Z = g(z, \varepsilon)$$

Vector field (Lie algbra),

$$G = dg/d\varepsilon\,|_{\varepsilon=0}\ ,$$

$$-G = dg^{-1}/d\varepsilon\,|_{\varepsilon=0}\ .$$

γ in Z

Pullback

Taylor expansion

$$\Gamma(Z) = g^{-1*}\gamma(z) = \gamma\left[g^{-1}(Z)\right] = \gamma(Z) - L_{G(Z)}\gamma(Z) + O(\varepsilon^2)$$

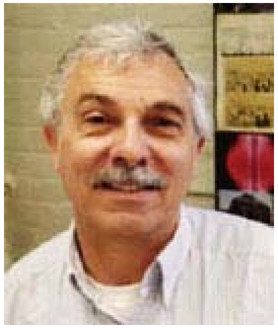
$$= \gamma(Z) - i_{G(Z)}d\gamma(Z) - d\left[\gamma \cdot G(Z)\right] + O(\varepsilon^2)\,.$$

Cartan’s formula

Insignificant

No need for Poisson bracket

0^{th} order gyrocenter coordinates



Catto, 1979

$$g_0 : z = (\mathbf{x}, \mathbf{v}, t) \mapsto \bar{Z} = (\bar{\mathbf{X}}, \bar{u}, \bar{w}, \bar{\theta}, t)$$

$$\mathbf{x} \equiv \bar{\mathbf{X}} + \rho(\bar{\mathbf{X}}, \mathbf{v}) \, , \, \bar{u} \equiv u(\bar{\mathbf{X}}, \mathbf{v}) \, , \, \bar{w} \equiv w(\bar{\mathbf{X}}, \mathbf{v}),$$

$$\sin\bar{\theta} \equiv -\mathbf{c}(\bar{\mathbf{X}}) \cdot \mathbf{e}_1(\bar{\mathbf{X}}) \, , \mathbf{v} = \mathbf{D}(\bar{\mathbf{X}}) + \bar{u}\mathbf{b}(\bar{\mathbf{X}}) + \bar{w}\mathbf{c}(\bar{\mathbf{X}}) \, .$$

$$u(y, \mathbf{v}_x)\mathbf{b}(y) \equiv [\mathbf{v}_x - \mathbf{D}(y)] \cdot \mathbf{b}(y) \; \mathbf{b}(y) \, ,$$

$$w(y, \mathbf{v}_x)\mathbf{c}(y, \mathbf{v}_x) \equiv [\mathbf{v}_x - \mathbf{D}(y)] \times \mathbf{b}(y) \times \mathbf{b}(y) \, ,$$

$$\mathbf{c}(y, \mathbf{v}_x) \cdot \mathbf{c}(y, \mathbf{v}_x) = 1,$$

$$\mathbf{a}(y, \mathbf{v}_x) \equiv \mathbf{b}(y) \times \mathbf{c}(y, \mathbf{v}_x) \, ,$$

$$\rho(y, \mathbf{v}_x) \equiv \frac{\mathbf{b}(y) \times [\mathbf{v}_x(y) - \mathbf{D}(y)]}{B_0(y)} .$$

$$\mathbf{D}(y) \equiv \frac{\mathbf{E}_0(y) \times \mathbf{B}_0(y)}{[B_0(y)]^2} \, , \quad \mathbf{b}(y) \equiv \frac{\mathbf{B}_0(y)}{B_0(y)} \, ,$$

$$\mathbf{v}_x(y) \equiv \mathbf{D}(y) + u(y, \mathbf{v}_x)\mathbf{b}(y) + w(y, \mathbf{v}_x)\mathbf{c}(y, \mathbf{v}_x) \, .$$

Lie perturbations

Cary, Littlejohn 1980s

Noncanonical coordinates: $\bar{Z} = (\bar{\mathbf{X}}, \bar{u}, \bar{w}, \bar{\theta})$

$$\gamma = \bar{\gamma}_0 + \bar{\gamma}_1 + \bar{\gamma}_2 + \ldots,$$

$$\frac{\partial \bar{\gamma}_0}{\partial \bar{\theta}} = 0, \quad \frac{\partial \bar{\gamma}_1}{\partial \bar{\theta}} \neq 0,$$

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1, \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1,$$

$$\mathbf{E}_0 \sim \frac{\mathbf{v} \times \mathbf{B}_0}{c}, \ \mathbf{E}_1 \sim \varepsilon_1 \frac{\mathbf{v} \times \mathbf{B}_0}{c}, \ \mathbf{B}_1 \sim \varepsilon_1 \mathbf{B}_0,$$

$$\left(|\rho| \frac{\nabla E_0}{E_0}, \frac{1}{\Omega E_0} \frac{\partial E_0}{\partial t} \right) \sim \left(|\rho| \frac{\nabla B_0}{B_0}, \frac{1}{\Omega B_0} \frac{\partial B_0}{\partial t} \right) \sim \varepsilon_0 \, ,$$

$$\left(|\rho| \frac{\nabla E_1}{E_1}, \frac{1}{\Omega E_1} \frac{\partial E_1}{\partial t} \right) \sim \left(|\rho| \frac{\nabla B_1}{B_1}, \frac{1}{\Omega B_1} \frac{\partial B_1}{\partial t} \right) \sim 1 \, ,$$

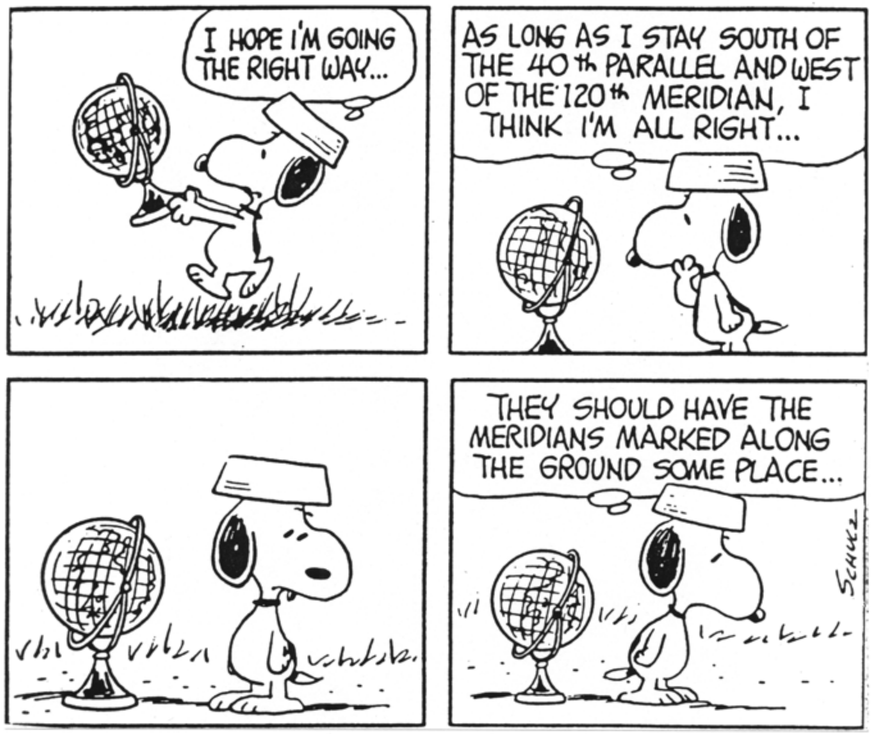
$$\begin{aligned} \gamma(Z) &= \gamma_0(Z) + \gamma_1(Z) + O(\varepsilon^2) \, , \\ \gamma_0(Z) &= \bar{\gamma}_0(Z) \, , \\ \gamma_1(Z) &= \bar{\gamma}_1(Z) - i_{G_1(Z)} d\gamma_0(Z) - dS_1 \, , \\ \gamma_2(Z) &= \bar{\gamma}_2(Z) - L_{G_1(Z)} \bar{\gamma}_1(Z) \\ &\quad + \left(\frac{1}{2} L^2_{G_1(Z)} - L_{G_2(Z)} \right) \gamma_0(Z) + dS_2 \, . \end{aligned}$$

$g: \ \bar{Z} \rightarrow Z = g(\bar{Z})$ such that $\partial\gamma / \partial\theta = 0$.

$G_1, \ G_2, \ S_1, \ S_2$

Gyrocenter gauges

Perturbation techniques — quest of good coordinates



Peanuts by Charles Schulz. Reprint permitted by UFS, Inc.

γ in the 0th order gyrocenter coordinates

$$\gamma = (\mathbf{A}+\mathbf{v})\cdot d\mathbf{x} - \left[\frac{v^2}{2}+\phi\right]dt = \bar{\gamma}_0 + \bar{\gamma}_1 + O(\varepsilon^2),$$

$$\bar{\gamma}_0 = \left(A_0 + \bar{u}\mathbf{b} + \mathbf{D}\right)\cdot d\bar{\mathbf{X}} + \frac{\bar{w}^2}{2B_0}d\bar{\theta} - \left(\frac{\bar{u}^2+\bar{w}^2+D^2}{2}+\phi_0\right)dt,$$

$$\begin{aligned}\bar{\gamma}_1 = &\left[\frac{\bar{w}}{B_0}\nabla\mathbf{a}\cdot\left(\bar{u}\mathbf{b}+\frac{\bar{w}\mathbf{c}}{2}\right)+\frac{1}{2}\rho\cdot\nabla\mathbf{B}_0\times\rho-\frac{\bar{w}}{B_0}\nabla\mathbf{D}\cdot\mathbf{a}+\mathbf{A}_1(\bar{\mathbf{X}}+\rho)\right]\cdot d\bar{\mathbf{X}}\\
&+\left[-\frac{\bar{w}^3}{2B_0^3}\mathbf{a}\cdot\nabla\mathbf{B}_0\cdot\mathbf{b}+\frac{\bar{w}}{B_0}\mathbf{A}_1(\bar{\mathbf{X}}+\rho)\cdot\mathbf{c}\right]d\bar{\theta}+\left[\frac{1}{\bar{w}}\mathbf{A}_1(\bar{\mathbf{X}}+\rho)\cdot\mathbf{a}\right]d\bar{\mu}\\
&-\left[\phi_1(\bar{\mathbf{X}}+\rho)+\rho\cdot\frac{\partial\mathbf{D}}{\partial t}-\frac{1}{2}\rho\cdot\nabla\mathbf{E}_0\cdot\rho-\left(\bar{u}\mathbf{b}+\frac{\bar{w}\mathbf{c}}{2}\right)\cdot\frac{\bar{w}}{B_0}\frac{\partial\mathbf{a}}{\partial t}\right]dt.\end{aligned}$$

$$Z = g_1(\bar{Z},\varepsilon),\ \frac{dg_1}{d\varepsilon}\Big|_{\varepsilon=0} = G_1(\bar{Z}),$$

$$\gamma_1(Z) = \bar{\gamma}_1(Z) - i_{G_1(Z)}d\gamma_0(Z) + dS_1(Z),$$

γ under coordinate perturbation

$$
\begin{aligned}
\gamma_1(Z) = \Big[& \mathbf{G}_{1\mathbf{X}} \times \mathbf{B}^{\dagger} - G_{1u}\mathbf{b} + \nabla S_1 + \frac{w}{B_0}\nabla\mathbf{a}\cdot\left(u\mathbf{b} + \frac{w\mathbf{c}}{2}\right) + \frac{1}{2}\rho\cdot\nabla\mathbf{B}_0\times\rho \\
& - \frac{w}{B_0}\nabla\mathbf{D}\cdot\mathbf{a} + \mathbf{A}_1(\mathbf{X}+\rho)\Big]\cdot d\mathbf{X} + \left[\mathbf{G}_{1\mathbf{X}}\cdot\mathbf{b} + \frac{\partial S_1}{\partial u}\right]du + \Big[G_{1\theta} + \frac{\partial S_1}{\partial \mu} + \\
& + \frac{1}{w}\mathbf{A}_1(\mathbf{X}+\rho)\cdot\mathbf{a}\Big]dw + \Big[-G_{1\mu} + \frac{\partial S_1}{\partial\theta} - \frac{w^3}{2B_0^3}\mathbf{a}\cdot\nabla\mathbf{B}_0\cdot\mathbf{b} \\
& + \frac{w}{B_0}\mathbf{A}_1(\mathbf{X}+\rho)\cdot\mathbf{c}\Big]d\theta + \Big[-\mathbf{E}_0^{\dagger}\cdot\mathbf{G}_{1\mathbf{X}} + uG_{1u} + B_0G_{1\mu} + \frac{\partial S_1}{\partial t} - \phi_1(\mathbf{X}+\rho) \\
& - \rho\cdot\frac{\partial\mathbf{D}}{\partial t} + \frac{1}{2}\rho\cdot\nabla\mathbf{E}_0\cdot\rho + \left(u\mathbf{b} + \frac{w\mathbf{c}}{2}\right)\cdot\frac{w}{B_0}\frac{\partial\mathbf{a}}{\partial t}\Big]dt\,.
\end{aligned}
$$

+

$\partial\gamma_1/\partial\theta = 0.$ $\Longrightarrow$ $G_1,\ S_1$

Freedoom
Gyro-center gauges

γ in the 1st order gyrocenter coordinate, $\partial\gamma/\partial\theta=0$,

$$\gamma(Z)=\gamma_0(Z)+\gamma_1(Z)\,,$$

$$\gamma_0=\left(A_0+u\mathbf{b}+\mathbf{D}\right)\cdot d\mathbf{X}+\frac{w^2}{2B_0}d\theta-\left[\frac{u^2+w^2+D^2}{2}+\phi_0\right]dt\,,$$

$$\gamma_1(Z)=-\frac{w^2}{2B_0}\mathbf{R}\cdot d\mathbf{X}-H_1dt\,,$$

Gyrocenter dynamics

$$H_1=\left(\mathbf{E}_{0\perp}^{\text{唵}}-\mathbf{B}_\perp\times u\mathbf{b}\right)\cdot\frac{w^2}{4B_0^2B_\parallel^{\square}}\nabla\mathbf{B}_0+\frac{w^2u}{4B_0}\mathbf{b}\cdot\nabla\times\mathbf{b}$$

$$-\frac{w^2}{4B_0^2}\left(\nabla\cdot\mathbf{E}_0-\mathbf{bb}:\nabla\mathbf{E}_0\right)-\frac{w^2}{2B_0}R_0+\left\langle\psi_1\right\rangle$$

$$\mathbf{R}\equiv\nabla\mathbf{c}\cdot\mathbf{a}\,,\;\;R_0\equiv-\frac{\partial\mathbf{c}}{\partial t}\cdot\mathbf{a}\,,$$

Gyro-gauge invariant

$$R\longrightarrow R'+\nabla\delta(X)\,,$$

$$\theta\longrightarrow\theta'+\delta(X)\,.$$

$$R=(R_0,\mathbf{R}),\quad X=(t,\mathbf{X}),$$

$$\nabla=(-\partial/\partial t,\nabla).$$

$$\psi_1\equiv\phi_1(\mathbf{X}+\rho)-\left(\frac{\mathbf{E}_0^\dagger\times\mathbf{b}+\mathbf{B}_\perp^\dagger u}{B_\parallel^\dagger}+u\mathbf{b}+w\mathbf{c}\right)\cdot\mathbf{A}_1(\mathbf{X}+\rho),$$

$$\langle\alpha\rangle\equiv\frac{1}{2\pi}\int_0^{2\pi}\alpha d\theta\,,\;\;\tilde{\alpha}\equiv\alpha-\langle\alpha\rangle\,.$$

$$\mathbf{B}^\dagger\equiv\nabla\times\left(\mathbf{A}_0+u\mathbf{b}+\mathbf{D}\right),$$

$$\mathbf{E}_0^\dagger\equiv\mathbf{E}_0-\nabla\left[\mu B_0+\frac{D^2}{2}\right]-u\frac{\partial\mathbf{b}}{\partial t}-\frac{\partial\mathbf{D}}{\partial t}\,.$$

2^{nd} order

$$\begin{aligned}
\gamma_2 &= -\langle\,\psi_2\,\rangle\, dt\,, \\
\psi_2 &\equiv \frac{1}{2}\mathbf{E}_{0\perp}\cdot\Big[\Big(\mathbf{G}_1^{\text{唵}}\times\mathbf{B}_1\Big)\times\mathbf{b}\Big] - \frac{1}{2}(\,u\mathbf{b}+w\mathbf{c}\,)\cdot\Big(\mathbf{G}_1\times\mathbf{B}_1\Big) + \mathbf{G}_1^{\text{唵}}\cdot\mathbf{E}_1 \\
\mathbf{G}_1^{\dagger} &\equiv \mathbf{G}_{1\mathbf{x}} + G_{1w}\frac{\partial}{\partial\mu} + G_{1\theta}\frac{\partial}{\partial\theta}, \\
\mathbf{E}_1^{\dagger} &\equiv -\nabla\phi_1 - \frac{\partial\mathbf{A}_1}{\partial t} - \nabla\langle\,\psi_1\,\rangle.
\end{aligned}$$

Gyrocenter dynamics

$i_\tau d\gamma = 0\,.$

Curvature drift

$\mathbf{E}\times\mathbf{B},\ \nabla B$ drift

∇D & curvature $\mathbf{D}$ drift

drift by spacetime inhomogeneties of $\mathbf{E}_0$

$$\frac{d\mathbf{X}}{dt} = \frac{\mathbf{B}^{*}}{B_\parallel^{*}}\left(u + \frac{\mu}{2}\mathbf{b}\cdot\nabla\times\mathbf{b}\right) - \frac{\mathbf{b}\times\mathbf{E}}{B_\parallel},$$

$$\frac{du}{dt} = \frac{\mathbf{B}^{*}\cdot\mathbf{E}}{B_\parallel^{\dagger}},$$

$$\frac{d\theta}{dt} = B_0 + \mathbf{R}\cdot\frac{d\mathbf{X}}{dt} - R_0 + \frac{\mathbf{E}_0\cdot\nabla B_0}{B_0^2} + \frac{u}{2}\mathbf{b}\cdot\nabla\times\mathbf{b}$$

$$-\frac{1}{2B_0}\left[\nabla\cdot\mathbf{E}_0 - \mathbf{bb}:\nabla\mathbf{E}_0\right] + \frac{\partial}{\partial\mu}\left\langle\psi_1+\psi_2\right\rangle$$

$$\frac{d\mu}{dt} = 0\,,\ \mu \equiv \frac{w^2}{2B_0},$$

Banos drift

$$\mathbf{B}^{*} \equiv \nabla\times\left(\mathbf{A}_0 + u\mathbf{b} + \mathbf{D}\right), B_\parallel^{*} = \mathbf{B}^{*}\cdot\mathbf{b},$$

$$\mathbf{E}^{\dagger} \equiv \mathbf{E}_0 - \nabla\left[\mu B_0 + \frac{D^2}{2} + \left\langle\psi_1+\psi_2\right\rangle\right] - u\frac{\partial\mathbf{b}}{\partial t} - \frac{\partial\mathbf{D}}{\partial t}$$

Gyrokinetic equations

$$\boxed{L_{\tau} F = i_{\tau} dF = 0.}$$

$$\frac{dZ_j}{dt}\frac{\partial F}{\partial Z_j} = 0\,, \left(0 \le j \le 6\right).$$

$$F = \left\langle F \right\rangle.$$

$$\boxed{\frac{\partial}{\partial \theta}\left(\frac{dZ}{dt}\right) = 0\,,}$$

$$\frac{\partial \left\langle F \right\rangle}{\partial t} + \frac{d\mathbf{X}}{dt}\cdot \nabla_{\mathbf{x}} \left\langle F \right\rangle + \frac{du}{dt}\frac{\partial \left\langle F \right\rangle}{\partial u} = 0\,,$$

Gyrokinetic field theory

$$\boxed{\Omega \equiv -\frac{1}{3!m^3} d\gamma \wedge d\gamma \wedge d\gamma \,,}$$

$$\downarrow$$

$$\begin{aligned}
\Omega &= B_{\|}^{\dagger} dx^1 \wedge dx^2 \wedge dx^3 \wedge du \wedge d\mu \wedge d\theta \\
&+\left[A^{\text{唵}}_{j,t} b_i - A_{i,j} u - b_j H_{,i} \right] dt \wedge dx^j \wedge dx^i \wedge du \wedge d\mu \wedge d\theta \\
&+\left[A^{\text{唵}}_{i,j} H_{,l} + A_{i,j} A_{l,t} \right] dx^j \wedge dx^i \wedge dt \wedge d\mu \wedge d\theta \\
&-A^{\dagger}_{i,j} b_l H_{,\mu} dx^j \wedge dx^i \wedge dt \wedge du \wedge d\mu \,.
\end{aligned}$$

γ

$$\downarrow$$

$$d * dA = 4\pi * j \,,$$

$$j^{\alpha}(x) = \int_{x'} \int_{\pi^{-1}(x')} F\Omega \wedge \frac{\delta\gamma(x')}{\delta A_{\alpha}(x)} \,, (\alpha = 0,1,2,3).$$

Gyrokinetic system without FLR effect

Model

$$\gamma = \mathbf{A}^{\dagger} \cdot d\mathbf{X} + \mu d\theta - H dt\,,$$

$$H = \frac{u^2 + w^2 + D^2}{2} + \phi_0\,.$$

$$\mathbf{j} = \mathbf{j}_g + \mathbf{j}_M\,,$$

Gyrocenter current

$$\mathbf{j}_g = \int_{x'} \delta\left(x - x'\right) \int_{\pi^{-1}(x')} F\Omega \wedge \frac{\partial \gamma(x')}{\partial A_i(x)}$$

$$= \int F \left(\frac{\mathbf{E}^{\text{唵}} \times \mathbf{b} + \mathbf{B}\ u}{B_{\parallel}^{\dagger}} \right) B_{\parallel}^{\dagger} du \wedge d\mu \wedge d\theta\,.$$

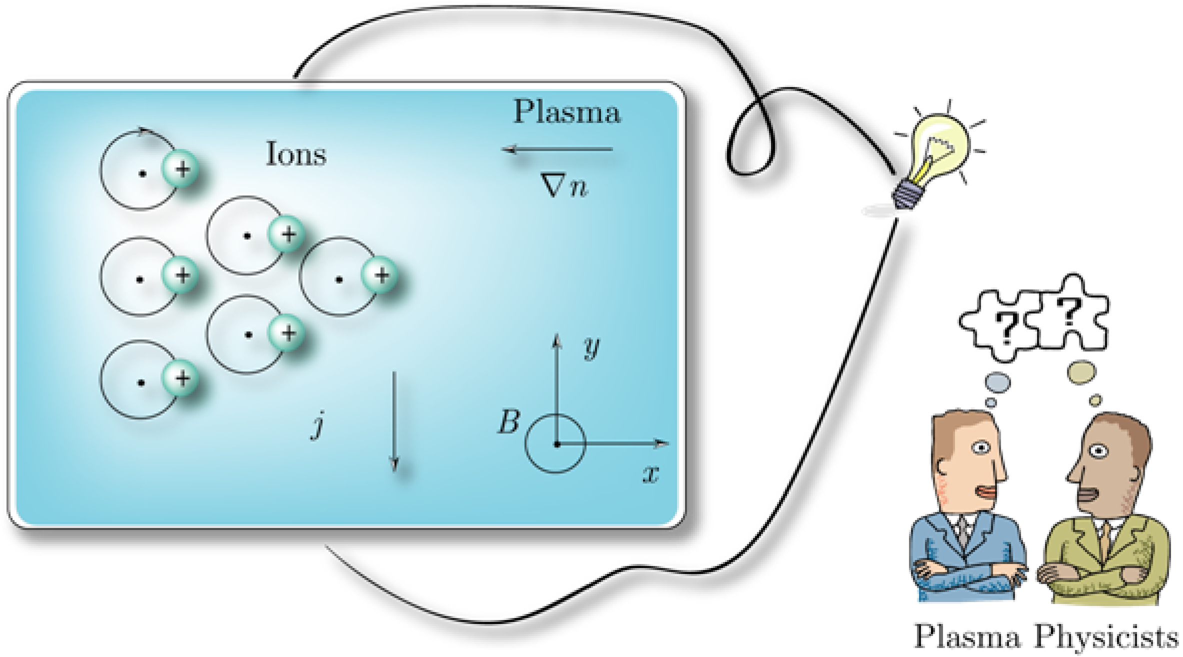
All drifts except Banos

Dia-magnetic current

$$\mathbf{j}_M = -\frac{\partial}{\partial x^j}\left[\int_{x'} \delta\left(x - x'\right) \int_{\pi^{-1}(x')} F\Omega \wedge \frac{\partial \gamma(x')}{\partial A_{i,j}(x)} \right]$$

$$= -\nabla \times \left[\mathbf{b} \int \mu F B_{\parallel}^{\dagger} du \wedge d\mu \wedge d\theta \right].$$

Physics of 2nd order field theory — Spitzer paradox



$$\boxed{\mathbf{j}_y = \underbrace{\mathbf{j}_g}_{0} + \mathbf{j}_M = \left(\mathbf{b} \times \frac{\nabla p}{B}\right)_y}$$

Gyrokinetic system with a time-independent backgroud

$$\gamma = \mathbf{A}^{\dagger} \cdot d\mathbf{X} + \mu d\theta - H dt,$$

$$H = \frac{u^2 + w^2 + D^2}{2} + \phi_0 + \langle \psi_1 \rangle.$$

- Exact energy conservation
- Inhomogeneous EM background

$$\langle \psi_1 \rangle = \langle \phi_1(\mathbf{X} + \rho) \rangle$$

$$s_1 = \frac{1}{B_0} \int \tilde{\phi}_1(\mathbf{X} + \rho) d\theta \equiv \frac{1}{B_0} \tilde{\phi}_1^{(1)},$$

$$\langle \psi_2 \rangle = -\frac{1}{2} \left\langle \nabla \tilde{\phi}_1 \cdot \nabla \tilde{\phi}_1^{(1)} \right\rangle \times \frac{\mathbf{b}}{B_\|^{\dagger} \bar{B}_0} - \frac{1}{2B_0} \left\langle \frac{\partial \tilde{\phi}_1^2}{\partial \mu} \right\rangle,$$

$$n(x) = \int_{x'} \delta\left(x - x'\right) \int_{\pi^{-1}(x')} F\Omega \wedge \left[\frac{\partial \langle \psi_1 \rangle}{\partial \phi_1(x)} + \frac{\partial \langle \psi_2 \rangle}{\partial \phi_1(x)} \right]$$

$$= \int \left[F + \frac{\partial F}{\partial \mu} \frac{\tilde{\phi}_1}{B_0} + \nabla F \times \frac{\nabla \tilde{\phi}_1^{(1)}}{B_\|^{\dagger} \bar{B}_0} \cdot \mathbf{b} + \frac{\partial F}{\partial u} \frac{\mathbf{B}^{\dagger} \cdot \nabla \tilde{\phi}_1^{(1)}}{B_\| \bar{B}_0} \right]$$

$$\times \delta\left[\mathbf{x} - \mathbf{x}' - \rho(\mathbf{x}') \right] B_\|^{\dagger} d^3x du d\mu d\theta .$$

Pullback transformation

Pullback of distribution function

Particle distribution

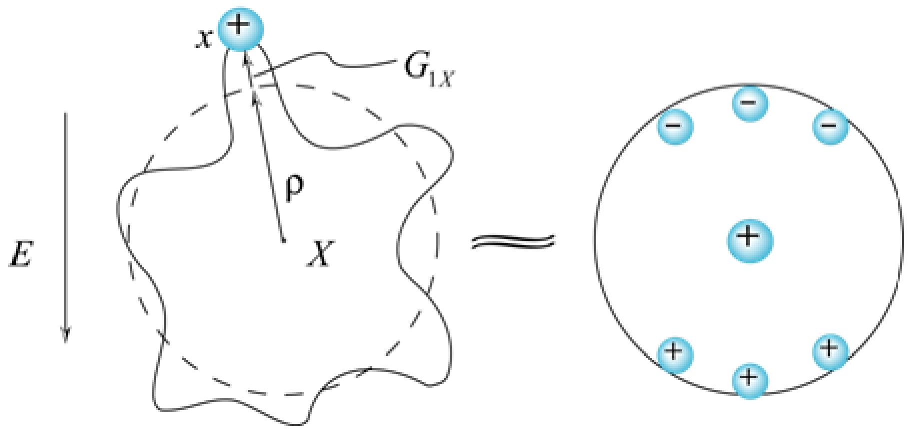
Don't know $f(z)$.

Gyrocenter distribution

pullback from $F(Z)$:

$$f(z) = g^* [F(Z)] = F(g(z)).$$

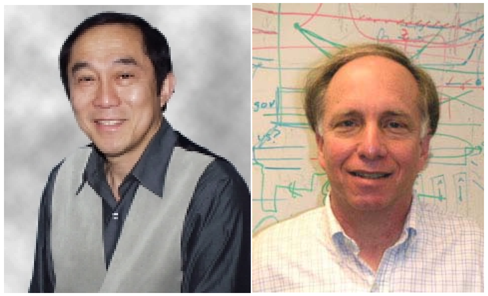
Physics of 2$^{\text{nd}}$ order field theory — polarization density



$$\boxed{x = X + \rho + G_{1X}\ ,}$$

$$\boxed{p = \frac{e}{2\pi}\int G_{1X}\Big|_{Z\to g_0(z)} d\theta = \frac{-e}{\Omega^2}\nabla_{\perp}\phi\,.}$$

$$\boxed{G_{1X} = \frac{1}{B}\times\nabla S_1\,.}$$



Lee (1983) used different method.
Friedman (1981) in different context.

Gyrokinetic system with a self-consistent time-dependent background

Two-field theory --- $\phi_0(x,t),\ \phi_1(x,t)$

need two field equations

$$\boxed{\nabla^2(\phi_0 + \phi_1) = -4\pi \sum_s en}$$

$$\frac{\delta\gamma}{\delta\phi_0} = 0 \longrightarrow$$

$$\boxed{\begin{aligned} n(x) = \int \Bigg\{ & \frac{\mathbf{E}_{0\perp}}{B_0}\left(1 - \frac{B_\parallel^\dagger}{B_0}\right) - \frac{1}{B_0} u\mathbf{b} \times \nabla \times (u\mathbf{b} + \mathbf{D}) \\ & - \frac{1}{B_0}\nabla_\perp\left[\mu B_0 + \frac{D^2}{2} + \langle\phi_1\rangle\right]\Bigg\} F du d\mu d\theta . \end{aligned}}$$

$$\frac{\delta\gamma}{\delta\phi_1} = 0 \longrightarrow$$

$$\boxed{\begin{aligned} n(x) &= \int_{x'} \delta\left(x - x'\right) \int_{\pi^{-1}(x')} F\Omega \wedge \left[\frac{\partial\langle\psi_1\rangle}{\partial\phi_1(x)} + \frac{\partial\langle\psi_2\rangle}{\partial\phi_1(x)}\right] \\ &= \int \left[F + \frac{\partial F}{\partial\mu}\frac{\tilde{\phi}_1}{B_0} + \nabla F \times \frac{\nabla\tilde{\phi}_1^{(1)}}{B_\parallel^{\text{唵}}\bar{B}_0} \cdot \mathbf{b} + \frac{\partial F}{\partial u}\frac{\mathbf{B}^\dagger \cdot \nabla\tilde{\phi}_1^{(1)}}{B_\parallel \bar{B}_0}\right] \\ &\quad \times \delta\left[\mathbf{x} - \mathbf{x}' - \rho(\mathbf{x}')\right] B_\parallel^\dagger d^3x du d\mu d\theta . \end{aligned}}$$

Conclusions