A new derivation of the plasma susceptibility tensor for a hot magnetized plasma without infinite sums of products of Bessel functions

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The susceptibility tensor of a hot, magnetized plasma is conventionally expressed in terms of infinite sums of products of Bessel functions. For applications where the particle's gyroradius is larger than the wavelength, such as alpha particle dynamics interacting with lower-hybrid waves, and the focusing of charged particle beams using a solenoidal field, the infinite sums converge slowly. In this paper, a new derivation of the plasma susceptibility tensor is presented which exploits a symmetry in the particle's orbit to simplify the integration along the unperturbed trajectories. As a consequence, the infinite sums appearing in the conventional expression are replaced by definite double integrals over one gyroperiod, and the cyclotron resonances of all orders are captured by a single term. Furthermore, the double integrals can be carried out and expressed in terms of Bessel functions of complex order, in agreement with expressions deduced previously using the Newburger sum rule. From this new formulation, it is straightforward to derive the asymptotic form of the full hot plasma susceptibility tensor for a gyrotropic but otherwise arbitrary plasma distribution in the large gyroradius limit. These results are of more general importance in the numerical evaluation of the plasma susceptibility tensor. Instead of using the infinite sums occurring in the conventional expression, it is only necessary to evaluate the Bessel functions once according to the new expression, which has significant advantages, especially when the particle's gyroradius is large and the conventional infinite sums converge slowly. Depending on the size of the gyroradius, the computational saving enabled by this representation can be several orders-of-magnitude. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2769968]

I. INTRODUCTION

The susceptibility tensor $\boldsymbol{\chi}$ of a hot, uniform, magnetized plasma is conventionally expressed[1] in terms of infinite sums of products of Bessel functions, i.e.,

$$\boldsymbol{\chi} = \frac{2\pi\omega_p^2}{\omega\Omega}\int_0^\infty p_\perp dp_\perp \int_{-\infty}^\infty dp_\parallel \left[\mathbf{e}_\parallel \mathbf{e}_\parallel \frac{\Omega}{\omega}\left(\frac{1}{p_\parallel}\frac{\partial f_0}{\partial p_\parallel} - \frac{1}{p_\perp}\frac{\partial f_0}{\partial p_\perp}\right)p_\parallel^2 + \sum_{n=-\infty}^{\infty} \frac{\Omega p_\perp U}{\omega - k_\parallel v_\parallel - n\Omega}\mathbf{T}_n \right], \tag{1}$$

where

$$\mathbf{T}_n \equiv \begin{pmatrix} \frac{n^2 J_n^2}{z^2} & \frac{inJ_nJ_n'}{z} & \frac{nJ_n^2 p_\parallel}{zp_\perp} \\ -\frac{inJ_nJ_n'}{z} & (J_n')^2 & -\frac{iJ_nJ_n'p_\parallel}{p_\perp} \\ \frac{nJ_n^2 p_\parallel}{zp_\perp} & \frac{iJ_nJ_n'p_\parallel}{p_\perp} & \frac{J_n^2 p_\parallel^2}{p_\perp^2} \end{pmatrix}, \tag{2}$$

and

$$z \equiv \frac{k_\perp v_\perp}{\Omega}, \tag{3}$$

$$U \equiv \frac{\partial f_0}{\partial p_\perp} + \frac{k_\parallel}{\omega}\left(v_\perp \frac{\partial f_0}{\partial p_\parallel} - v_\parallel \frac{\partial f_0}{\partial p_\perp}\right). \tag{4}$$

All symbols in Eqs. (1) and (2) have their usual meaning as defined in Ref. 1, i.e., $f_0(p_\perp, p_\parallel)$ is the particle distribution function with normalization $2\pi\int_{-\infty}^{\infty}dp_\parallel\int_0^\infty dp_\perp p_\perp f_0(p_\perp,p_\parallel)=1$; ω is the (complex) oscillation frequency with $\mathrm{Im}\,\omega>0$ corresponding to temporal growth; $\mathbf{k}=\mathbf{k}_\perp+k_\parallel\mathbf{e}_\parallel$ is the wave vector of the perturbations; $\mathbf{B}=B\mathbf{e}_\parallel$ is the uniform applied magnetic field; $\Omega=qB/mc$ is the gyrofrequency, where q and m are the particle charge and mass, respectively, and c is the speed of light *in vacuo*; and $\omega_p=(4\pi n_0 q^2/m)^{1/2}$, where n_0 is the number density. In addition, $J_n(z)$ is the Bessel function of the first kind of order n, and J_n' denotes $(d/dz)J_n(z)$.

The infinite sums in Eq. (1) converge with a reasonable speed for small gyroradius, i.e., $|z|\ll 1$. However, there are applications where the gyroradius is comparable to or larger than the wavelength. One well-known example involves alpha particle dynamics interacting with lower-hybrid waves.[2–8] Alpha particle dynamics plays an important role in the process of lower-hybrid current drive[9] and heating for burning plasmas. In this application, the gyroradius of the alpha particles is typically much larger than the wavelength of the lower-hybrid waves. Another example can be found in the focusing of charged particle beams by a solenoidal field in particle accelerators[10] and ion-beam-driven high energy density physics experimental devices.[11] In these systems, the gyroradius of the charged particles is comparable to the transverse size of the system, and larger than the wavelength

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of collective excitations with transverse mode numbers larger than one. For these applications with $|z| \gg 1$, it is not practical to use Eq. (1) to calculate the plasma susceptibility. This is because the infinite sums in Eq. (1) converge slowly for large z, which can be easily seen from the asymptotic form of $J_n(z)$ for large z,

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{1/2}\left[\cos\left(z - \frac{\nu}{2}\pi - \frac{\pi}{4}\right) - \frac{4\nu^2 - 1}{8z}\sin\left(z - \frac{\nu}{2}\pi - \frac{\pi}{4}\right)\right] + \cdots . \tag{5}$$

Equation (5) implies

$$\frac{J_{n+1}}{J_n} \sim \frac{(n+1)^2}{n^2} \sim 1,$$

$$\frac{\mathbf{T}_{n+1}}{\mathbf{T}_n} \sim 1,$$

for large z and n. Fortunately, this difficulty can be avoided by using the following surprising new sum rule (this is surprising considering the fact that Bessel functions had been extensively studied for three centuries and by the most talented mathematical minds) for products of Bessel functions discovered by Newberger in 1982[12]

$$\sum_{n=-\infty}^{\infty} \frac{J_n^2(z)}{a-n} = \frac{\pi}{\sin \pi a} J_{-a}(z) J_a(z). \tag{6}$$

Every infinite sum in Eqs. (1) and (2) can be reduced to a single term using Eq. (6) and its variations, as indicated previously by Swanson.[13]

In the standard derivation of the plasma susceptibility,[1] the infinite sums in Eq. (1) are brought into the calculation by adopting the expansion

$$\exp[iz \sin \phi] = \sum_{n=-\infty}^{\infty} \exp[in\phi] J_n(z) \tag{7}$$

for the purpose of carrying out the orbit integral along the unperturbed trajectories. According to Stix,[1] this technique was first introduced by Montgomery and Tidman.[14] However, if every infinite sum in the final expression for the susceptibility defined in Eq. (1) can be reduced to a single term, then the expansion in Eq. (7) may not be necessary after all, and an alternate approach may lead directly to the more compact form for the plasma dielectric tensor. In this paper, we show that this is indeed the case. We give a new derivation and a new expression for the plasma susceptibility without using infinite sums and Newberger's sum rule.

This new result is fundamentally due to a symmetry in the particle's orbit that can be exploited to simplify the integration along the unperturbed trajectories. This simplification replaces the necessity of using Eq. (7). A similar technique was used by Weiss to derive the plasma susceptibility for magnetized plasmas with an isotropic velocity distribution.[15] The derivation given in the present paper is for general *anisotropic* velocity distribution function, which is critical for many important applications to laboratory and space plasmas. A detailed comparison between our method and results with those of Ref. 15 will be given in Sec. III. The paper is organized as follows: In Sec. II, we describe the symmetry that simplifies the integration along unperturbed trajectories. In Sec. III, the derivation of the plasma susceptibility without using infinite sums is presented. As a simple but important application of the new result, the asymptotic form of the full hot plasma susceptibility for large z, is calculated for the first time for non-Maxwellian particle velocity distributions that are gyrotropic but otherwise arbitrary.

II. SYMMETRY IN INTEGRATION ALONG UNPERTURBED TRAJECTORIES

For the linearized Vlasov-Maxwell equations in a constant magnetic field $\mathbf{B} = B\mathbf{e}_z = B\mathbf{e}_\parallel$, the perturbed distribution function is obtained by integrating along the unperturbed orbits,[1]

$$f_1(\mathbf{r},\mathbf{p},t) = -qe^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\left\{\int_0^\infty d\tau e^{i\beta} E_x U \cos(\phi + \Omega\tau) + E_y U \sin(\phi + \Omega\tau) + E_z\left[\frac{\partial f_0}{\partial p_\parallel} - V\cos(\phi - \theta + \Omega\tau)\right]\right\}, \tag{8}$$

$$V \equiv \frac{k_\perp}{\omega}\left(v_\perp \frac{\partial f_0}{\partial p_\parallel} - v_\parallel \frac{\partial f_0}{\partial p_\perp}\right), \tag{9}$$

$$\beta \equiv -z[\sin(\phi - \theta + \Omega\tau) - \sin(\phi - \theta)] + \omega_k \tau, \tag{10}$$

$$\omega_k \equiv \omega - k_\parallel v_\parallel, \tag{11}$$

where $\mathbf{k} = k_\parallel \mathbf{e}_\parallel + k_\perp \cos\theta \mathbf{e}_x + k_\perp \sin\theta \mathbf{e}_y$, and $\mathrm{Im}\,\omega > 0$. Without loss of generality, we choose $\theta = 0$. The cases for $\theta \neq 0$ can be obtained easily by a rotation.[1] When $\theta = 0$, the three terms in the orbit integral in Eq. (8) can be reduced to a single term by means of the following equations:

$$g(\phi,z) \equiv \int_0^\infty \exp[-iz\sin(\phi + \Omega\tau) + i\omega_k\tau]d\tau = \frac{1}{\Omega}\int_0^\infty \exp[-iz\sin(\phi + s) + ias]ds, \tag{12}$$

$$s \equiv \Omega\tau, \quad a \equiv \frac{\omega_k}{\Omega} = \frac{\omega - k_\parallel v_\parallel}{\Omega}, \tag{13}$$

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$$\frac{\partial g}{\partial \phi} = \frac{1}{\Omega}\int_0^\infty \exp[-iz\sin(\phi+s)+ias][-iz\cos(\phi+s)]ds, \tag{14}$$

$$\frac{\partial g}{\partial z} = \frac{1}{\Omega}\int_0^\infty \exp[-iz\sin(\phi+s)+ias][-i\sin(\phi+s)]ds. \tag{15}$$

In terms of $g(\phi,z)$, the perturbed distribution function can be expressed as

$$f_1(\mathbf{r},\mathbf{p},t) = -qe^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}e^{iz\sin\phi}\left\{\frac{E_xU}{-iz}\frac{\partial g}{\partial\phi} + \frac{E_yU}{-i}\frac{\partial g}{\partial z} + E_z\left[\frac{\partial f_0}{\partial p_\parallel}g - \frac{V}{-iz}\frac{\partial g}{\partial\phi}\right]\right\}. \tag{16}$$

The standard approach in completing the orbit integral in Eq. (12) is to use Eq. (7) to expand it into an infinite sum of Bessel functions, and the resulting plasma susceptibility is given by Eq. (1). Here, we adopt a different approach by exploiting an important symmetry in Eq. (12). The symmetry of interest is the discrete symmetry associated with the definition of gyrophase ϕ, i.e.,

$$g(\phi,z) = g(\phi+2\pi,z). \tag{17}$$

From Eq. (17), we obtain

$$\begin{aligned} g(\phi,z) &= \frac{1}{\Omega}\int_0^\infty \exp[-iz\sin(\phi+s+2\pi) + ia(s+2\pi)]e^{-i2\pi a}ds \\ &= \frac{1}{\Omega}e^{-i2\pi a}\int_{2\pi}^\infty \exp[-iz\sin(\phi+t)+ia(t)]dt \\ &= e^{-i2\pi a}\left[g - \frac{1}{\Omega}\int_0^{2\pi}\exp[-iz\sin(\phi+t)+iat]dt\right], \end{aligned} \tag{18}$$

which gives

$$\begin{aligned} g(\phi,z) &= \frac{1}{\Omega}\frac{1}{1-e^{i2\pi a}}\int_0^{2\pi}\exp[-iz\sin(\phi+\gamma)+ia\gamma]d\gamma \\ &= \frac{c_0}{2\pi}\int_0^{2\pi}\exp[-iz\sin(\phi+\gamma)+ia\gamma]d\gamma, \end{aligned} \tag{19}$$

where

$$c_0 \equiv \frac{-e^{-ia\pi}\pi}{i\Omega\sin\pi a}. \tag{20}$$

There are two advantages of Eq. (19), compared with its conventional form using the infinite sum of Bessel functions. First of all, Eq. (19) replaces the infinite sum by a definite integral over one gyroperiod, whose numerical calculation can be much more efficient. Second, Eq. (19) explicitly displays the cyclotron resonances of all orders by the $\sin\pi a$ term in the denominator of c_0. The resonance condition is

$$\sin\pi a = 0, \quad \text{or equivalently } \omega - k_\parallel v_\parallel = n\Omega, \tag{21}$$

where n is an integer.

III. SUSCEPTIBILITY WITHOUT INFINITE SUMS AND THE ASYMPTOTIC FORM FOR LARGE *z*

To calculate the susceptibility, we need to take the velocity moment of f_1 to obtain the perturbed current in terms of the perturbed electric field. Some algebraic manipulation gives

$$\begin{aligned} \mathbf{j} &\equiv -\frac{i\omega}{4\pi}\boldsymbol{\chi}\cdot\mathbf{E} \\ &= q\int p_\perp dp_\perp dp_\parallel d\phi(v_\parallel\mathbf{e}_\parallel + v_\perp\cos\phi\mathbf{e}_x + v_\perp\sin\phi\mathbf{e}_y)f_1 \\ &= j_x\mathbf{e}_x + j_y\mathbf{e}_y + j_\parallel\mathbf{e}_\parallel, \end{aligned} \tag{22}$$

where

$$j_x = -e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}q^2 2\pi\int p_\perp dp_\perp dp_\parallel v_\perp\left[E_xUG_{33} + E_yUG_{32} + E_\parallel\left(\frac{\partial f_0}{\partial p_\parallel}G_{31} - VG_{33}\right)\right], \tag{23}$$

$$j_y = -e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}q^2 2\pi\int p_\perp dp_\perp dp_\parallel v_\perp\left[E_xUG_{23} + E_yUG_{22} + E_\parallel\left(\frac{\partial f_0}{\partial p_\parallel}G_{21} - VG_{23}\right)\right], \tag{24}$$

$$j_\parallel = -e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}q^2 2\pi\int p_\perp dp_\perp dp_\parallel v_\parallel\left[E_xUG_{13} + E_yUG_{12} + E_\parallel\left(\frac{\partial f_0}{\partial p_\parallel}G_{11} - VG_{13}\right)\right], \tag{25}$$

and

$$G_{ij} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{iz\sin\phi}\begin{pmatrix} g & i\frac{\partial g}{\partial z} & \frac{i}{z}\frac{\partial g}{\partial\phi} \\ g\sin\phi & i\frac{\partial g}{\partial z}\sin\phi & \frac{i}{z}\frac{\partial g}{\partial\phi}\sin\phi \\ g\cos\phi & i\frac{\partial g}{\partial z}\cos\phi & \frac{i}{z}\frac{\partial g}{\partial\phi}\cos\phi \end{pmatrix}. \tag{26}$$

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The susceptibility tensor $\boldsymbol{\chi}$ can therefore be expressed as

$$\boldsymbol{\chi} = \frac{2\pi\omega_p^2}{\omega\Omega}\int p_\perp dp_\perp dp_\parallel \mathbf{S}, \tag{27}$$

$$\mathbf{S} \equiv -i\Omega\begin{pmatrix} p_\perp UG_{33} & p_\perp UG_{32} & \frac{\partial f_0}{\partial p_\parallel}p_\perp G_{31} - p_\perp VG_{33} \\ p_\perp UG_{23} & p_\perp UG_{22} & \frac{\partial f_0}{\partial p_\parallel}p_\perp G_{21} - p_\perp VG_{23} \\ p_\parallel UG_{13} & p_\parallel UG_{12} & \frac{\partial f_0}{\partial p_\parallel}p_\parallel G_{11} - p_\parallel VG_{13} \end{pmatrix}. \tag{28}$$

The susceptibility χ given by Eq. (27) is expressed in terms of double definite integrals over one gyroperiod of the form $\int_0^{2\pi}d\gamma\int_0^{2\pi}d\phi\cdots$, whereas the conventional result is expressed in terms of infinite sums of products of Bessel functions. Obviously, Eq. (27) is preferable for the purpose of numerical calculation, especially in circumstances where the infinite sums in Eq. (1) converge slowly for large z.

It turns out that the double integrals of the form $\int_0^{2\pi}d\gamma\int_0^{2\pi}d\phi\cdots$ in every element of G can be carried out using the familiar integral representation of a single Bessel function,

$$J_m(x) = \frac{1}{2\pi}\int_0^{2\pi} d\alpha \exp[-im\alpha + ix\sin\alpha], \tag{29}$$

where m is an integer, and the following less familiar but famous integral representation of the products of Bessel functions due to Cauchy,[16]

$$J_{-\mu}(z)J_\mu(z) = \frac{2}{\pi}\int_0^{\pi/2} d\theta J_0(2z\cos\theta)\cos(2\mu\theta) = \frac{1}{2\pi}e^{i\mu\pi}\int_0^{2\pi} d\beta J_0\left(2z\sin\frac{\beta}{2}\right)e^{-i\mu\beta}. \tag{30}$$

For example, it follows that

$$\begin{aligned} G_{11} &= \frac{1}{2\pi}\int_0^{2\pi} d\phi g e^{iz\sin\phi} = \frac{c_0}{2\pi}\int_0^{2\pi} d\phi \frac{1}{2\pi}\int_0^{2\pi} d\gamma \exp[-iz\sin(\phi+\gamma) + ia\gamma + iz\sin\phi] \\ &= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{1}{2\pi}\int_0^{2\pi} d\phi \exp[-iz\sin(\phi+\gamma) + iz\sin\phi] \\ &= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{1}{2\pi}\int_0^{2\pi} d\phi \exp\left[2iz\sin\left(-\frac{\gamma}{2}\right)\cos\left(-\frac{\gamma}{2}-\phi\right)\right] = \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}J_0\left(2z\sin\frac{\gamma}{2}\right) = -\frac{\pi}{i\Omega\sin\pi a}J_{-a}(z)J_a(z), \end{aligned} \tag{31}$$

where $a=(\omega-k_\parallel v_\parallel)/\Omega$ and $z=k_\perp v_\perp/\Omega$. Detailed calculations of all the other elements of G are given in the Appendix. The final result is

$$\mathbf{G} = \begin{pmatrix} -\frac{\pi}{i\Omega\sin\pi a}J_{-a}J_a & -\frac{\pi}{2\Omega\sin\pi a}\frac{\partial}{\partial z}(J_{-a}J_a) & \frac{1}{iz\Omega}\left(1-\frac{\pi a}{\sin\pi a}J_{-a}J_a\right) \\ \frac{\pi}{2\Omega\sin\pi a}\frac{\partial}{\partial z}(J_{-a}J_a) & \frac{i}{\Omega}\left(\frac{\pi}{\sin\pi a}J'_{-a}J'_a + \frac{a}{z^2}\right) & \frac{\pi a}{2z\Omega\sin\pi a}\frac{\partial}{\partial z}(J_{-a}J_a) \\ \frac{1}{iz\Omega}\left(1-\frac{\pi a}{\sin\pi a}J_{-a}J_a\right) & -\frac{\pi a}{2z\Omega\sin\pi a}\frac{\partial}{\partial z}(J_{-a}J_a) & \frac{a}{iz^2\Omega}\left(1-\frac{\pi a}{\sin\pi a}J_{-a}J_a\right) \end{pmatrix},$$

or equivalently,

$$\mathbf{G} = \begin{pmatrix} -\frac{1}{i\Omega a}Q & -\frac{1}{2\Omega a}Q' & \frac{1}{iz\Omega}(1-Q) \\ \frac{1}{2\Omega a}Q' & \frac{i}{\Omega}\left(\frac{\pi}{\sin\pi a}J'_{-a}J'_a + \frac{a}{z^2}\right) & \frac{1}{2z\Omega}Q' \\ \frac{1}{iz\Omega}(1-Q) & -\frac{1}{2z\Omega}Q' & \frac{a}{iz^2\Omega}(1-Q) \end{pmatrix}, \tag{32}$$

where

$$Q \equiv \frac{\pi a}{\sin\pi a}J_{-a}(z)J_a(z), \quad Q' = \frac{\pi a}{\sin\pi a}\frac{\partial}{\partial z}(J_{-a}J_a). \tag{33}$$

The dependence on V in the last column of $\mathbf{S}$ in Eq. (28) can be factored out to give a compact expression for the plasma susceptibility, i.e.,

$$\chi = \frac{\omega_p^2}{\omega\Omega}\int 2\pi p_\perp dp_\perp dp_\parallel \times\left[\mathbf{e}_\parallel\mathbf{e}_\parallel\frac{\Omega}{\omega}\frac{p_\parallel}{p_\perp}\left(p_\perp\frac{\partial f_0}{\partial p_\parallel} - p_\parallel\frac{\partial f_0}{\partial p_\perp}\right) + p_\perp U\mathbf{T}\right], \tag{34}$$

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$$\mathbf{T} \equiv \begin{pmatrix} \frac{a}{z^2}(Q-1) & \frac{-i}{2z}Q' & \frac{1}{z}(Q-1)\frac{p_\parallel}{p_\perp} \\ \frac{i}{2z}Q' & \frac{\pi}{\sin \pi a}J'_{-a}J'_a + \frac{a}{z^2} & -\frac{ia}{2}Q' \\ \frac{1}{z}(Q-1)\frac{p_\parallel}{p_\perp} & \frac{i}{2a}Q' & \frac{Q}{a}\left(\frac{p_\parallel}{p_\perp}\right)^2 \end{pmatrix}. \tag{35}$$

To evaluate the plasma susceptibility according to Eqs. (34) and (35), it is only necessary to evaluate the Bessel function factors once, whereas the infinite sums of products of Bessel functions are needed to be calculated if using the conventional expression in Eq. (1). Equations (1) and (35) are particularly advantageous when $|z| \gg 1$ and the infinite sums converge slowly. Depending on the value of z, the computational savings enabled by using this representation can be several orders-of-magnitude.

As discussed in Sec. I, Weiss[15] used a somewhat similar method to calculate the plasma susceptibility for isotropic velocity distributions. In Ref. 15, a solution of the Vlasov equation is guessed without integration along unperturbed orbits. By contrast, the method we have adopted is to start from an integration along unperturbed orbits and reduce the integration to one gyroperiod by utilizing a symmetry intrinsic to the orbits of the particles. Obviously, this method is more systematic and transparent. In addition, the susceptibility tensor in Ref. 15 is expressed in terms of polarized components, whereas our results are expressed in conventional Cartesian coordinates, which can be easily compared with the conventional form using infinite sums of Bessel functions.

To demonstrate a simple but important application of the result given in Eq. (34), we calculate the asymptotic form of the plasma susceptibility for $|z| \to \infty$. It is only necessary to determine the asymptotic form of $\mathbf{T}$ for large z, which can be easily calculated from the asymptotic form of $J_a(z)$ displayed in Eq. (5). We obtain

$$\mathbf{T} \sim \frac{\mathbf{T}_1}{z} + \frac{\mathbf{T}_2}{z^2} + \cdots, \tag{36}$$

where

$$\mathbf{T}_1 = \begin{pmatrix} 0 & 0 & -\frac{p_\parallel}{p_\perp} \\ 0 & \frac{A}{a} & -\frac{iA}{\pi a}\cos 2z \\ -\frac{p_\parallel}{p_\perp} & \frac{iA}{\pi a}\cos 2z & \frac{A}{\pi a}(\sin 2z + \cos a\pi)\left(\frac{p_\parallel}{p_\perp}\right)^2 \end{pmatrix}, \tag{37}$$

$$\mathbf{T}_2 = \begin{pmatrix} a & -\frac{iA}{\pi}\cos 2z & \frac{A}{\pi}(\sin 2z + \cos a\pi)\frac{p_\parallel}{p_\perp} \\ \frac{iA}{\pi}\cos 2z & -\frac{A(4a^2+3)}{4\pi a}\cos 2z + 1 & \frac{iA}{2\pi a}(4a^2 \sin 2z + \cos a\pi) \\ \frac{A}{\pi}(\sin 2z + \cos a\pi)\frac{p_\parallel}{p_\perp} & -\frac{iA}{2\pi a}(4a^2 \sin 2z + \cos a\pi) & \frac{A(4a^2-1)}{4\pi a}\cos 2z\left(\frac{p_\parallel}{p_\perp}\right)^2 \end{pmatrix}, \tag{38}$$

and

$$A \equiv \frac{\pi a}{\sin \pi a}. \tag{39}$$

What is retained in Eq. (36) are the two leading orders of magnetic field effects for particles with large gyroradius. Obviously, this result is not accessible from the conventional expression for $\boldsymbol{\chi}$ in Eq. (1) using infinite sums.

IV. CONCLUSIONS AND FUTURE WORK

We have shown that the susceptibility $\boldsymbol{\chi}$ of a hot, magnetized plasma can be derived without using infinite sums of Bessel functions. The infinite sums appearing in the conventional expression for $\boldsymbol{\chi}$ are replaced by definite double integrals over one gyroperiod. Furthermore, the double integrals can be carried out and expressed in terms of Bessel functions of complex order. These results are of importance for the numerical evaluation of the plasma susceptibility tensor. Instead of using the infinite sums over Bessel functions according to the conventional expression in Eq. (1), it is only necessary to evaluate the Bessel functions once according to the new result given in Eq. (34). For applications with large z, such as alpha-particle dynamics interacting with low-hybrid waves, and the focusing of charged particle beams using a solenoidal magnetic field, the infinite sums in Eq. (1) converge slowly, and the new results in Eqs. (27)–(34) obviously have significant advantages. From Eq. (34), it is straightforward to derive the asymptotic form for the plasma susceptibility for large z, which is not accessible from the conventional representation of $\boldsymbol{\chi}$ in terms of infinite sums of products of Bessel functions. Previous treatments of the large

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$k_\perp \rho$ asymptotic limit were focused on electrostatic waves for thermal distributions of particles.[13,17,18]

The basic technique developed in this paper may be applicable to other plasma physics problems as well. In particular, we expect that calculations in gyrokinetic theory for general plasma waves[19–21] can be significantly simplified using similar methods.

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APPENDIX: CALCULATION OF THE *G* MATRIX

The double integral over $\int_0^{2\pi} d\phi \int_0^{2\pi} d\gamma \cdots$ for every element of **G** can be carried out and expressed in terms of Bessel functions. For example, G_{11} is given by Eq. (31). All the other elements of **G** can be calculated by using similar methods. Some straightforward algebra gives

$$\begin{aligned}
G_{21} &= \frac{1}{2\pi}\int_0^{2\pi} d\phi g \sin\phi e^{iz\sin\phi} = \frac{c_0}{2\pi}\int_0^{2\pi} d\phi \frac{1}{2\pi}\int_0^{2\pi} d\gamma \exp[-iz\sin(\phi+\gamma)+ia\gamma+iz\sin\phi]\sin\phi \\
&= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{1}{2\pi}\int_0^{2\pi} d\psi \exp\left[2iz\sin\left(-\frac{\gamma}{2}\right)\cos\psi\right]\sin\left(\psi-\frac{\gamma}{2}\right) \\
&= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{1}{2\pi}\int_0^{2\pi} d\psi \exp\left[2iz\sin\left(-\frac{\gamma}{2}\right)\cos\psi\right]\cos\psi\sin\frac{-\gamma}{2} \\
&= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{1}{2i}\frac{\partial}{\partial z}\left\{\frac{1}{2\pi}\int_0^{2\pi} d\psi \exp\left[2iz\sin\left(-\frac{\gamma}{2}\right)\cos\psi\right]\right\} = \frac{\pi}{2\Omega\sin\pi a}\frac{\partial}{\partial z}[J_{-a}(z)J_a(z)],
\end{aligned} \tag{A1}$$

$$\begin{aligned}
G_{12} &= \frac{i}{2\pi}\int_0^{2\pi} d\phi \frac{\partial g}{\partial z} e^{iz\sin\phi} = \frac{\partial}{\partial z}\left[\frac{i}{2\pi}\int_0^{2\pi} d\phi g e^{iz\sin\phi}\right] - \frac{i}{2\pi}\int_0^{2\pi} d\phi g e^{iz\sin\phi} i\sin\phi = \frac{\partial}{\partial z}[iG_{11}] + G_{21} = \frac{i}{2}\frac{\partial}{\partial z}G_{11} \\
&= -\frac{\pi}{2\Omega\sin\pi a}\frac{\partial}{\partial z}[J_{-a}(z)J_a(z)] = -G_{21},
\end{aligned} \tag{A2}$$

$$\begin{aligned}
G_{31} &= \frac{1}{2\pi}\int_0^{2\pi} d\phi g \cos\phi e^{iz\sin\phi} = \frac{c_0}{2\pi}\int_0^{2\pi} d\phi \frac{1}{2\pi}\int_0^{2\pi} d\gamma \exp[-iz\sin(\phi+\gamma)+ia\gamma+iz\sin\phi]\cos\phi \\
&= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{1}{2\pi}\int_0^{2\pi} d\psi \exp\left[2iz\sin\left(-\frac{\gamma}{2}\right)\cos\psi\right]\cos\left(\psi-\frac{\gamma}{2}\right) \\
&= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{1}{2\pi}\int_0^{2\pi} d\psi \exp\left[2iz\sin\left(-\frac{\gamma}{2}\right)\cos\psi\right]\cos\psi\cos\frac{\gamma}{2} \\
&= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{-1}{iz}\frac{\partial}{\partial\gamma}\left\{\frac{1}{2\pi}\int_0^{2\pi} d\psi \exp\left[2iz\sin\left(-\frac{\gamma}{2}\right)\cos\psi\right]\right\} \\
&= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{-1}{iz}\frac{\partial}{\partial\gamma}J_0\left(2z\sin\frac{\gamma}{2}\right) = \frac{1}{iz\Omega}\left[1-\frac{\pi a}{\sin\pi a}J_{-a}(z)J_a(z)\right],
\end{aligned} \tag{A3}$$

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and

$$G_{13} = \frac{i}{z}\frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{\partial g}{\partial \phi} e^{iz \sin \phi}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} d\phi g \cos \phi e^{iz \sin \phi} = G_{31}. \tag{A4}$$

To calculate G_{23}, G_{32}, G_{22}, G_{33} we make use of the following simple variations of Eq. (29):

$$J_0'(x) = \frac{i}{2\pi}\int_0^{2\pi} d\alpha \exp[ix \sin \alpha]\sin \alpha$$
$$= \frac{i}{2\pi}\int_0^{2\pi} d\alpha \exp[ix \cos \alpha]\cos \alpha, \tag{A5}$$

$$J_0''(x) = \frac{-1}{2\pi}\int_0^{2\pi} d\alpha \exp[ix \sin \alpha]\sin^2 \alpha$$
$$= \frac{-1}{2\pi}\int_0^{2\pi} d\alpha \exp[ix \cos \alpha]\cos^2 \alpha, \tag{A6}$$

and

$$J_0 = J_0\left(2z \sin \frac{\gamma}{2}\right), \quad J_0'' + \frac{J_0'}{2z \sin(\gamma/2)} + J_0 = 0, \tag{A7}$$

$$\frac{\partial J_0}{\partial z} = J_0' 2 \sin\left(\frac{\gamma}{2}\right), \tag{A8}$$

$$\frac{\partial^2 J_0}{\partial z^2} = J_0'' 4 \sin^2\left(\frac{\gamma}{2}\right), \tag{A9}$$

$$\frac{\partial J_0}{\partial \gamma} = J_0' z \cos\left(\frac{\gamma}{2}\right), \tag{A10}$$

$$\frac{\partial^2 J_0}{\partial \gamma^2} = J_0'' z^2 \cos^2\left(\frac{\gamma}{2}\right) - J_0' \frac{z}{2} \sin\left(\frac{\gamma}{2}\right), \tag{A11}$$

$$J_0'' \sin^2 \frac{\gamma}{2} = \frac{1}{4}\frac{\partial^2 J_0}{\partial z^2}, \tag{A12}$$

$$J_0'' \cos^2 \frac{\gamma}{2} = \frac{1}{z^2}\frac{\partial^2 J_0}{\partial \gamma^2} + \frac{1}{4z}\frac{\partial J_0}{\partial z}, \tag{A13}$$

$$\frac{\partial^2 J_0}{\partial z \partial \gamma} = J_0'' 2z \sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right) + \frac{\partial J_0}{\partial \gamma}\frac{1}{z}. \tag{A14}$$

In Eqs. (A7)–(A14) and in the subsequent analysis, the argument of J_0 is $2z \sin(\gamma/2)$. For the elements of G_{23}, G_{32}, G_{22}, and G_{33}, we obtain

$$G_{23} = \frac{c_0}{2\pi}\int_0^{2\pi} d\phi \frac{1}{2\pi}\int_0^{2\pi} d\gamma \exp[-iz \sin(\phi+\gamma) + ia\gamma + iz \sin \phi]\cos(\phi+\gamma)\sin \phi$$
$$= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{-1}{2\pi}\int_0^{2\pi} d\psi \exp\left[2iz \sin\left(-\frac{\gamma}{2}\right)\cos \psi\right]\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right)$$
$$= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}(-1)J_0 \sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right) = \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{1}{2z}\frac{\partial^2 J_0}{\partial z \partial \gamma} = \frac{1}{2z\Omega}\frac{\pi a}{\sin \pi a}\frac{\partial}{\partial z}[J_{-a}(z)J_a(z)] = \frac{a}{z}G_{21}, \tag{A15}$$

$$G_{32} = \frac{c_0}{2\pi}\int_0^{2\pi} d\phi \frac{1}{2\pi}\int_0^{2\pi} d\gamma \exp[-iz \sin(\phi+\gamma) + ia\gamma + iz \sin \phi]\sin(\phi+\gamma)\cos \phi$$
$$= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{-1}{2\pi}\int_0^{2\pi} d\psi \exp\left[2iz \sin\left(-\frac{\gamma}{2}\right)\cos \psi\right]\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right) = -G_{23}, \tag{A16}$$

$$G_{22} = \frac{c_0}{2\pi}\int_0^{2\pi} d\phi \frac{1}{2\pi}\int_0^{2\pi} d\gamma \exp[-iz \sin(\phi+\gamma) + ia\gamma + iz \sin \phi]\sin(\phi+\gamma)\sin \phi$$
$$= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{-1}{2\pi}\int_0^{2\pi} d\psi \exp\left[2iz \sin\left(-\frac{\gamma}{2}\right)\cos \psi\right]\left[1 + \frac{1}{2}\cos \gamma + \frac{1}{2}\cos 2\psi - 2\cos^2 \psi\right]$$
$$= G_{11} + G_{33} + \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{1}{2\pi}\int_0^{2\pi} d\psi \exp\left[2iz \sin\left(-\frac{\gamma}{2}\right)\cos \psi\right][-2\cos^2 \psi]$$

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$$= G_{11} + G_{33} + \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma} 2J_0'' = G_{11} + G_{33} + \frac{2c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\left[\frac{1}{4}\frac{\partial^2 J_0}{\partial z^2} + \frac{1}{z^2}\frac{\partial^2 J_0}{\partial \gamma^2} + \frac{1}{4z}\frac{\partial J_0}{\partial z}\right]$$

$$= G_{11} + G_{33} + 2c_0\frac{1}{4}\left[\frac{\partial^2}{\partial z^2} + \frac{1}{z}\frac{\partial}{\partial z}\right][e^{ia\pi}J_{-a}(z)J_a(z)] + \frac{1}{z^2}\frac{2c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{\partial^2 J_0}{\partial \gamma^2} = \frac{i}{\Omega}\left[\frac{\pi}{\sin \pi a}J_{-a}'(z)J_a'(z) + \frac{a}{z^2}\right], \tag{A17}$$

and

$$G_{33} = \frac{c_0}{2\pi}\int_0^{2\pi} d\phi \frac{1}{2\pi}\int_0^{2\pi} d\gamma \exp[-iz\sin(\phi+\gamma) + ia\gamma + iz\sin\phi]\cos(\phi+\gamma)\cos\phi$$

$$= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\frac{1}{2\pi}\int_0^{2\pi} d\psi \exp\left[2iz\sin\left(-\frac{\gamma}{2}\right)\cos\psi\right]\left[\cos^2\psi - \sin^2\left(\frac{\gamma}{2}\right)\right]$$

$$= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\left\{-J_0''\left[2z\sin\left(\frac{\gamma}{2}\right)\right] - J_0\left[2z\sin\left(\frac{-\gamma}{2}\right)\right]\sin^2\left(\frac{\gamma}{2}\right)\right\}$$

$$= \frac{c_0}{2\pi}\int_0^{2\pi} d\gamma e^{ia\gamma}\left[-\frac{1}{z^2}\frac{\partial^2 J_0}{\partial \gamma^2}\right] = \frac{a}{i\Omega z^2}\left[1 - \frac{\pi a}{\sin \pi a}J_{-a}(z)J_a(z)\right] = \frac{a}{z}G_{31}. \tag{A18}$$

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