Gyrokinetic perpendicular dynamics

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Gyrokinetic perpendicular dynamics, an important component not systematically considered in previous gyrokinetic theories, is identified and developed. A ''distribution function'' S and its governing gyrokinetic equation are introduced to describe the gyrokinetic perpendicular dynamics. The complete treatment of the perpendicular current rendered by the gyrokinetic perpendicular dynamics enables one to recover the compressional Alfvén wave from the gyrokinetic model. From the viewpoint of gyrokinetic theory, the physics of the compressional Alfvén wave is the polarization current at second order. Therefore, in a low frequency gyrokinetic system, the compressional Alfvén wave is naturally decoupled from the shear Alfvén wave and drift wave. In the gyrocenter coordinates, the gyrophase dependent parts of the distribution function S and $\tilde{f}$ are decoupled from the gyrophase independent part $\bar{f}$. Introducing the gyrokinetic perpendicular dynamics also extends the gyrokinetic model to arbitrary frequency modes. As an example, the Bernstein wave is recovered from the gyrokinetic model. The gyrokinetic perpendicular dynamics uncovered here emphasizes that the spirit of gyrokinetic reduction is to decouple the gyromotion from the particle's gyrocenter orbit motion, instead of averaging out the gyromotion.
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I. PHYSICS OF GYROKINETIC PERPENDICULAR DYNAMICS

The electrostatic gyrokinetic model was originally derived by Rutherford and Frieman[1] and Taylor and Hastie[2] for low frequency modes ($\omega \ll \Omega$). Since then, gyrokinetic theory has been significantly advanced and its importance for magnetized plasmas, especially for magnetic fusion, has been greatly appreciated. The gyrokinetic system was first extended to electromagnetic modes by Catto *et al.*[3] using the guiding center coordinates[4] and independently by Antonsen *et al.*[5] Nonlinear electrostatic gyrokinetic equations for small amplitude perturbations were then derived by Frieman and Chen,[6] Lee,[7] and later by Dubin *et al.*[8] and Yang *et al.*[9] using the Hamilton Lie perturbation method, and by Hahm[10] using the phase space Lagrangian Lie perturbation method. Recently, Hahm *et al.*[11] and Brizard[12–14] developed the first nonlinear electromagnetic gyrokinetic system. Meanwhile, a gyrokinetic system valid for both long wavelength and short wavelength modes (without using the ballooning representation) was investigated by Qin *et al.*[15–18] The importance of gyrokinetic Maxwell equations was later realized. Lee[7] first discovered the difference between the gyrocenter density and the particle density in the Poisson equation, which was further studied by Dubin *et al.*[8] The gyrokinetic effect in the parallel Ampere's law was then investigated by Hahm *et al.*[11]

Parallel to Littlejohn's Hamiltonian method for guiding center motion,[19–21] the gyrokinetic formalism using the Lie perturbation method has provided theoretical treatment which is more systematic and comprehensive, and has been applied to many important problems in magnetic fusion plasma. For example, particle simulation based on the gyrokinetic model[7,22–28] has proved to be an effective method to study electrostatic turbulence and the associated transport phenomena. Electromagnetic gyrokinetic models have also been used to study drift waves, shear Alfvén waves, and the coupling between them.[15–18] However, previous gyrokinetic theories using the Lie perturbation method have not been able to recover the well-known compressional Alfvén wave. This can be attributed to the lack of a systematic treatment for the plasma perpendicular response in gyrokinetic models. For a kinetic system, the kinetic equation can be viewed as a theoretical description for the response of the plasma, in terms of particle density and flow, to the electromagnetic field. When the Maxwell equations are included, the system of equations is complete. Since the Maxwell equations are always the same in a chosen coordinate system, the reason that the compressional Alfvén wave is not recoverable from previous gyrokinetic models must lie in the gyrokinetic equation itself. In this paper, we develop the perpendicular gyrokinetic dynamics, an important component not systematically considered in previous gyrokinetic theories. A ''distribution function'' S and a gyrokinetic equation for it are introduced to describe the gyrokinetic perpendicular dynamics. To systematically derive our electromagnetic gyrokinetic system, and especially the gyrokinetic perpendicular dynamics, we use the phase space Lagrangian Lie perturbation method.

The complete treatment of the perpendicular current rendered by the gyrokinetic perpendicular dynamics enables us to recover the compressional Alfvén wave from the gyrokinetic model. It turns out that the physics of the perpendicular current for the compressional Alfvén wave is the polarization drift, which is of the order $O(\epsilon_\omega^2)$, where $\epsilon_\omega \equiv \omega/\Omega$. The compressional Alfvén wave in tokamaks usually has higher frequency than the shear Alfvén wave and the drift wave. It can cause numerical instabilities in simulations targeted at the shear Alfvén waves and the drift waves. In fluid simulations, implicit schemes are often required to eliminate these numerical instabilities. The recovery of the compressional Alfvén wave in the gyrokinetic model shows that, in the low frequency gyrokinetic system, the compressional Alfvén wave is naturally decoupled from the shear Alfvén waves and the drift waves. Therefore, implicit schemes are not necessary for gyrokinetic simulations.

Introducing the gyrokinetic perpendicular dynamics also extends the gyrokinetic model to arbitrary frequency modes. The gyrokinetic theory for arbitrary frequency modes was first studied by Chen and Tsai.[29,30] It was then investigated by Lee *et al.*,[31] Chiu,[32] Brizard,[14] and applied to the problem of fast wave cyclotron resonance by Lashmore-Davies and Dendy.[33–35] The basic method was based on the point of view that averaging over gyrophase is equivalent to taking the zeroth component of the Fourier series in gyrophase, and keeping all the Fourier components extends the gyrokinetic model to arbitrary frequency.[29] However, all of these theoretical models use the guiding center coordinates in which the particle's gyromotion is not decoupled from the rest of the particle dynamics when time-dependent perturbations exist in the system. Therefore, Fourier decomposition in gyrophase has to be carried out for the field variables, the Vlasov equation, and the Maxwell equations. This results in a coupled equation system for all the gyrophase harmonics. On the other hand, keeping all the gyrophase harmonics and the coupling terms guarantees that the coupled system is fully equivalent to the original Vlasov equation and capable of recovering all the results obtained by the conventional Vlasov–Maxwell method, such as the compressional Alfvén wave, which has not been recovered by other gyrokinetic approaches. In our approach, we use the gyrocenter coordinates instead of the guiding center coordinates. The gyrophase independent part and the gyrophase dependent part are naturally separated in the gyrocenter coordinates. Consequently, the mathematical formalism is much more transparent and general. For example, the methods developed here apply to arbitrary wavelength modes. We do not assume the ballooning representation; the background inhomogeneity can be treated completely. In a sense, the extension of gyrokinetic theory to arbitrary frequency rendered by the gyrokinetic perpendicular dynamics reported here can be viewed as the Hamiltonian (geometric) counterpart for the theory initiated by Chen and Tsai.

The insight into the gyrokinetic perpendicular dynamics uncovered here clarifies the understanding of gyrokinetic reduction. What gyrokinetic theory offers is a simplified version of the Vlasov–Maxwell system by utilizing the fact that the particle's gyroradius is much smaller than the scale length of the magnetic field: $\epsilon_{B_0+B_1} \equiv |\rho/L_{B_0+B_1}| \ll 1$. As long as $\epsilon_{B_0+B_1}$ is small, we are able to construct a gyrocenter coordinate system in which the particle's gyromotion is decoupled from the rest of the particle dynamics. The existence of the gyrocenter coordinates does not depend on the mode frequency directly. Therefore even when the mode frequency is comparable to the cyclotron frequency, $\epsilon_\omega \sim 1$, we can still take advantage of the gyrocenter coordinates and simplify the kinetic system. On the other hand, information about the gyromotion is obviously important for cyclotron waves. How does the gyrokinetic model provide this information? It will be shown that, for arbitrary frequency linear analysis and nonlinear low frequency analysis, the distribution function in the gyrocenter coordinate system does not depend on the gyrophase. All the information about the gyromotion is contained in the ''distribution function'' S introduced in the construction of the gyrocenter coordinates. We emphasize that the spirit of gyrokinetic theory is to decouple the gyromotion from the particle's gyrocenter orbit motion, instead of averaging it out. As a matter of fact, for the compressional Alfvén wave in the framework of gyrokinetic theory, information about the gyromotion is important. If we simply average out the gyromotion, the compressional Alfvén wave will be ''averaged out'' as well.

In the gyrocenter coordinate system, the gyrokinetic equation consists of two components, the gyrokinetic equation for f and that for S, and it is fully equivalent to the Vlasov equation. To be accurate, the gyrokinetic equation (including both components) is the Vlasov equation in the gyrocenter coordinate system. The Vlasov equation, in its general (geometric) form $\{F,H_E\}=0$, is coordinate independent. The simplest and often-used coordinate system for the 6D phase space is the particle coordinates $(\mathbf{x},\mathbf{v})$ (representing all coordinate systems with decoupled configuration space coordinates and velocity space coordinates). As different choices of the 6D phase space coordinate system, the guiding center coordinate system $(\mathbf{X},U,\mu,\xi)$ and the gyrocenter coordinate system $(\bar{\mathbf{X}},\bar{U},\bar{\mu},\xi)$ have the same ability as the particle coordinate system to describe the Vlasov equation. It is important to notice that the previously derived gyrokinetic equations for f alone is not the Vlasov equation in the gyrocenter coordinate system. Only when the gyrokinetic equation for S is also included do we obtain the Vlasov equation in the gyrocenter coordinate system. This is one of the motivations of this paper. From this point of view, Chen and Tsai's equation system is the representation of the Vlasov equation in the guiding center coordinate if all the gyrophase harmonics and coupling terms are kept. A single gyrokinetic equation for f is not the Vlasov equation in either the guiding center coordinate system or the gyrocenter coordinate system.

Even though all coordinate systems are geometrically equivalent, the algebra involved is different depending on the specific problems being studied. For applications in magnetized plasmas, the advantage of the gyrocenter coordinate system lies at the fact that in this coordinate system the fast time scale gyromotion is decoupled from the particle's gyrocenter orbit dynamics. For low frequency electrostatic modes and shear Alfvén modes, the gyromotion is not important and is naturally decoupled from the system as if it completely ''averaged out.'' On the other hand, general frequency mode and compressional Alfvén modes can be easily recovered by including the gyrokinetic perpendicular dynamics in the gyrocenter coordinate system, since the gyrocenter orbit motion is independent of the gyromotion. The current numerical codes and particle simulation codes based on gyrocenter orbit integration for low frequency electrostatic and shear Alfvén mode can be extended to general frequency by appropriately adding in the component of perpendicular dynamics.

The paper is organized as follows. In Secs. II, III, and IV, the gyrokinetic system is developed using the phase space Lagrangian Lie perturbation method, the mathematical background of which is briefly introduced in the Appendix. In Sec. II, we construct the gyrocenter coordinates by a symplectic transformation from the guiding center coordinates. Then, in Sec. III, the Gyrokinetic Maxwell equations are discussed. In Sec. IV, the gyrokinetic equations, including that for the ''distribution function'' S responsible for the gyrokinetic perpendicular dynamics, are developed in the gyrocenter coordinates. As examples of many possible applications of the gyrokinetic perpendicular dynamics, the compressional Alfvén wave and the Bernstein wave are recovered, respectively, in Secs. V and VII, and the Gyrokinetic perpendicular Ohm's law for shear Alfvén waves is derived in Sec. VI. In the last section, we summarize and discuss future work.

II. SYMPLECTIC GYROCENTER TRANSFORMATION

To establish our gyrokinetic system for arbitrary wavelength, electromagnetic perturbations, we start from Littlejohn's guiding center theory. When

$$\epsilon_{B_0} \equiv \left| \frac{\rho}{L_{B_0}} \right| \ll 1, \tag{1}$$

we can construct a set of noncanonical phase space coordinates in which the gyromotion is decoupled from the rest of the particle dynamics to any order in ϵ_{B_0}. This special set of coordinates is called the guiding center coordinates. The underlying method is to look at the perturbation of the phase space Lagrangian when ϵ_{B_0} is small, and introduce a near identity coordinate transformation such that, in the new coordinate system, the gyromotion is decoupled. We will summarize the basic results without derivation. The mathematical background of this theory is briefly introduced in the Appendix.

The equilibrium is assumed to be magnetostatic. The guiding center transformation T_{GC}, which transforms the particle coordinate into the guiding center coordinate, is given by[20,21,12,14]

$$\begin{aligned}
&\mathbf{X} = \mathbf{x} - \boldsymbol{\rho}_0, \\
&U = v_\parallel + \frac{c}{e}\mu_0 \mathbf{b}\cdot\nabla\times\mathbf{b} + O(\epsilon_{B_0}^2), \\
&\mu = \mu_0 - \frac{mc}{e}\mu_0 \frac{v_\parallel}{B}\mathbf{b}\cdot\nabla\times\mathbf{b} + O(\epsilon_{B_0}^2), \\
&\xi = \theta - \boldsymbol{\rho}_0\cdot\mathbf{R} - \frac{mc}{e}\frac{v_\parallel}{4B}(\hat{\boldsymbol{\rho}}_0\hat{\boldsymbol{\rho}}_0 - \hat{\mathbf{v}}_\perp\hat{\mathbf{v}}_\perp):\nabla\mathbf{b} + O(\epsilon_{B_0}^2),
\end{aligned} \tag{2}$$

where $(\mathbf{x}, v_\parallel, v_\perp, \theta)$ are the usual local particle coordinates. $\boldsymbol{\rho}_0$, defined in particle coordinates, is the usual gyroradius. θ is chosen such that $\hat{\mathbf{v}}_\perp = -e/|e|(\mathbf{e}_x \sin\theta + \mathbf{e}_y \cos\theta)$. $\mathbf{e}_x$ and $\mathbf{e}_y$ are two perpendicular directions in the configuration space, and $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{b})$ is a right-handed orthogonal frame.

In the extended guiding center coordinates $(\mathbf{X}, U, \mu, \xi, w, t)$, the extended phase space Lagrangian is[20,21,12,14]

$$\begin{aligned}
\gamma_E &= \hat{\gamma}_E - H_E\, d\tau \\
&= \left(\frac{e}{c}\mathbf{A} + mU\mathbf{b} - \mu\frac{mc}{e}\mathbf{W}\right)\cdot d\mathbf{X} + \frac{mc}{e}\mu\, d\xi - w\, dt \\
&\quad - (H - w)\, d\tau,
\end{aligned} \tag{3}$$

where species subscripts are temporarily suppressed. $\mathbf{X}$ is the configuration component of the guiding center coordinate, U is the parallel velocity, μ is the magnetic moment, ζ is the gyrophase angle, and

$$\mathbf{W} = \mathbf{R} + \frac{\mathbf{b}}{2}(\mathbf{b}\cdot\nabla\times\mathbf{b}), \quad \mathbf{R} = (\nabla\mathbf{e}_1)\cdot\mathbf{e}_2, \quad \mathbf{b} = \mathbf{B}/B. \tag{4}$$

$\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors in two arbitrarily chosen perpendicular directions, and $\mathbf{e}_1$ and $\mathbf{e}_2$ are perpendicular to each other. The regular phase space is extended to include the time coordinate t and its conjugate coordinate energy w. $\hat{\gamma}_E$ gives the extended symplectic structure, $H_E = H - w$ is the extended Hamiltonian, and H is the regular Hamiltonian defined as

$$H = \frac{mU^2}{2} + \mu B.$$

The corresponding Poisson bracket is obtained by inverting the symplectic structure $\hat{\gamma}_{Eij}$,[20,21,12]

$$\begin{aligned}
\{F,G\} &= \frac{e}{mc}\left(\frac{\partial F}{\partial\xi}\frac{\partial G}{\partial\mu} - \frac{\partial F}{\partial\mu}\frac{\partial G}{\partial\xi}\right) - \frac{c\mathbf{b}}{eB_\parallel^*}\cdot\left[\left(\nabla F + \mathbf{W}\frac{\partial F}{\partial\xi}\right)\right. \\
&\quad \left.\times\left(\nabla G + \mathbf{W}\frac{\partial G}{\partial\xi}\right)\right] + \frac{\mathbf{B}^*}{mB_\parallel^*}\cdot\left[\left(\nabla F + \mathbf{W}\frac{\partial F}{\partial\xi}\right)\frac{\partial G}{\partial U}\right. \\
&\quad \left. - \left(\nabla G + \mathbf{W}\frac{\partial G}{\partial\xi}\right)\frac{\partial F}{\partial U}\right] + \left(\frac{\partial F}{\partial w}\frac{\partial G}{\partial t} - \frac{\partial F}{\partial t}\frac{\partial G}{\partial w}\right),
\end{aligned} \tag{5}$$

where

$$\mathbf{B}^* = \mathbf{B} + \frac{cmU}{e}\nabla\times\mathbf{b}, \quad B_\parallel^* = \mathbf{b}\cdot\mathbf{B}^*. \tag{6}$$

The guiding center coordinate system in a static magnetic field is illustrated in Fig. 1.

When the perturbed electromagnetic field is introduced, the extended phase space Lagrangian is perturbed accordingly:[11,12]

$$\begin{aligned}
\gamma_E &= \gamma_{E0} + \gamma_{E1}, \\
\gamma_{E1} &= \left[\frac{e}{c}\mathbf{A}_1(T_{\mathrm{GC}}^{-1}\mathbf{X}, t)\cdot d(T_{\mathrm{GC}}^{-1}\mathbf{X})\right] - e\phi_1(T_{\mathrm{GC}}^{-1}\mathbf{X}, t)\, d\tau,
\end{aligned} \tag{7}$$

where T_{GC}^{-1} is the inverse of the guiding center transformation:

$$T_{\mathrm{GC}}^{-1}\mathbf{X} = \mathbf{X} + \boldsymbol{\rho}_0 + \boldsymbol{\rho}_1 + O(\epsilon_B^2), \tag{8}$$

where

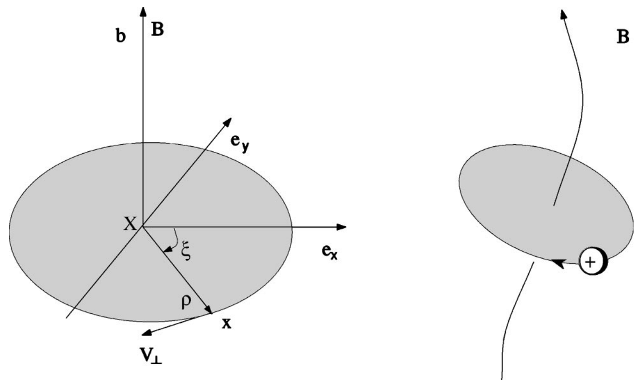

FIG. 1. Guiding center coordinate system.

$$\boldsymbol{\rho}_0 \equiv \frac{c}{e}\sqrt{\frac{2m\mu}{B}}\hat{\boldsymbol{\rho}}_0. \tag{9}$$

(See Fig. 1.) $\hat{\boldsymbol{\rho}}_0$ is the unit vector pointing from the guiding center to the particle's physical position, and $\boldsymbol{\rho}_1$ is the next order correction. To derive the linear gyrokinetic equation, we usually do not need higher orders of the guiding center transformation, because the guiding center transformation is the transformation from the particle ''physical coordinate'' in an equilibrium magnetic field to the ''guiding center coordinate'' in the same equilibrium magnetic field. No perturbed field is involved in this transformation. However, for nonlinear gyrokinetic formalisms, the background and perturbed fields cannot be separated very well; therefore it is necessary to keep the $\boldsymbol{\rho}_1$ term. Our current formalism is a linear one. The leading order expression,

$$T_{\mathrm{GC}}^{-1}\mathbf{X} = \mathbf{X} + \boldsymbol{\rho}_0, \tag{10}$$

will be sufficient for our purpose.

In tokamak geometry, $L_B \sim R_0$. The background (equilibrium) FLR (finite Larmor radius) effect is represented by the small parameter ϵ_B and ignored in linear gyrokinetic theory. Important FLR effects come from the perturbed magnetic field whose wavelength could be much shorter than the scale length of the equilibrium structure and could be comparable to the particle gyroradius. This FLR effect is represented by the parameter: $\epsilon_\delta = |k\rho|$. In general, we keep the FLR effects on the perturbed field to at least $O(\epsilon_\delta^2)$.

Expanding $d(T_{\mathrm{GC}}^{-1}\mathbf{X})$, we obtain:

$$\gamma_{E1} = \frac{e}{c}\mathbf{A}_1(\mathbf{X}+\boldsymbol{\rho}_0,t)\cdot\left[(1+\nabla\boldsymbol{\rho}_0)\cdot d\mathbf{X} + \frac{\partial\boldsymbol{\rho}_0}{\partial\mu}d\mu + \frac{\partial\boldsymbol{\rho}_0}{\partial\xi}d\xi\right] - e\phi_1(\mathbf{X}+\boldsymbol{\rho}_0,t)d\tau. \tag{11}$$

The essence of the Lie perturbation method is to introduce a near identity transformation from the equilibrium guiding center coordinates $\mathbf{Z}=(\mathbf{X},U,\mu,\xi,w,t)$ to the gyrocenter coordinates $\bar{\mathbf{Z}}=(\bar{\mathbf{X}},\bar{U},\bar{\mu},\bar{\xi},\bar{w},\bar{t})$ when the perturbed field is present such that the transformed extended phase space Lagrangian $\bar{\gamma}$ can be gyrophase independent.

We emphasize that there are three different coordinate systems appearing in our formalism. $(\mathbf{x},\mathbf{v})$ is the particle ''physical'' coordinate system. $\mathbf{Z}=(\mathbf{X},U,\mu,\xi,w,t)$ is the (extended) ''guiding center'' coordinate system in an equilibrium magnetic field. When the time-dependent electromagnetic field is introduced, we use the (extended) ''gyrocenter'' coordinate system $\bar{\mathbf{Z}}=(\bar{\mathbf{X}},\bar{U},\bar{\mu},\bar{\xi},\bar{w},\bar{t})$ to describe the gyrocenter motion. Among other things, the most well-known difference between the guiding center motion and the gyrocenter motion is the polarization drift motion due to the time-dependent electrical perturbation. We are following Brizard[12] in using the terms ''gyrocenter'' and ''guiding center'' to distinguish these two different coordinate systems.

For the transformation

$$\bar{Z}^i = (e^{\mathbf{G}}\mathbf{Z})^i \approx Z^i + G^i(\mathbf{Z}), \tag{12}$$

the leading order transformed extended phase space Lagrangian is:

$$\bar{\gamma}_{E1} = \gamma_{E1} - i_{\mathbf{G}}\omega_{E0} + dS = \hat{\bar{\gamma}}_{E1} - \bar{H}_{E1}\,d\tau, \tag{13}$$

where $\omega_{E0} = d\gamma_{E0}$, S is the gauge function, and $i_{\mathbf{G}}\omega_{E0}$ is the interior product between the vector field $\mathbf{G}$ and the two form ω_{E0}. There are several ways to make $\hat{\bar{\gamma}}_E$ and $\bar{H}_E\,d\tau$ gyrophase independent. We will choose $\mathbf{G}$ and S such that the transformation is symplectic, that is, there is no perturbation on the symplectic structure,

$$\hat{\bar{\gamma}}_{E1} = 0. \tag{14}$$

Other nonsymplectic transformations are also possible. Generally nonsymplectic transformations are more algebraically involved. We will use the symplectic transformation throughout this paper.

This symplectic transformation will transfer the perturbation into the Hamiltonian. Since we choose not to change the time variable t, $G^t = 0$. Other components of $\mathbf{G}$ are solved for from $\hat{\bar{\gamma}}_{E1} = 0$:

$$\begin{aligned}
\mathbf{G}^{\mathbf{X}} &= -\frac{c}{eB_\parallel^*}\mathbf{b}\times\left(\frac{e}{c}\mathbf{A}_1 + \nabla S\right) - \frac{\mathbf{B}^*}{mB_\parallel^*}\frac{\partial S}{\partial U} + O(\epsilon_B),\\
G^U &= \frac{\mathbf{B}^*}{mB_\parallel^*}\cdot\left(\frac{e}{c}\mathbf{A}_1 + \nabla S\right) + O(\epsilon_B),\\
G^\mu &= \frac{e}{mc}\left(\frac{e}{c}\mathbf{A}_1\cdot\frac{\partial\boldsymbol{\rho}_0}{\partial\xi} + \frac{\partial S}{\partial\xi}\right),\\
G^\xi &= -\frac{e}{mc}\left(\frac{e}{c}\mathbf{A}_1\cdot\frac{\partial\boldsymbol{\rho}_0}{\partial\mu} + \frac{\partial S}{\partial\mu}\right) + O(\epsilon_B),\\
G^w &= -\frac{\partial S}{\partial t}.
\end{aligned} \tag{15}$$

The transformed Hamiltonian is

$$\bar{H}_{E1} = H_{E1} - G^i\frac{\partial H_{E0}}{\partial x^i} + G^w = e\phi_1(\bar{\mathbf{X}}+\boldsymbol{\rho}_0,t) - \frac{e}{c}\mathbf{A}_1(\bar{\mathbf{X}}+\boldsymbol{\rho}_0,t)\cdot\{\bar{\mathbf{X}}+\boldsymbol{\rho}_0,H_{E0}\} - \{S,H_{E0}\}, \tag{16}$$

in which

$$\{\bar{\mathbf{X}}+\boldsymbol{\rho}_0,H_{E0}\} = \bar{\mathbf{V}} + \mathbf{v}_d, \tag{17}$$

where

$$\overline{\mathbf{V}} = \overline{\mathbf{V}}_\perp + \overline{U}\mathbf{b}, \quad \overline{\mathbf{V}}_\perp = \{\boldsymbol{\rho}_0, H_{E0}\}. \tag{18}$$

In the calculation related to the gyrocenter transformation, we will only keep the lowest order in terms of ϵ_B, because the background FLR effects normally are not important.

Here we encounter the second choice in the process of constructing gyrocenter coordinates. We choose

$$\overline{H}_{E1} = \left\langle e\phi_1(\overline{\mathbf{X}}+\boldsymbol{\rho}_0,t) - \overline{\mathbf{V}}\cdot\frac{e}{c}\mathbf{A}_1(\overline{\mathbf{X}}+\boldsymbol{\rho}_0,t)\right\rangle, \tag{19}$$

where $\langle\ \rangle$ represents the gyrophase averaging operation. This leads to the equation determining the gauge function S:

$$\begin{aligned}\{S,H_{E0}\} &= \Omega\frac{\partial S}{\partial \xi} + \frac{\partial S}{\partial t} + \frac{\partial S}{\partial \mathbf{X}}\cdot\{\mathbf{X},H_{E0}\} + \frac{\partial S}{\partial U}\{U,H_{E0}\} \\ &= e\tilde{\phi}_1(\overline{\mathbf{X}}+\boldsymbol{\rho}_0,t) - \frac{e}{c}\widetilde{\overline{\mathbf{V}}\cdot\mathbf{A}_1}(\overline{\mathbf{X}}+\boldsymbol{\rho}_0,t),\end{aligned} \tag{20}$$

where $\tilde{\phi}_1(\overline{\mathbf{X}}+\boldsymbol{\rho}_0,t)$ and $\widetilde{\overline{\mathbf{V}}\cdot\mathbf{A}_1}(\overline{\mathbf{X}}+\boldsymbol{\rho}_0,t)$ are the gyrophase dependent parts of $\phi_1(\overline{\mathbf{X}}+\boldsymbol{\rho}_0,t)$ and $\overline{\mathbf{V}}\cdot\mathbf{A}_1(\overline{\mathbf{X}}+\boldsymbol{\rho}_0,t)$, respectively:

$$\begin{aligned}\tilde{\phi}_1(\overline{\mathbf{X}}+\boldsymbol{\rho}_0,t) &= \phi_1(\overline{\mathbf{X}}+\boldsymbol{\rho}_0,t) - \langle\phi_1(\overline{\mathbf{X}}+\boldsymbol{\rho}_0,t)\rangle, \\ \widetilde{\overline{\mathbf{V}}\cdot\mathbf{A}_1}(\overline{\mathbf{X}}+\boldsymbol{\rho}_0,t) &= \overline{\mathbf{V}}\cdot\mathbf{A}_1(\overline{\mathbf{X}}+\boldsymbol{\rho}_0,t) - \langle\overline{\mathbf{V}}\cdot\mathbf{A}_1(\overline{\mathbf{X}}+\boldsymbol{\rho}_0,t)\rangle.\end{aligned} \tag{21}$$

Further study in the forthcoming sections will reveal the fundamental roles of H_{E1} and S in many unexplored area of gyrokinetic theory. In particular, we will see that S is the central piece of the theory for arbitrary frequency modes, compressional Alfvén waves, and the gyrokinetic Ohm's law.

Since the transformation we have chosen is symplectic, $\hat{\gamma}_{E1}=0$, the Poisson bracket in the gyrocenter coordinates is the same as that in the guiding center coordinates, which is given by Eq. (5). After obtaining the desired gyrocenter coordinate, we will "push" objects on the original particle coordinates onto the new coordinates. The objects of physical interest here are the Maxwell equations and the Vlasov equation.

We will use $\mathbf{A}$ and ϕ to notate the perturbed field hereafter; the subscript "1" will be dropped. Unless clarity requires us to use the barred notation, we will also drop the bars for the gyrocenter coordinates hereafter, and use $\mathbf{Z}=(\mathbf{X},U,\ \mu,\ \xi)$ to denote the 6D gyrocenter coordinates.

III. GYROKINETIC MAXWELL EQUATIONS

Before introducing the gyrokinetic equations, we look at the gyrokinetic Maxwell equations. The gyrokinetic Maxwell equations are as important as the gyrokinetic equation itself. The differences between different versions of the gyrokinetic equation can usually be resolved when the corresponding gyrokinetic Maxwell equations are taken into account in the appropriate coordinate systems.

The Poisson equation is

$$-\nabla^2\phi(\mathbf{r},t) = 4\pi\sum_j e\int d^3\mathbf{v} f(\mathbf{r},\mathbf{v},t) + \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\mathbf{A}(\mathbf{r},t), \tag{22}$$

where

$$\int d^3\mathbf{v} f(\mathbf{r},\mathbf{v},t) = \int d^6\mathbf{Z}[T_{\mathrm{GY}}^* f](\mathbf{Z},t)\,\delta(T_{\mathrm{GC}}^{-1}\mathbf{X}-\mathbf{r}). \tag{23}$$

Ampere's law is

$$\nabla\times(\nabla\times\mathbf{A}(\mathbf{r},t)) = \frac{4\pi}{c}\sum_j e\int d^3\mathbf{v}\mathbf{v} f(\mathbf{r},\mathbf{v},t), \tag{24}$$

where

$$\int d^3\mathbf{v}\mathbf{v} f(\mathbf{r},\mathbf{v},t) = \int d^6\mathbf{Z}\mathbf{V}_{\mathrm{GC}}(\mathbf{Z})[T_{\mathrm{GY}}^* f](\mathbf{Z},t)\,\delta(T_{\mathrm{GC}}^{-1}\mathbf{X}-\mathbf{r}). \tag{25}$$

In the above equations, $d^6\mathbf{Z}$ is understood to be $(B_\parallel^*/m)d^3\mathbf{X}\,dU\,d\mu\,d\xi$. T_{GY}^* is the pull-back transformation, which transforms the perturbed distribution f in the gyrocenter coordinates into that in the guiding center coordinates. T_{GC}^{-1} is the inverse of T_{GC} that transforms the particle physical coordinates $(\mathbf{r},\mathbf{v})$ into the guiding center coordinates. We assume the guiding center transformation T_{GC} and the corresponding pull-back transformation T_{GC}^*, and the gyrocenter transformation T_{GY} and the corresponding pull-back transformation T_{GY}^* are one–one onto (bijective). Generally for a macroscopic quantity $Q(\mathbf{r})$ in the particle coordinates, we have[10,12,8,11]

$$\begin{aligned}Q(\mathbf{r}) &= \int Q(\mathbf{r},\mathbf{v}) f_P(\mathbf{r},\mathbf{v},t)d^3\mathbf{v} \\ &= \int \delta(\mathbf{x}-\mathbf{r})Q(\mathbf{z})f_P(\mathbf{z},t)d^6\mathbf{z}.\end{aligned} \tag{26}$$

In the guiding center coordinate $\mathbf{Z}=(\mathbf{X},U,\mu,\xi)$,

$$Q(\mathbf{r}) = \int [T_{\mathrm{GC}}^{*-1}Q](\mathbf{Z}) f_{\mathrm{GC}}(\mathbf{Z},t)\,\delta(T_{\mathrm{GC}}^{-1}\mathbf{X}-\mathbf{r})d^6\mathbf{Z}. \tag{27}$$

Replacing $f_{\mathrm{GC}}(\mathbf{Z},t)$ by its pull-back from the gyrocenter coordinate, we get

$$Q(\mathbf{r}) = \int [T_{\mathrm{GC}}^{*-1}Q](\mathbf{Z})[T_{\mathrm{GY}}^* f_{\mathrm{GY}}](\mathbf{Z},t)\,\delta(T_{\mathrm{GC}}^{-1}\mathbf{X}-\mathbf{r})d^6\mathbf{Z}. \tag{28}$$

The pull-back transformation from the gyrocenter coordinates to the guiding center coordinates is easily obtained from the expression for G given by Eq. (15),

$$\begin{aligned}T_{\mathrm{GY}}^* F &= F + L_G F \\ &= F - \frac{\mathbf{b}}{B}\times\left(\mathbf{A}+\frac{c}{e}\nabla S\right)\cdot\nabla F + \frac{e}{mc}\mathbf{b}\cdot\left(\mathbf{A}+\frac{c}{e}\nabla S\right)\frac{\partial F}{\partial U} \\ &\quad + \frac{e}{mc}\left[\frac{e}{c}\mathbf{A}\cdot\frac{\partial\boldsymbol{\rho}_0}{\partial\xi} + \frac{\partial S}{\partial\xi}\right]\frac{\partial F}{\partial\mu} + O(\epsilon_B),\end{aligned} \tag{29}$$

where $L_G F$ represents the Lie derivative of F with respect to the vector field $\mathbf{G}$. The pull-back transformation T_{GY}^* lies at the center of the gyrokinetic theory. Lee[7] first discovered its physical effects for electrostatic modes. By this discovery, several basic difficulties in both gyrokinetic theory and gyrokinetic simulation are solved. This problem was further studied by Dubin *et al.*,[8] Hahm *et al.*,[11] Hahm,[10] and Brizard.[12] To be historically complete, we note that one piece of the pull-back transformations appeared in the gyrokinetic equation for ballooning modes derived by Catto *et al.*[3] and Antonsen *et al.*[5] in the form

$$\frac{e}{mc}\frac{\partial F}{\partial \mu}\left\{\left[\phi-\frac{U}{c}A_\parallel\right]J_0+\frac{V_\perp}{k_\perp c}B_\parallel\right\}.$$

However, this expression was only introduced as a mathematical convenience. As an important component of our gyrokinetic theory, the pull-back transformation in the Maxwell equations will enable us to recover many classical results, such as the compressional Alfvén wave, from a purely gyrokinetic approach. Naturally, it brings in kinetic effects as well.

IV. GYROKINETIC EQUATIONS AND GYROKINETIC PERPENDICULAR DYNAMICS

Now we are ready to obtain the linear gyrokinetic equation. In the (extended) gyrocenter coordinates $(\mathbf{X},U,\mu,\xi,w,t)$, the distribution function F satisfies the Vlasov equation:

$$\{F,H_E\}=\frac{\partial F}{\partial t}+\{F,H\}=\frac{\partial F}{\partial t}+\dot{\mathbf{X}}\frac{\partial F}{\partial \mathbf{X}}+\dot{U}\frac{\partial F}{\partial U}+\dot{\xi}\frac{\partial F}{\partial \xi}=0. \tag{30}$$

The linear gyrokinetic equation in its geometric form (coordinate independent form) can be written as:

$$\{f,H_E\}+\{F_0,H_{E1}\}=0, \tag{31}$$

or

$$\frac{\partial f}{\partial t}+\{f,H_0\}=-\{F_0,H_1\}, \tag{32}$$

where

$$F=F_0+f,$$
$$H_0=\frac{mU^2}{2}+\mu B, \tag{33}$$
$$H_1=\left\langle e\phi_1(\mathbf{X}+\boldsymbol{\rho}_0,t)-\frac{e}{c}\mathbf{V}\cdot\mathbf{A}_1(\mathbf{X}+\boldsymbol{\rho}_0,t)\right\rangle.$$

Let $f=\overline{f}+\tilde{f}$ and $\overline{f}=\langle f\rangle$. Since $\dot{\mathbf{X}}$, $\dot{U}$, $\dot{\xi}$ and $\{F_0,H_1\}$ are gyrophase independent, gyrophase averaging gives

$$\frac{\partial \overline{f}}{\partial t}+\dot{\mathbf{X}}\frac{\partial \overline{f}}{\partial \mathbf{X}}+\dot{U}\frac{\partial \overline{f}}{\partial U}=-\{F_0,H_1\}, \tag{34}$$

and

$$\frac{\partial \tilde{f}}{\partial t}+\dot{\mathbf{X}}\frac{\partial \tilde{f}}{\partial \mathbf{X}}+\dot{U}\frac{\partial \tilde{f}}{\partial U}+\dot{\xi}\frac{\partial \tilde{f}}{\partial \xi}=0. \tag{35}$$

The equation for $\tilde{f}$ is homogeneous and does not depend on the perturbed field. For the initial value problem, $\tilde{f}$ is purely a residual of the gyrophase dependent part of the initial perturbation. The physics for linear eigenmode analysis requires $\tilde{f}=0$ when the linear driving $\boldsymbol{\phi}=0$ and $\mathbf{A}=0$. Since $\tilde{f}$ does not depend on the field perturbation, $\tilde{f}=0$ for any field perturbations. Therefore we conclude

$$f=\overline{f}. \tag{36}$$

The distribution function f only contains the gyrophase independent part, and the gyrokinetic equation is valid for arbitrary frequency and wavelength as long as the gyrocenter coordinates exist.

In the coordinates $(\mathbf{X},U,\mu,\xi,\omega,t)$, the linear gyrokinetic equation is

$$\frac{\partial f}{\partial t}+(U\mathbf{b}+\mathbf{v}_d)\cdot\nabla f-\frac{1}{m}\mathbf{b}\cdot\nabla H_0\frac{\partial f}{\partial U}$$
$$=\frac{c}{eB}\mathbf{b}\cdot(\nabla F_0\times\nabla H_1)-\frac{1}{m}\mathbf{b}\cdot\left(\nabla F_0\frac{\partial H_1}{\partial U}-\nabla H_1\frac{\partial F_0}{\partial U}\right). \tag{37}$$

Another set of gyrocenter coordinates $(\mathbf{X},\epsilon,\mu,\xi,t)$ is often used. ϵ is the total energy in the unperturbed field, that is,

$$\epsilon=H_0=\frac{mU^2}{2}+\mu B_0. \tag{38}$$

In this set of gyrocenter coordinates, the linear gyrokinetic equation is:

$$\frac{\partial f}{\partial t}+(U\mathbf{b}+\mathbf{v}_d)\cdot\nabla f=\left(\frac{c\mathbf{b}}{eB}\times\nabla F_0\right)\cdot\nabla H_1$$
$$+\frac{\partial F_0}{\partial \epsilon}(U\mathbf{b}+\mathbf{v}_d)\cdot\nabla H_1. \tag{39}$$

An alternative form of this equation is written in terms of the nonadiabatic part of f,

$$g=f-H_1\frac{\partial F_0}{\partial \epsilon}, \tag{40}$$

$$\frac{\partial g}{\partial t}+(U\mathbf{b}+\mathbf{v}_d)\cdot\nabla g=\left(\frac{c\mathbf{b}}{eB}\times\nabla F_0\cdot\nabla-\frac{\partial F_0}{\partial \epsilon}\frac{\partial}{\partial t}\right)H_1. \tag{41}$$

The fact that f contains no gyrophase dependent part in the gyrocenter coordinates does not imply that f is gyrophase independent in the particle coordinates. The important information about gyromotion is carried by the gauge function S whose governing equation is:[14]

$$\{S,H_{E0}\}=\Omega\frac{\partial S}{\partial \xi}+\frac{\partial S}{\partial t}+\frac{\partial S}{\partial \mathbf{X}}\cdot\{\mathbf{X},H_{E0}\}+\frac{\partial S}{\partial U}\{U,H_{E0}\}$$
$$=e\tilde{\phi}_1(\mathbf{X}+\boldsymbol{\rho}_0,t)-\frac{e}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}_1}(\mathbf{X}+\boldsymbol{\rho}_0,t). \tag{42}$$

This equation plays the same role for perpendicular dynamics as the gyrokinetic equation does for the parallel dynamics. In this sense, S, as a function of phase space and time, can be viewed as a "distribution function," even though its dimension is [energy][time]. Contrary to the gyrophase independent function f, S is gyrophase dependent, and it can be solved for in a formal form when

$$\epsilon_\omega=\frac{\omega}{\Omega}\ll 1.$$

Let

$$S=S^{(0)}+\epsilon_\omega S^{(1)}+\epsilon_\omega^2 S^{(2)}+\cdots, \tag{43}$$

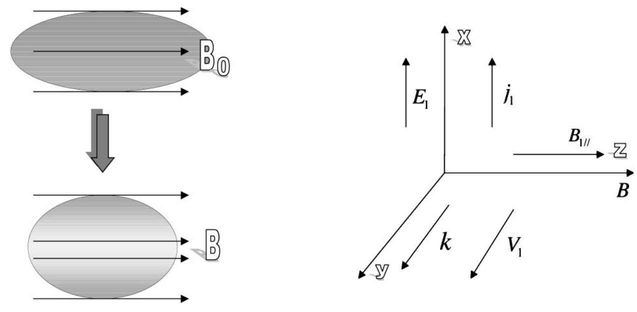

FIG. 2. Compressional Alfvén wave.

and

$$e\left(\tilde{\phi}-\frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right)=e\tilde{\phi}(\mathbf{X}+\boldsymbol{\rho}_0,t)-\frac{e}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}(\mathbf{X}+\boldsymbol{\rho}_0,t).$$

To $O(\epsilon_\omega^0)$,

$$\Omega\frac{\partial S^{(0)}}{\partial\xi}=e\left(\tilde{\phi}-\frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right),\qquad S^{(0)}=\frac{e}{\Omega}\int\left(\tilde{\phi}-\frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right)d\xi. \tag{44}$$

To $O(\epsilon_\omega^1)$,

$$\Omega\frac{\partial S^{(1)}}{\partial\xi}=-\frac{dS^{(0)}}{dt},\qquad S^{(1)}=-\frac{e}{\Omega^2}\frac{d}{dt}\int\int\left(\tilde{\phi}-\frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right)d\xi\,d\xi. \tag{45}$$

To $O(\epsilon_\omega^2)$,

$$\Omega\frac{\partial S^{(2)}}{\partial\xi}=-\frac{dS^{(1)}}{dt},\qquad S^{(2)}=\frac{e}{\Omega^3}\frac{d^2}{dt^2}\int\int\int\left(\tilde{\phi}-\frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right)d\xi\,d\xi\,d\xi. \tag{46}$$

Therefore, we have

$$S=\sum_{n=0}^{\infty}(-\epsilon_\omega)^n\frac{e}{\Omega^{n+1}}\frac{d^n}{dt^n}\int^{(n+1)}\left[\tilde{\phi}-\frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right]d\xi^{n+1}, \tag{47}$$

where

$$\frac{d}{dt}\equiv\frac{\partial}{\partial t}+\{\mathbf{X},H_{E0}\}\cdot\frac{\partial}{\partial\mathbf{X}}+\{U,H_{E0}\}\frac{\partial}{\partial U} \tag{48}$$

is the total gyrokinetic time derivative and is gyrophase independent.

As discussed before, the purpose of solving the gyrokinetic equation is to obtain the charge and current responses, in terms of the electromagnetic field, such that the Maxwell equations are complete. We have seen that the distribution function f described by the usual gyrokinetic equation, Eq. (37), cannot provide all the information about the plasma responses. The pull-back transformation appearing in the gyrokinetic Maxwell equations requires the solution for S, which is governed by Eq. (42). The effects of S, i.e., the gyrokinetic perpendicular dynamics, are present in all the Maxwell equations, although its effect in the perpendicular Ampere's law is most prominent and has not been realized before. The perpendicular Ampere's law can formally be written as:

$$\begin{aligned}(\nabla\times\nabla\times\mathbf{A})_\perp=&\sum_j e\int\mathbf{v}_d\,\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})f(\mathbf{Z})d^6\mathbf{Z}\\&+\sum_j e\int\mathbf{V}_\perp[\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})\\&-\delta(\mathbf{X}-\mathbf{r})]f(\mathbf{Z})d^6\mathbf{Z}+\sum_j e\int(\mathbf{V}_\perp+\mathbf{v}_d)\\&\times\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})L_{\mathbf{G}}F_0\,d^6\mathbf{Z},\end{aligned} \tag{49}$$

where L_G is given in Eq. (29), and S by Eq. (42).

Besides the compressional Alfvén wave, gyrokinetic perpendicular Ohm's Law and Bernstein wave discussed in the next three sections, other applications of the gyrokinetic perpendicular dynamics include the gyrokinetic MHD theory, which will be reported in future publications.

V. COMPRESSIONAL ALFVÉN WAVE

In this section, we use the simplest example of the compressional Alfvén wave in a homogeneous magnetized plasma to demonstrate the essence of gyrokinetic perpendicular dynamics. $\mathbf{B}$ is in the $\mathbf{e}_z$ direction. For simplicity, we let $\mathbf{k}=k_y e_y$. The magnetohydrodynamic (MHD) results for the compressional Alfvén wave indicate that the magnetic perturbation is in the parallel direction, the electrical perturbation and current perturbation are in the $\mathbf{e}_x$ direction, and the plasma displacement is in the $\mathbf{e}_y$ direction. From the kinetic point of view, we can choose (see Fig. 2)

$$\phi=0\quad\text{and}\quad\mathbf{A}=A_x\mathbf{e}_x. \tag{50}$$

The gyrokinetic equation is:

$$\frac{\partial f}{\partial t}+U\mathbf{b}\cdot\nabla f=0. \tag{51}$$

Assuming A_x, $f\propto e^{ik_y y-i\omega t}$, we have $f=0$.

Interesting physics is found in the gyrocenter pull-back transformation and the gyrokinetic Ampere's law. In order to obtain the necessary pull-back transformation to $O(\epsilon_\omega^2)$, we need to solve for $\partial S/\partial\xi$ from Eq. (42). Note that $\{\mathbf{X},H_{E0}\}=U\mathbf{b}$ and $\{U,H_{E0}\}=0$ in a homogeneous equilibrium. Also,

$$\frac{\partial S}{\partial\mathbf{X}}\cdot\{\mathbf{X},H_{E0}\}=0,$$

when $\mathbf{k}=\mathbf{k}_\perp$. Let

$$S=S^{(0)}+\epsilon_\omega S^{(1)}+\epsilon_\omega^2 S^{(2)}+O(\epsilon_\omega^3). \tag{52}$$

As derived in Sec. IV,

$$\Omega \frac{\partial S}{\partial \xi} \approx \Omega \frac{\partial S^{(0)} + S^{(1)} + S^{(2)}}{\partial \xi} + O(\epsilon_\omega^3)$$

$$= e\left(\tilde{\phi} - \frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right) - \frac{e}{\Omega}\frac{\partial}{\partial t}\int\left(\tilde{\phi} - \frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right)d\xi + \frac{e}{\Omega^2}\frac{\partial^2}{\partial t^2}\int\int\left(\tilde{\phi} - \frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right)d\xi\, d\xi + O(\epsilon_\omega^3). \tag{53}$$

From the general form of the gyrocenter pull-back transformation

$$T_{\mathrm{GY}}^* F = F + L_{\mathbf{G}} F = F - \frac{\mathbf{b}}{B}\left(\mathbf{A}_1 + \frac{c}{e}\nabla S\right)\cdot\nabla F + \frac{e}{mc}\mathbf{b}\cdot\left(\mathbf{A}_1 + \frac{c}{e}\nabla S\right)\frac{\partial F}{\partial U} + \frac{e}{mc}\left[\frac{e}{c}\mathbf{A}_1\cdot\frac{\partial \boldsymbol{\rho}_0}{\partial \xi} + \frac{\partial S}{\partial \xi}\right]\frac{\partial F}{\partial \mu} + O(\epsilon_B), \tag{54}$$

we have:

$$[T_{\mathrm{GY}}^*(F_0+f)]_1 = f + \frac{e}{mc}\frac{\partial F_0}{\partial \mu}\left\{-\frac{e v_\perp^2}{\Omega^2 c}B_\parallel - \frac{e}{\Omega^2}\frac{\partial}{\partial t}\int\left(\tilde{\phi} - \frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right)d\xi + \frac{e}{\Omega^3}\frac{\partial^2}{\partial t^2}\int\int\left(\tilde{\phi} - \frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right)d\xi\, d\xi\right\}. \tag{55}$$

In the above derivation, we have used the following expressions for the gyro-average:

$$H_1 = \left\langle e\phi(\mathbf{X}+\boldsymbol{\rho}_0, t) - \frac{e}{c}\mathbf{V}\cdot\mathbf{A}(\mathbf{X}+\boldsymbol{\rho}_0, t)\right\rangle$$

$$= eJ_0\left(\frac{k_\perp v_\perp}{\Omega}\right)\left[\phi(\mathbf{X},t) - \frac{1}{c}UA_\parallel(\mathbf{X},t)\right] + e\frac{v_\perp}{k_\perp c}J_1\left(\frac{k_\perp v_\perp}{\Omega}\right)\mathbf{b}\cdot(\nabla\times\mathbf{A}_\perp) \tag{56}$$

$$\approx e\left[\phi(\mathbf{X},t) - \frac{1}{c}UA_\parallel(\mathbf{X},t) + \frac{v_\perp^2}{c\Omega}B_\parallel(\mathbf{X},t)\right],$$

$$\frac{\partial \boldsymbol{\rho}_0}{\partial \xi} = \frac{\mathbf{V}_\perp}{\Omega}. \tag{57}$$

We need to express the perturbed density and the perturbed current in the Maxwell equations in terms of the perturbed fields. They can be derived from the general form of Eq. (26).

$$n_1 = \left\{\int [T_{\mathrm{GY}}^*(F_0+f)](\mathbf{Z})\,\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})d^6\mathbf{Z}\right\}_1$$

$$= \int \delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})f(\mathbf{Z})d^6\mathbf{Z} + \int \delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})\frac{e}{mc}\frac{\partial F_0}{\partial \mu} \times\left\{-\frac{e v_\perp^2}{\Omega^2 c}B_\parallel - \frac{e}{\Omega^2}\frac{\partial}{\partial t}\int\left(\tilde{\phi} - \frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right)d\xi + \frac{e}{\Omega^3}\frac{\partial^2}{\partial t^2}\int\int\left(\tilde{\phi} - \frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right)d\xi\, d\xi\right\}d^6 Z. \tag{58}$$

Detailed calculation shows that the quasi-neutrality condition

$$\sum_j (en_1)_j = 0$$

is degenerate. It gives no information about the dispersion relation.

The perpendicular Ampere's law is needed to complete the equation system. For this purpose, it is necessary to obtain the perturbed perpendicular flow:

$$n_0\mathbf{v}_{1\perp} = \left\{\int \mathbf{V}_\perp[T_{\mathrm{GY}}^*(F_0+f)](\mathbf{Z})\,\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})d^6\mathbf{Z}\right\}_1 \tag{59}$$

$$= \int \mathbf{V}_\perp\,\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})f(\mathbf{Z})d^6\mathbf{Z} + \int \mathbf{V}_\perp\,\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r}) \times\frac{e}{mc}\frac{\partial F_0}{\partial \mu}\left\{\frac{-e v_\perp^2}{\Omega^2 c}B_\parallel - \frac{e}{\Omega^2}\frac{\partial}{\partial t}\int\left(\tilde{\phi} - \frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right)d\xi + \frac{e}{\Omega^3}\frac{\partial^2}{\partial t^2}\int\int\left(\tilde{\phi} - \frac{1}{c}\widetilde{\mathbf{V}\cdot\mathbf{A}}\right)d\xi\, d\xi\right\}d^6\,\mathbf{Z}.$$

Here we denote the macroscopic flow by $\mathbf{v}$, which was used differently in previous sections as the notation for the particle velocity in the particle coordinates. However, there is no confusion because the meaning of $\mathbf{v}$ is usually clear from the context. We will ignore the FLR effects, and use the expression for the particle perpendicular velocity

$$\mathbf{V}_\perp = -V_\perp[\sin(\xi)\mathbf{e}_x + \cos(\xi)\mathbf{e}_y]. \tag{60}$$

Finally, the perturbed perpendicular flow is

$$n_0\mathbf{v}_{1\perp} = \int \mathbf{V}_\perp\,\delta(\mathbf{X}-\mathbf{r})\frac{e}{mc}\frac{\partial F_0}{\partial \mu}\left\{\frac{e}{\Omega^2}\frac{\partial}{\partial t}\int\frac{1}{c}\mathbf{V}_\perp\cdot\mathbf{A}_\perp\,d\xi - \frac{e}{\Omega^3}\frac{\partial^2}{\partial t^2}\int\int\frac{1}{c}\mathbf{V}_\perp\cdot\mathbf{A}_\perp\,d\xi\,d\xi\right\}d^6\mathbf{Z}$$

$$= \frac{-i\omega n_0 e^2 B}{m^2c^2\Omega^2}A_x\mathbf{e}_y + \frac{\omega^2 n_0 e^2 B}{m^2c^2\Omega^3}A_x\mathbf{e}_x$$

$$= \frac{n_0 c}{B^2}\mathbf{E}\times\mathbf{B} + \frac{n_0 mc^2}{eB^2}\frac{\partial \mathbf{E}_\perp}{\partial t}. \tag{61}$$

It is obvious that the **E×B** flow does not contribute to the perpendicular current. The perpendicular current is generated by the ion polarization drift,

$$\mathbf{j}=\sum_j (en_0\mathbf{v}_{1\perp})_j \approx \frac{n_0 m_i c}{B^2}\omega^2 \mathbf{A}_x \mathbf{e}_x . \tag{62}$$

The perpendicular Ampere's law $(\nabla\times\nabla\times\mathbf{A})_\perp = 4\pi/c\,\mathbf{j}_\perp$ gives:

$$k_y^2 A_x = \frac{4\pi n_0 m_i}{B^2}\omega^2 A_x, \quad \text{or} \quad \omega^2 = k_y^2 v_A^2. \tag{63}$$

This is the compressional Alfvén wave. The key point of this gyrokinetic version of the compressional Alfvén wave is the perturbed perpendicular current. To $O(\epsilon_\omega)$, the perpendicular flow is the **E×B** flow, which gives no current. Therefore we have to go to $O(\epsilon_\omega^2)$. The current to this order is the current generated by the polarization drift in the perturbed electromagnetic field. We recall that the physical effect of the gyrocenter pull-back transformation in the Poisson equation is the polarization density,[7] and in the parallel Ampere's law is the skin depth current.[11] What we have found out here is the physical effect of the gyrocenter pull-back transformation in the perpendicular Ampere's law—the polarization current. Actually, the polarization density in the Poisson equation is also an effect of the gyrokinetic perpendicular dynamics. It is easy to verify that keeping the solution of S to order $O(\epsilon_\omega^2)$ in the pull-back term of the Poisson equation gives the polarization density.

Heuristically speaking, to the leading order, the polarization drift is

$$\mathbf{v}_p = \frac{mc^2}{eB^2}\frac{\partial \mathbf{E}_\perp}{\partial t}, \tag{64}$$

where higher order terms associated with $\mathbf{v}_d\cdot\nabla$ have been neglected. For the electrostatic modes and the shear Alfvén modes,

$$\mathbf{v}_p = -\frac{mc^2}{eB^2}\frac{\partial \nabla_\perp \phi}{\partial t}, \tag{65}$$

and it is important in the Poisson equation. For the compressional Alfvén modes, the polarization drift is

$$\mathbf{v}_p = -\frac{mc}{eB^2}\frac{\partial^2 \mathbf{A}_\perp}{\partial t^2}, \tag{66}$$

and it is important in the perpendicular Ampere's law.

As usual, the linear current response can be expressed in terms of the plasma susceptibility χ (and equivalently the dielectric tensor ϵ). From this point of view, Eq. (62) can be alternatively interpreted as

$$\chi_{xx} = -\frac{4\pi n_0 m_i c^2}{B^2}. \tag{67}$$

In the above derivation, we have assumed that $\epsilon_\omega \equiv \omega/\Omega \ll 1$ for the compressional Alfvén wave to obtain the solution for S. Since $\omega \sim k_y v_A$, when k_y increases sufficiently,[36] ω can become comparable to Ω. We note that the equation for S, Eq. (42), can be solved without the assumption that $\epsilon_\omega \ll 1$. In the current case, Eq. (42) is simplified into:

$$\Omega\frac{\partial S}{\partial \xi} + \frac{\partial S}{\partial t} = -e\frac{\mathbf{V}_\perp\cdot\mathbf{A}_\perp}{c} = \frac{e}{c}V_\perp \sin(\xi) A_x . \tag{68}$$

This an ordinary differential equation for S, the solution for which is

$$S = \left(\frac{eV_\perp}{c\Omega}A_x\right)\frac{\cos(\xi)+i\bar{\omega}\sin(\xi)}{(\bar{\omega}+1)(\bar{\omega}-1)} + c_1 e^{i\xi\bar{\omega}}, \tag{69}$$

where $\bar{\omega}\equiv\omega/\Omega$. As a choice of gauge, c_1 is set to zero. This is also because $c_1 e^{i\xi\bar{\omega}}$ is independent of the linear drive, and it corresponds to the initial condition. Therefore, for linear eigenmode analysis, c_1 can be set to zero. Substituting the solution for S into the pull-back transformation, we obtain the perpendicular flow

$$\begin{aligned} n_0\mathbf{v}_{1\perp} &= \int \mathbf{V}_\perp \delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})\frac{e}{cB}\frac{\partial F_0}{\partial \mu}(-V_\perp A_x) \\ &\quad \times\left[\sin(\xi) - \frac{-\sin(\xi)+i\bar{\omega}\cos(\xi)}{(\bar{\omega}+1)(\bar{\omega}-1)}\right] d^6\mathbf{Z} \\ &= \frac{n_0 e A_x}{cm}\left[\left(\frac{-\bar{\omega}^2}{-\bar{\omega}^2+1}\right)\mathbf{e}_x + \frac{i\bar{\omega}}{(\bar{\omega}+1)(\bar{\omega}-1)}\mathbf{e}_y\right]. \end{aligned} \tag{70}$$

Obviously, the first term is the polarization flow; the second term is the **E×B** flow. This result is valid for general frequency. Because of the term $-\bar{\omega}^2+1$ in the denominator, the **E×B** currents of ions and electrons do not cancel with each other. It is easy to see that Eq. (70) recovers the low frequency result when $\bar{\omega}\ll 1$.

VI. PERPENDICULAR OHM'S LAW

The gyrokinetic perpendicular Ohm's law is seldom discussed. The lack of a perpendicular Ohm's law is fundamentally due to the lack of perpendicular dynamics in the previous gyrokinetic theory. The basic analytic formalism of perpendicular dynamics introduced can be used to derive a perpendicular Ohm's law. The single most important step is to obtain the perpendicular flow. In the normal procedure, the perpendicular current needs to be related to the first perpendicular moment of the gyrokinetic equation for f. But we notice immediately that the $\int \mathbf{V}_\perp\, d^3\mathbf{v}$ operation on the gyrokinetic equation provides us with no information at all, because all the quantities appearing in the gyrokinetic equation are gyrophase independent. The result of this operation is $0=0$. Therefore, deriving the gyrokinetic Ohm's law essentially means obtaining an expression for the perpendicular current from the gyrokinetic pull-back transformation. In this section, we consider the gyrokinetic perpendicular Ohm's law for the shear Alfvén modes ($\mathbf{A}_\perp = 0$).

First for the equilibrium,[37]

$$\mathbf{j}_{0\perp}=\sum_j e\int (\mathbf{V}_\perp+\mathbf{v}_d)F_0\,\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})d^6\mathbf{Z}$$

$$\approx\sum_j e\int [\mathbf{V}_\perp F_0\boldsymbol{\rho}_0\cdot\nabla\,\delta(\mathbf{X}-\mathbf{r})+\mathbf{v}_dF_0\,\delta(\mathbf{X}-\mathbf{r})]d^6\mathbf{Z}$$

$$=\sum_j e\int \left\{-\frac{\partial}{\partial x_l}(V_{\perp n}\rho_{0l})F_0+\mathbf{v}_dF_0\right\}d^3\mathbf{V}$$

$$=c\left[-\nabla\times\left(\mathbf{b}\frac{P_\perp}{B}\right)+\frac{\mathbf{b}}{B}\times\left(P_\perp\frac{\nabla B}{B}-P_\parallel\mathbf{b}\times\nabla\mathbf{b}\right)\right]$$

$$=c\left[\frac{\mathbf{b}}{B}\times\nabla P_\perp+\left(\mathbf{b}\times\frac{\nabla B}{B^2}-\nabla\times\frac{\mathbf{b}}{B}\right)P_\perp+\frac{P_\parallel}{B}\nabla\times\mathbf{b}\right]$$

$$=\frac{c}{B}[\mathbf{b}\times\nabla P_\perp+(P_\parallel-P_\perp)\nabla\times\mathbf{b}] \tag{71}$$

where we have kept only the leading order FLR effect and

$$P_\perp\equiv\sum_j\int\frac{V_\perp^2}{2}F_0\,d^3\mathbf{V},$$
$$P_\parallel\equiv\sum_j\int U^2F_0\,d^3\mathbf{V}. \tag{72}$$

For the perturbed part,

$$n_0\mathbf{v}_{1\perp}=\int (\mathbf{V}_\perp+\mathbf{v}_d)f\,\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})d^6\mathbf{Z}$$
$$+\int\mathbf{V}_\perp\,\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})L_GF_0\,d^6\mathbf{Z}, \tag{73}$$

where $\mathbf{v}_d$ is neglected for the L_GF_0 part of the pull-back transformation. For the first term, we have

$$\int (\mathbf{V}_\perp+\mathbf{v}_d)f\,\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})d^6\mathbf{Z}$$
$$=\frac{c}{eB}\left[\mathbf{b}\times\nabla\int\frac{mV_\perp^2}{2}f\,d^3\mathbf{V}\right.$$
$$\left.+\int m\left(U^2-\frac{V_\perp^2}{2}\right)f\,d^3\mathbf{V}\nabla\times\mathbf{b}\right]. \tag{74}$$

Unlike the fluid approach, we do not introduce any closure scheme to close the perpendicular and parallel energy moment. In kinetic theory, this term is simply determined by the distribution function solved for from the gyrokinetic equation.

For the term $\int\mathbf{V}_\perp\,\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})L_{\mathbf{G}}F_0\,d^6\mathbf{Z}$, we have

$$L_{\mathbf{G}}F_0=\frac{e}{mc}A_\parallel(\mathbf{X}+\boldsymbol{\rho}_0,t)\frac{\partial F}{\partial U}+\frac{e}{mc}\frac{\partial S}{\partial\xi}\frac{\partial F_0}{\partial\mu}. \tag{75}$$

We will keep terms up to $O(\epsilon_\omega^2)$ and $O(\rho_0k_\perp)$ in $\partial S/\partial\xi$,

$$\frac{\partial S}{\partial\xi}=\frac{e}{\Omega}\left(\tilde{\phi}-\frac{1}{c}\widetilde{UA_\parallel}\right)-\frac{e}{\Omega^2}\frac{d}{dt}\int\left(\tilde{\phi}-\frac{1}{c}\widetilde{UA_\parallel}\right)d\xi$$
$$=\frac{e}{\Omega}\left[\boldsymbol{\rho}_0+\frac{i\omega}{\Omega}\rho_0(\mathbf{e}_x\sin\xi+\mathbf{e}_y\cos\xi)\right]\cdot\nabla\left(\phi-\frac{1}{c}UA_\parallel\right), \tag{76}$$

where the curved geometry term in d/dt is neglected,

$$\frac{d}{dt}\equiv\frac{\partial}{\partial t}+\mathbf{v}_d\cdot\nabla\approx\frac{\partial}{\partial t}. \tag{77}$$

Therefore to $O(\rho_0k_\perp)$,

$$\int\mathbf{V}_\perp\,\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})L_{\mathbf{G}}F_0\,d^6\mathbf{Z}$$
$$=\frac{e}{mc}\int\frac{\partial F_0}{\partial U}\mathbf{V}_\perp[\,\delta(\mathbf{X}-\mathbf{r})\boldsymbol{\rho}_0\cdot\nabla A_\parallel+\boldsymbol{\rho}_0\cdot\nabla\,\delta(\mathbf{X}-\mathbf{r})A_\parallel]$$
$$\times d^6\mathbf{Z}+\frac{e}{B}\int\frac{\partial F_0}{\partial\mu}\mathbf{V}_\perp\,\delta(\mathbf{X}-\mathbf{r})\left[\left(\tilde{\phi}-\frac{1}{c}\widetilde{UA_\parallel}\right)\right.$$
$$\left.-\frac{1}{\Omega}\frac{d}{dt}\int\left(\tilde{\phi}-\frac{1}{c}\widetilde{UA_\parallel}\right)d\xi\right]d^6\mathbf{Z}. \tag{78}$$

The first term is obviously zero.

$$\int\mathbf{V}_\perp\,\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})L_{\mathbf{G}}F_0d^6\mathbf{Z}$$
$$=-\frac{e}{T}\int F_0\mathbf{V}_\perp\left[\boldsymbol{\rho}_0+\frac{i\omega}{\Omega}\rho_0(\mathbf{e}_x\sin\xi+\mathbf{e}_y\cos\xi)\right]$$
$$\cdot\boldsymbol{\nabla}\phi(\mathbf{X})d^3\mathbf{V}$$
$$=-\frac{e}{\Omega m}\int F_0\frac{mV_\perp^2}{2T}\left(\epsilon_{ij3}-\frac{i\omega}{\Omega}\delta_{ij}\right)\frac{\partial\phi}{\partial x_j}d^3\mathbf{V} \tag{79}$$
$$=-\frac{n_0c}{B}\left[\nabla\phi\times\mathbf{b}-\frac{i\omega}{\Omega}\nabla\phi\right]=\frac{n_0c}{B}\mathbf{E}_\perp\times\mathbf{b}+\frac{n_0c}{B\Omega}\frac{\partial\mathbf{E}_\perp}{\partial t}.$$

Finally, the perpendicular flow is

$$(n_0\mathbf{v}_{1\perp})_j=\left\{\frac{c}{eB}\left[\mathbf{b}\times\nabla\int\frac{mV_\perp^2}{2}f\,d^3\mathbf{V}\right.\right.$$
$$\left.\left.+\int m\left(U^2-\frac{V_\perp^2}{2}\right)f\,d^3\mathbf{V}\nabla\times\mathbf{b}\right]\right\}_j$$
$$+\frac{n_{0j}c}{B}\mathbf{E}_\perp\times\mathbf{b}+\left(\frac{n_0c}{B\Omega}\right)_j\frac{\partial\mathbf{E}_\perp}{\partial t}. \tag{80}$$

Clearly the second term in above equation is the $\mathbf{E}\times\mathbf{B}$ drift term, and the last term is the polarization drift term. The macroscopic flow of the plasma is that of the ions; to the leading order it is

$$\mathbf{v}_{1\perp}=\frac{c}{B}\mathbf{E}_\perp\times\mathbf{b}. \tag{81}$$

This is the perpendicular component of the ideal MHD Ohm's law. One would argue that this equation can be written down directly from the $\mathbf{E}\times\mathbf{B}$ drift. However, we emphasize that while this argument is physically correct, but the result here is systematically derived from the gyrokinetic theory. Our rigorous derivation confirms that, to the leading order, the $\mathbf{E}\times\mathbf{B}$ drift is indeed the plasma flow. However, the

$\mathbf{E}\times\mathbf{B}$ drift does not generate any current. The perpendicular current comes from the parallel and the perpendicular energy moment and the polarization drift.

$$\mathbf{j}_{1\perp}=\sum_j\left\{\frac{c}{B}\left[\mathbf{b}\times\nabla\int\frac{mV_\perp^2}{2}f\,d^3\mathbf{V}+\int m\left(U^2-\frac{V_\perp^2}{2}\right)\times f\,d^3\mathbf{V}\nabla\times\mathbf{b}\right]\right\}_j+\sum_j\left(\frac{en_0c}{B\Omega}\right)_j\frac{\partial\mathbf{E}_\perp}{\partial t}. \tag{82}$$

VII. BERNSTEIN WAVE

In this section we recover the Bernstein wave and use it as a second example of how to apply the general theory developed here to related problems.

We consider an electrostatic wave propagating in a homogeneous magnetized plasma with $\omega\sim\Omega$. Let $\mathbf{B}_0=B_0\mathbf{e}_z$ and $\mathbf{k}=k\mathbf{e}_x$. The solution for the linear gyrokinetic equation is degenerate because $k_\parallel=0$,

$$f=-\frac{e}{T}F_0\frac{-k_\parallel U}{\omega-k_\parallel U}\phi=0. \tag{83}$$

In this special case, f, the gyrophase independent part of the distribution function, does not play any role, and we focus on the gyrophase dependent part which is described by the ''distribution function'' S. The equation for S is

$$\begin{aligned}\{S,H_{E0}\}&=\Omega\frac{\partial S}{\partial\xi}+\frac{\partial S}{\partial t}=e\tilde{\phi}(\mathbf{X}+\boldsymbol{\rho}_0)\\&=e\left[e^{\boldsymbol{\rho}_0\cdot\nabla}-J_0\left(\frac{\boldsymbol{\rho}_0\cdot\nabla}{i}\right)\right]\phi.\end{aligned} \tag{84}$$

That is,

$$\frac{\partial S}{\partial\xi}-i\bar{\omega}S=\frac{e}{\Omega}[e^{i\rho k\cos\xi}-J_0(\rho k)]\phi, \tag{85}$$

where $\bar{\omega}=\omega/\Omega$. Using the identity

$$e^{\lambda\cos\xi}=\sum_{n=-\infty}^{\infty}I_n(\lambda)e^{in\xi}, \tag{86}$$

we solve for S,

$$S=\frac{e}{\Omega i\bar{\omega}}J_0\phi+\frac{e}{\Omega}\sum_{n=-\infty}^{\infty}\frac{I_n(i\rho k)}{i(n-\bar{\omega})}e^{in\xi}\phi. \tag{87}$$

We need only $\partial S/\partial\xi$ in the pull-back transformation,

$$\frac{\partial S}{\partial\xi}=\frac{e}{\Omega}\sum_{n=-\infty}^{\infty}\frac{nI_n(i\rho k)}{(n-\bar{\omega})}e^{in\xi}\phi. \tag{88}$$

The density response comes only from the pull-back transformation since $f=0$.

$$\begin{aligned}n_1&=\int J_0f\,d^3\mathbf{v}+\int\delta(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{r})\frac{e}{mc}\frac{\partial S}{\partial\xi}\frac{\partial F_0}{\partial\mu}d^6\mathbf{Z}\\&=\int[e^{\boldsymbol{\rho}_0\cdot\nabla}\delta(\mathbf{X}-\mathbf{r})]\frac{-e}{T}F_0\sum_{n=-\infty}^{\infty}\frac{nI_n(i\rho k)}{(n-\bar{\omega})}e^{in\xi}\phi\,d^6\mathbf{Z}.\end{aligned} \tag{89}$$

Using the facts that

$$\int[e^{\boldsymbol{\rho}_0\cdot\nabla}\delta(\mathbf{X}-\mathbf{r})]Q\,d^6\mathbf{Z}=\int\delta(\mathbf{X}-\mathbf{r})e^{-\boldsymbol{\rho}_0\cdot\nabla}Q\,d^6\mathbf{Z}, \tag{90}$$

and

$$\int_0^{2\pi}e^{i(m+n)\xi}d\xi=\delta_{m,-n}2\pi, \tag{91}$$

we have

$$n_1=\frac{2\pi}{(2\pi T/m)^{3/2}}\int\frac{-n_0e\phi}{T}\exp\left(-\frac{v_\parallel^2+v_\perp^2}{2T/m}\right)\times\sum_{n=-\infty}^{\infty}\frac{nI_{-n}(-i\rho k)I_n(i\rho k)}{(n-\bar{\omega})}v_\perp\,dv_\parallel\,dv_\perp. \tag{92}$$

The following properties of the Bessel function are needed to finish the integral:

$$\begin{aligned}&I_n(x)=i^{-n}J_n(ix),\\&J_{-n}(x)=(-1)^nJ_n(x)=J_n(-x),\\&\int_0^\infty te^{-pt^2}J_n^2(at)dt=\frac{1}{2p}e^{-a^2/2p}I_n\left(\frac{a^2}{2p}\right).\end{aligned} \tag{93}$$

Carrying out the algebra, we obtain

$$n_1=n_0\frac{e\phi}{T}\sum_{n=1}^{\infty}\frac{2n^2}{\left(\frac{\omega}{\Omega}\right)^2-n^2}\exp\left(-\frac{k^2T}{\Omega^2m}\right)I_n\left(\frac{k^2T}{\Omega^2m}\right). \tag{94}$$

Finally, the Poisson equation $-\nabla^2\phi=\Sigma_j4\pi(en_1)_j$ gives the dispersion relation,

$$1=\sum_j\frac{4\pi n_0e^2}{Tk^2}\sum_{n=1}^{\infty}\frac{2n^2}{\left(\frac{\omega}{\Omega}\right)^2-n^2}\exp\left(-\frac{k^2T}{\Omega^2m}\right)I_n\left(\frac{k^2T}{\Omega^2m}\right). \tag{95}$$

This is the Bernstein wave. As we can see,[38] this derivation from gyrokinetic theory is quite different and more straightforward, compared with the conventional method—integrating the Vlasov equation along the particle's unperturbed orbit in the particle coordinate.[39]

VIII. CONCLUSIONS AND FUTURE WORK

In this paper, the theory for gyrokinetic perpendicular dynamics is developed by introducing an extra ''distribution function'' and a gyrokinetic equation for it. Using this model, we have recovered the compressional Alfvén wave from a gyrokinetic approach. From the viewpoint of gyrokinetic theory, the physics of the compressional Alfvén wave is the polarization current at order $O(\epsilon_\omega^2)$. Therefore, in a low frequency gyrokinetic system, the compressional Alfvén wave is naturally decoupled from the shear Alfvén wave and the drift wave. In the gyrocenter coordinates, the gyrophase dependent parts of the distribution function S and $\tilde{f}$ are decoupled from the gyrophase independent part $\bar{f}$. The information about S is important, not only for waves at the cyclotron frequency, but also for low frequency waves, such as the compressional Alfvén wave. S and the corresponding gyrokinetic equation are responsible for the perpendicular dynamics. They produce the polarization density in the gyrokinetic Poisson equation and the polarization current in the gyrokinetic perpendicular Ampere's law. Introducing the gyrokinetic perpendicular dynamics also extends the gyrokinetic model to arbitrary frequency modes. As an example, the Bernstein wave is recovered from the gyrokinetic model. The gyrokinetic perpendicular dynamics uncovered here emphasizes that the spirit of gyrokinetic reduction is not averaging out the gyromotion, but rather decoupling the gyromotion from the particle's gyrocenter orbit motion. When necessary, the information about S can always be obtained easily.

Only linear theory is presented in this paper. Carrying out the analysis to the second order will give the nonlinear theory for small amplitude perturbations.[12,14] Since the construction of the gyrocenter coordinates depends on the perturbed field, the gyrocenter coordinates have to be defined to the second order as well. As a result, the gyrokinetic equation for S will become nonlinear. However, the basic features of the linear gyrokinetic perpendicular dynamics are still valid for nonlinear theory. For example, the gyrophase dependent and gyrophase independent parts of F are still decoupled. The nonlinear gyrokinetic equation is

$$\{F,H_E\}=\frac{\partial F}{\partial t}+\{F,H\}=\frac{\partial F}{\partial t}+\dot{\mathbf{X}}\frac{\partial F}{\partial \mathbf{X}}+\dot{U}\frac{\partial F}{\partial U}+\dot{\xi}\frac{\partial F}{\partial \xi}=0. \tag{96}$$

Let $F=\overline{F}+\tilde{F}$ and $\overline{F}=\langle F\rangle$. Since $\dot{\mathbf{X}}$, $\dot{U}$, and $\dot{\xi}$ are gyrophase independent, we have

$$\{\overline{F},H_E\}=\frac{\partial \overline{F}}{\partial t}+\{\overline{F},H\}=\frac{\partial \overline{F}}{\partial t}+\dot{\mathbf{X}}\frac{\partial \overline{F}}{\partial \mathbf{X}}+\dot{U}\frac{\partial \overline{F}}{\partial U}=0, \tag{97}$$

and

$$\{\tilde{F},H_E\}=\frac{\partial \tilde{F}}{\partial t}+\{\tilde{F},H\}=\frac{\partial \tilde{F}}{\partial t}+\dot{\mathbf{X}}\frac{\partial \tilde{F}}{\partial \mathbf{X}}+\dot{U}\frac{\partial \tilde{F}}{\partial U}+\dot{\xi}\frac{\partial \tilde{F}}{\partial \xi}=0. \tag{98}$$

If $\epsilon_\omega=\omega/\Omega\ll 1$, it is easy to prove that $\tilde{F}=0$ to any order in ϵ_ω.

On the other hand, the nonlinear gyrokinetic theory and the nonlinear gyrokinetic perpendicular dynamics can be developed, without the assumption of small perturbations, directly from the guiding center theory for the time-dependent electromagnetic field.[20] The difference between the guiding center coordinates and the gyrocenter coordinates is not necessary for this approach, and the perturbations can be arbitrary as long as the guiding center coordinates exist. A successful theory for nonlinear gyrokinetic perpendicular dynamics is the key to nonlinear gyrokinetic MHD and a gyrokinetic model for nonlinear cyclotron waves. Recent development in perturbation methods for Hamiltonian dynamics, such as the Berry–Hannay phase[40–45] and the geometric perturbation,[46] may be helpful for the development of a nonlinear gyrokinetic theory, especially for nonlinear gyrokinetic perpendicular dynamics.

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APPENDIX: PHASE SPACE LAGRANGIAN LIE PERTURBATION METHOD

The study of dynamics from the Hamiltonian point of view provides us with systematic methods to deal with complicated dynamic structures, especially the perturbation methods which have great value in celestial mechanics and in guiding center dynamics of charged particles in an electromagnetic field. Hamiltonian dynamics[47,48] is given by a symplectic structure ω and a Hamiltonian function H on an even dimensional manifold M^{2n}.

By definition, the symplectic structure ω is a closed nondegenerate differential two-form on M^{2n}:

$$\begin{aligned}&d\omega=0, \quad \text{and}\\ &\forall\, \boldsymbol{\xi}\epsilon T_xM \quad \text{and} \quad \boldsymbol{\xi}\neq 0,\ \exists\, \boldsymbol{\eta}\epsilon T_xM \quad \text{such that} \quad \omega(\boldsymbol{\xi},\boldsymbol{\eta})\neq 0.\end{aligned} \tag{A1}$$

The symplectic structure as a special two-form establishes an isomorphism between the tangent space T_xM and the cotangent space T_x^*M at any x on M, that is, for any $\boldsymbol{\xi}\epsilon T_xM$, we have $\omega(\cdot,\boldsymbol{\xi})$ which is an element in T_x^*M, and vice versa. Denoting the isomorphism from T_x^*M to T_xM as I, we obtain a correspondence between functions on M and vector fields on M^{2n}:

$$\begin{aligned}&\mathbf{X}_g=I\,dg,\\ &dg=\omega(\cdot,Idg)=\omega(\cdot,\mathbf{X}_g).\end{aligned} \tag{A2}$$

Therefore, the symplectic structure generates an algebraic structure for functions on M—the Poisson bracket,

$$\{f,g\}\equiv\omega(\mathbf{X}_g,\mathbf{X}_f). \tag{A3}$$

It can be shown that the Poisson bracket is skew symmetrical and satisfies the Jacobi identity (from $d\omega=0$):

$$\begin{aligned}&\{f,g\}=-\{g,f\},\\ &\{\{f,g\},h\}+\{\{h,f\},g\}+\{\{g,h\},f\}=0.\end{aligned} \tag{A4}$$

The set of functions on M with the Poisson bracket is thus a Lie algebra.

Usually we want to solve for the Hamiltonian flow corresponding to a Hamiltonian vector field $\mathbf{X}_H=I\,dH$. The Hamiltonian flow $\phi(t,x)$ is an one parameter group of transformations on M^{2n} satisfying

$$\begin{aligned}&\phi(0,x)=x,\\ &\phi(s,\phi(t,x))=\phi(s+t,x),\\ &\left.\frac{d}{dt}\right|_{t=0}\phi(t,x)=\mathbf{X}_H=(I\,dH)_x.\end{aligned} \tag{A5}$$

For such a flow we have

$$\left.\frac{d}{dt}\right|_{t=s}\phi(t,x)=\left.\frac{d}{d(t-s)}\right|_{t-s=0}\phi(t-s,\phi(s,x))$$
$$=(I\,dH)_{\phi(0,\phi(s,x))}=(I\,dH)_{\phi(s,x)}. \tag{A6}$$

Now the dynamics of a function F on M under the Hamiltonian flow is simply given by the Poisson bracket between F and H,

$$\{F,H\}=\omega^2(I\,dH,I\,dF)=(I\,dH)dF$$
$$=\left.\frac{d}{dt}\right|_{t=0}\phi(t,x)\nabla F=\left.\frac{d}{dt}\right|_{t=0}F(\phi(t,x)). \tag{A7}$$

Similarly, we have

$$\left.\frac{d}{dt}\right|_{t=s}F(\phi(t,x))=\left.\frac{d}{dt}\right|_{t-s=0}F(\phi(t-s,\phi(s,x)))$$
$$=\{F,H\}(\phi(s,x)). \tag{A8}$$

In a given local coordinate system,

$$\frac{\partial g}{\partial x^i}=\omega_{ij}\mathbf{X}_g^j,$$
$$\mathbf{X}_g^i=(\omega^{-1})^{ij}\frac{\partial g}{\partial x^j}, \tag{A9}$$
$$\{f,g\}=\omega_{ij}\mathbf{X}_g^i\mathbf{X}_f^j=\omega_{ij}(\omega^{-1})^{il}\frac{\partial g}{\partial x^l}(\omega^{-1})^{jm}\frac{\partial f}{\partial x^m}$$
$$=(\omega^{-1})^{ml}\frac{\partial f}{\partial x^m}\frac{\partial g}{\partial x^l}.$$

For many physical problems, the symplectic structure and the Hamiltonian function are given by the Poincaré–Cartan form

$$\gamma=\hat{\gamma}+H\,dz, \tag{A10}$$

which is a one-form on the space (M^{2n},τ). H is the Hamiltonian function. The symplectic structure on M^{2n} is obtained by the exterior derivative of the one-form $\hat{\gamma}$,

$$\omega=d\hat{\gamma}. \tag{A11}$$

For a charged particle moving in an electromagnetic field, the phase space is the extended eight-dimensional space $(\mathbf{v},\mathbf{r},w,t)$. The Poincaré–Cartan one-form (phase space Lagrangian) is:

$$\gamma_E=\left[\frac{e}{c}\mathbf{A}(\mathbf{r},t)+m\mathbf{v}\right]\cdot d\mathbf{r}-w\,dt-\left[\frac{1}{2}mv^2+e\phi(\mathbf{r},t)-w\right]dz. \tag{A12}$$

It is easy to show that starting from this one-form, we can recover the usual motion equation for a charged particle in an electromagnetic field.

One of the most useful features of the Hamiltonian formalism for dynamic systems is the systematic techniques available to deal with a perturbed system. When the system is not far away from a preferred situation which could have, for example, an exact solution or some symmetry properties, we can reconstruct coordinate system such that the good properties of the unperturbed system can be utilized.

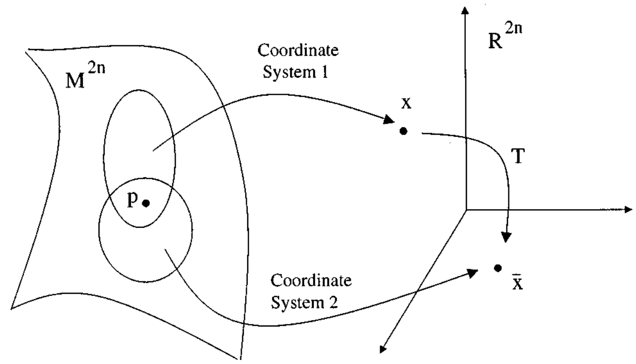

FIG. 3. Coordinate transformation as a map in R^{2n}.

The coordinate transformation for the phase space M^{2n} can be represented by a map in the R^{2n} space (see Fig. 3),

$$T:x\rightarrow\overline{x}. \tag{A13}$$

From the viewpoint of Lie perturbation methods,[49,50] such a map is generated by the flow due to a vector field $\mathbf{G}$. This flow can be formally written as

$$g_{\mathbf{G}}(t,x)=e^{t\mathbf{G}}x. \tag{A14}$$

The coordinate transformation T is the flow mapping when $t=1$,

$$T=e^{\mathbf{G}}. \tag{A15}$$

Perturbation theory considers transformations which are near identity. The vector field is small. Using the natural small parameter existing in the problem, ϵ, we express the coordinate transformation as

$$T=e^{\epsilon\mathbf{G}}. \tag{A16}$$

The push-forward and pull-back transformations are therefore

$$T_*=e^{-\epsilon L_{\mathbf{G}}},\qquad T^*=e^{\epsilon L_{\mathbf{G}}}, \tag{A17}$$

where $L_{\mathbf{G}}$ is the Lie derivative. The vector field $\mathbf{G}$ is chosen such that in the new coordinate system, the dynamic structure has some desired properties. However, it is generally only possible to satisfy these desired properties to $O(\epsilon)$. To go to higher order, we can introduce a series of transformations,[50]

$$T_n=e^{\epsilon^n\mathbf{G}_n}\quad n=1,2,\cdots. \tag{A18}$$

$\mathbf{G}_n$ is chosen to give the desired properties to $O(\epsilon^n)$. Therefore the overall transformation is

$$T=\cdots T_3T_2T_1. \tag{A19}$$

Its push-forward, pull-back, and their inverses are

$$T_*=\cdots T_{3*}T_{2*}T_{1*},\quad T_*^{-1}=T_{1*}^{-1}T_{2*}^{-1}T_{3*}^{-1}\cdots,$$
$$T^*=T_1^*T_2^*T_3^*\cdots,\quad T^{*-1}=\cdots T_3^{*-1}T_2^{*-1}T_1^{*-1}. \tag{A20}$$

When x is transformed to $\bar{x}=Tx$, the Poincaré–Cartan one-form γ is transformed to $\bar{\gamma}=T^{*-1}\gamma+dS$, where S is an arbitrary gauge function. Adding dS to $\bar{\gamma}$ does not change the particle dynamics because the symplectic structure is $d\gamma$ and $d\,dS=0$. The expression of $\bar{\gamma}$ in terms of ϵ can easily be calculated,[50]

$$\begin{aligned}
\tilde{\gamma}_0 &= \gamma_0 + dS_0, \\
\bar{\gamma}_1 &= \gamma_1 - L_1\gamma_0 + dS_1, \\
\bar{\gamma}_2 &= \gamma_1 - L_1\gamma_1 + (\tfrac{1}{2}L_1^2 - L_2)\gamma_0 + dS_2, \\
&\vdots .
\end{aligned} \tag{A21}$$

These equations are solved for the vector field $\mathbf{G}_n$ and the gauge function S_n order by order when the desired requirement for $\bar{\gamma}_n$ is imposed order by order. The Lie derivatives in the above equations can be simplified by use of the homotopy formula

$$L_{\mathbf{v}}\alpha = i_{\mathbf{v}}(d\alpha) + d(i_{\mathbf{v}}\alpha), \tag{A22}$$

where $i_{\mathbf{v}}\beta$ is the interior product between the vector field $\mathbf{v}$ and the form β. $d(i_{\mathbf{v}}\alpha)$ can be further absorbed into dS. For example, we will use the following expressions for $\bar{\gamma}_1$

$$\bar{\gamma}_1 = \gamma_1 - i_{\mathbf{G}_1}(d\gamma_0) + dS_1. \tag{A23}$$

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