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Periodically-focused solutions to the nonlinear Vlasov–Maxwell equations for intense charged particle beams

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Abstract

An intense nonneutral ion beam propagates in the z-direction through a periodic focusing quadrupole field with transverse focusing force, $\boldsymbol{F}_{\mathrm{foc}} = -\kappa_q(s)(x\hat{\boldsymbol{e}}_x - y\hat{\boldsymbol{e}}_y)$, on the beam ions. Here, the oscillatory lattice coefficient satisfies $\kappa_q(s+S) = \kappa_q(s)$, where $S=$ const. is the axial periodicity length. The model employs the Vlasov–Maxwell equations to describe the nonlinear evolution of the distribution function $f_{\mathrm{b}}(x,y,x',y',s)$ and the normalized self-field potential $\psi(x,y,s)$ in the transverse laboratory-frame phase space (x,y,x',y'). Using a third-order Hamiltonian averaging technique, a canonical transformation is employed with an expanded generating function which transforms away the rapidly oscillating terms, and leads to a Hamiltonian in the 'slow' transformed variables $(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}')$, with constant focusing coefficient $\kappa_{\mathrm{fq}} =$ const. © 1999 Elsevier Science B.V. All rights reserved.

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Periodic focusing accelerators and transport systems [1,2] have a wide range of applications ranging from basic scientific research, to applications such as heavy ion fusion, spallation neutron sources, tritium production, and nuclear waste treatment. Of particular importance, at high beam currents and charge densities, are the combined effects of the applied focusing field and the intense self fields produced by the beam space charge and current on determining detailed equilibrium, stability and transport properties [1]. Through analytical studies based on the nonlinear Vlasov–Maxwell equations, and numerical simulations using particle-in-cell models and nonlinear perturbative simulation techniques, considerable progress [3–11] has been made in developing an improved understanding of the collective processes and nonlinear beam dynamics characteristic of high-intensity beam propagation in periodic focusing and uniform focusing transport systems. However, despite the extensive literature on equilibrium and stability properties, until the present paper, the Kapchinskij–Vladimirskij (KV) beam equilibrium [3–6], including its recent generalization to a rotating beam in a periodic focusing solenoidal field [9], has been the *only* known periodically-focused solution to the nonlinear Vlasov–Maxwell equations for an intense beam propagating through an alternating-gradient quadrupole or solenoidal field configuration. While allowing for high space-charge intensity, the KV distribution is nonetheless of very limited practi-

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cal interest, particularly because the distribution function has a *highly-inverted* (and unphysical) distribution in phase space, and the corresponding density profile is *exactly uniform* in the beam interior.

It is therefore important to develop a framework based on the nonlinear Vlasov–Maxwell equations [5,9] that is able to investigate the equilibrium and stability properties of a far more general class of periodically-focused beam distribution functions. In a recent calculation [12], Channell has developed a third-order Hamiltonian averaging technique for investigating solutions to the nonlinear Vlasov–Maxwell equations for systems subject to a periodic external force. The formalism [12] uses a canonical transformation given by an expanded generating function to transform away the rapidly oscillating terms [12–14] and end up with a Hamiltonian $\mathscr{H}$ that depends only on 'slow' variables. The purpose of the present analysis is to apply this averaging technique to intense beam propagation through a periodic focusing lattice [14]. The expansion procedure is expected to be valid [14] for sufficiently small phase advance ($\sigma \lesssim 60°$, say).

We consider a thin ($r_b \ll S$), intense ion beam with characteristic radius r_b and axial momentum $\gamma_b m_b \beta_b c$ propagating in the z-direction through a periodic focusing quadrupole field with axial periodicity length S. Here, $(\gamma_b - 1)m_b c^2$ is the directed axial kinetic energy of the beam ions, $\gamma_b = (1-\beta_b^2)^{-1/2}$ is the relativistic mass factor, $V_b = \beta_b c$ is the average axial velocity, $+Z_b e$ and m_b are the ion charge and rest mass, respectively, and c is the speed of light in *vacuo*. The axial momentum spread is assumed to be negligibly small, and the ion motion in the beam frame is assumed to be nonrelativistic. We introduce the scaled time variable $s = \beta_b ct$, and the (dimensionless) transverse velocities $x' = dx/ds$ and $y' = dy/ds$. For a thin beam, the applied focusing force on a beam particle is taken to be $\boldsymbol{F}_{\rm foc} = -\kappa_q(s)[x\hat{\boldsymbol{e}}_x - y\hat{\boldsymbol{e}}_y]$, where (x,y) is the transverse displacement from the beam axis. The oscillating lattice coefficient $\kappa_q(s+S) = \kappa_q(s)$ is defined by $\kappa_q(s) = Z_b eB'_q(s)/\gamma_b m_b \beta_b c^2$, where $S = \text{const.}$ is the axial periodicity length, and $\int_0^S ds\kappa_q(s) = 0$. Within the context of these assumptions, the beam dynamics in the transverse, laboratory-frame phase space (x,y,x',y') is described self-consistently by the nonlinear Vlasov–Maxwell equations for the distribution function $f_b(x,y,x',y',s)$ and the (dimensionless) self-field potential $\psi(x,y,s) = Z_b e\phi(x,y,s)/\gamma_b^3 m_b \beta_b^2 c^2$, which can be expressed as [5,9]

$$\left\{ \frac{\partial}{\partial s} + x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} - \left(\kappa_q(s)\,x + \frac{\partial\psi}{\partial x} \right) \frac{\partial}{\partial x'} - \left(-\kappa_q(s)\,y + \frac{\partial\psi}{\partial y} \right) \frac{\partial}{\partial y'} \right\} f_b = 0\,, \tag{1}$$

and

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)\psi = -\frac{2\pi K_b}{N_b}\int dx'dy' f_b\,. \tag{2}$$

Here, $\phi(x,y,s)$ is the electrostatic potential, $n_b(x,y,s) = \int dx'dy' f_b(x,y,x',y',s)$ is the number density of the beam ions, and the constants, $K_b = 2N_b Z_b^2 e^2/\gamma_b^3 m_b \beta_b^2 c^2$ and $N_b = \int dx dy dx' dy' f_b$, are the self-field perveance and the number of beam ions per unit axial length, respectively.

A direct calculation of kinetic equilibrium and stability properties [5–11] from Eqs. (1) and (2) is considerably complicated by the fact that the lattice coefficient $\kappa_q(s)$ is an oscillatory function of s. In the present analysis, we make use of Channell's third-order Hamiltonian averaging technique [12] to transform from laboratory-frame variables (x,y,x',y') to 'slow' variables (X,Y,X',Y'), with a new Hamiltonian $\mathscr{H}(X,Y,X',Y',s)$. The formalism employs a canonical transformation given by an expanded generating function to transform away the rapidly oscillating terms [12–14]. The laboratory-frame Hamiltonian is formally expressed as

$$\begin{aligned} H(x,y,x',y',s) &= \epsilon\hat{H}(x,y,x',y',s) \\ &= \epsilon\Big[\tfrac{1}{2}(x'^2 + y'^2) + \tfrac{1}{2}\kappa_q(s)(x^2 - y^2) \\ &\quad + \psi(x,y,s) \Big]\,, \end{aligned} \tag{3}$$

where $\hat{H}$ is defined by Eq. (3), and ϵ is a small dimensionless parameter proportional to the strength of the focusing field. We introduce a near-identity canonical transformation [12] where the expanded generation function is defined by

$$S(x,y,X',Y',s) = xX' + yY' + \sum_{n=1}^{\infty} \epsilon^n S_n(x,y,X',Y',s) . \qquad (4)$$

The transformed Hamiltonian in the new variables is given by

$$\mathscr{H}(X,Y,X',Y',s) = H(x,y,x',y',s) + \frac{\partial}{\partial s} S(x,y,X',Y',s) , \qquad (5)$$

or equivalently, expressing $\mathscr{H} = \sum_{n=1}^{\infty} \epsilon^n \mathscr{H}_n(X,Y,X',Y',s)$,

$$\sum_{n=1}^{\infty} \epsilon^n \mathscr{H}_n(X,Y,X',Y',s) = \epsilon\Big[\tfrac{1}{2}(x'^2+y'^2) + \tfrac{1}{2}\kappa_q(s)(x^2-y^2) + \psi(x,y,s)\Big] + \sum_{n=1}^{\infty} \epsilon^n \frac{\partial}{\partial s} S_n(x,y,X',Y',s) . \qquad (6)$$

To determine the transformed Hamiltonian, note that the variables (x,y,x',y') occurring on the right-hand sides of Eqs. (5) and (6) are to be expressed in terms of (X,Y,X',Y',s), i.e., $x = x(X,Y,X',Y',s)$, etc. The coordinate transformation generated by Eq. (4) is given by

$$X = \frac{\partial S}{\partial X'} = x + \sum_{n=1}^{\infty} \epsilon^n \frac{\partial}{\partial X'} S_n(x,y,X',Y',s) , \qquad (7)$$

$$x' = \frac{\partial S}{\partial x} = X' + \sum_{n=1}^{\infty} \epsilon^n \frac{\partial}{\partial x} S_n(x,y,X',Y',s) , \qquad (8)$$

with similar expressions for $Y = \partial S/\partial Y'$ and $y' = \partial S/\partial y$. Eqs. (7) and (8) are to be solved iteratively for $x(X,Y,X',Y',s)$, $x'(X,Y,X',Y',s)$, etc.

Analysis of Eqs. (6)–(8) to determine the slowly-varying Hamiltonian $\mathscr{H}(X,Y,X',Y',s)$ correct to order ϵ^3 proceeds as follows. We solve Eq. (6) order by order for $\mathscr{H}_n$. Because the generating function $S_n(x,y,X',Y',s)$ is arbitrary and unspecified, we use this freedom to choose S_n to cancel any rapidly oscillating contributions to $\mathscr{H}_n$, so that the resulting expression for $\mathscr{H}_n$ is *slowly varying*, order by order. In Eq. (6), we expand $x = X + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots$, $x' = X' + \epsilon x_1' + \epsilon^2 x_2' + \epsilon^3 x_3' + \cdots$, etc., and $\psi(x,y,s) = \psi(X,Y,s) + \epsilon(x_1\partial/\partial X + y_1 \partial/\partial Y)\psi(X,Y,s) + \cdots$. The coordinate transformation is also determined *iteratively* by solving Eqs. (7) and (8) for (x_n, y_n, x_n', y_n').

The detailed solution to Eqs. (6)–(8) will be presented elsewhere [14] correct to order ϵ^3. We summarize here the definitions of the averages over the lattice function $\kappa_q(s)$ that occur in the analysis. Assuming that $\int_0^S ds\kappa_q(s) = 0$, and that $\kappa_q(s)$ has *odd* half-period symmetry with $\kappa_q(s - S/2) = -\kappa_q[-(s-S/2)]$, the key definitions [14] are

$$\alpha_q(s) = \int_0^s ds\kappa_q(s) , \quad \langle\alpha_q\rangle = \frac{1}{S}\int_0^S ds\alpha_q(s) ,$$

$$\beta_q(s) = \frac{1}{S}\int_0^s ds\big[\alpha_q(s) - \langle\alpha_q\rangle\big] ,$$

$$\langle\beta_q\rangle = \frac{1}{S}\int_0^S ds\beta_q(s) = 0 ,$$

$$\delta_q(s) = \alpha_q^2(s) - 2\kappa_q(s)\beta_q(s) ,$$

$$\langle\delta_q\rangle = \frac{1}{S}\int_0^S ds\big[3\alpha_q^2(s) - 2\langle\alpha_q\rangle^2\big] ,$$

$$\kappa_{\mathrm{fq}} = \langle\delta_q\rangle - \langle\alpha_q\rangle^2 = \frac{3}{S}\int_0^S ds\big[\alpha_q^2(s) - \langle\alpha_q\rangle^2\big] . \qquad (9)$$

Following the procedure outlined above, we solve Eqs. (6)–(8) order by order, correct to order ϵ^3. Without presenting algebraic details [14], the slowly varying Hamiltonian $\mathscr{H} = \epsilon\mathscr{H}_1 + \epsilon^2\mathscr{H}_2 + \epsilon^3\mathscr{H}_3$ is found to be

$$\mathscr{H}\big(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}',s\big) = \tfrac{1}{2}(\tilde{X}'^2 + \tilde{Y}'^2) + \tfrac{1}{2}\kappa_{\mathrm{fq}}(\tilde{X}^2 + \tilde{Y}^2) + \psi(\tilde{X},\tilde{Y},s) , \qquad (10)$$

where we have set $\epsilon = 1$. Here, we have introduced the additional (canonical) fiber transformation [15] to shifted velocity coordinates defined by

$$\tilde{X} = X , \quad \tilde{Y} = Y ,$$
$$\tilde{X}' = X' - \langle\alpha_q\rangle X , \quad \tilde{Y}' = Y' + \langle\alpha_q\rangle Y . \qquad (11)$$

Similarly, we calculate $x = X + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3$, $x' = X' + \epsilon x_1' + \epsilon^2 x_2' + \epsilon^3 x_3'$, etc. Setting $\epsilon = 1$, this gives [14]

$$x(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}',s) = [1-\beta_q(s)]\tilde{X} + 2\left(\int_0^s ds\beta_q(s)\right)\tilde{X}',$$

$$y(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}',s) = [1+\beta_q(s)]\tilde{Y} - 2\left(\int_0^s ds\beta_q(s)\right)\tilde{Y}', \tag{12}$$

and

$$\begin{aligned} x'(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}',s) &= [1+\beta_q(s)]\tilde{X}' + \Big\{-\alpha_q(s) + \langle\alpha_q\rangle \\ &\quad + \langle\alpha_q\rangle\beta_q(s) - \alpha_q(s)\beta_q(s) \\ &\quad - \left(\int_0^s ds[\delta_q(s) - \langle\delta_q\rangle]\right)\Big\}\tilde{X} \\ &\quad + \left(\int_0^s ds\beta_q(s)\right)\frac{\partial}{\partial\tilde{X}}\left(\tilde{X}\frac{\partial\psi}{\partial\tilde{X}} - \tilde{Y}\frac{\partial\psi}{\partial\tilde{Y}}\right), \end{aligned}$$

$$\begin{aligned} y'(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}',s) &= [1-\beta_q(s)]\tilde{Y}' + \Big\{\alpha_q(s) - \langle\alpha_q\rangle \\ &\quad + \langle\alpha_q\rangle\beta_q(s) - \alpha_q(s)\beta_q(s) \\ &\quad - \left(\int_0^s ds[\delta_q(s) - \langle\delta_q\rangle]\right)\Big\}\tilde{Y} \\ &\quad - \left(\int_0^s ds\beta_q(s)\right)\frac{\partial}{\partial\tilde{Y}}\left(\tilde{Y}\frac{\partial\psi}{\partial\tilde{Y}} - \tilde{X}\frac{\partial\psi}{\partial\tilde{X}}\right), \end{aligned} \tag{13}$$

correct to order ϵ^3. Here, the relative size of the various terms in Eqs. (12) and (13) stand in the ratio

$$\alpha_q(s), \langle\alpha_q\rangle \qquad \text{:Terms of order } \epsilon,$$

$$\beta_q(s) \qquad \text{:Terms of order } \epsilon^2,$$

$$\langle\alpha_q\rangle\beta_q(s),\ \alpha_q(s)\beta_q(s),\ \left(\int_0^s ds\beta_q(s)\right),$$

$$\left(\int_0^s ds[\delta_q(s) - \langle\delta_q\rangle]\right)\text{: Terms of order } \epsilon^3. \tag{14}$$

Because the focusing coefficient is constant ($\kappa_{fq} =$ const.) and isotropic in the $\tilde{X}$–$\tilde{Y}$ plane in the transformed Hamiltonian defined in Eq. (10), there is enormous simplification in analyzing kinetic equilibrium and stability properties in the transformed variables $(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}')$. The nonlinear Vlasov–Maxwell equations for the distribution function $F_b(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}',s)$ and self-field potential $\psi(\tilde{X},\tilde{Y},s)$ in the ‘slow’ variables are given by [14]

$$\begin{aligned} &\left\{\frac{\partial}{\partial s} + \tilde{X}'\frac{\partial}{\partial\tilde{X}} + \tilde{Y}'\frac{\partial}{\partial\tilde{Y}} - \left(\kappa_{fq}\tilde{X} + \frac{\partial}{\partial\tilde{X}}\psi\right)\frac{\partial}{\partial\tilde{X}'}\right. \\ &\quad \left. - \left(\kappa_{fq}\tilde{Y} + \frac{\partial}{\partial\tilde{Y}}\psi\right)\frac{\partial}{\partial\tilde{Y}'}\right\}F_b = 0, \end{aligned} \tag{15}$$

$$\left(\frac{\partial^2}{\partial\tilde{X}^2} + \frac{\partial^2}{\partial\tilde{Y}^2}\right)\psi = -\frac{2\pi K_b}{N_b}\int d\tilde{X}'d\tilde{Y}'F_b, \tag{16}$$

where $\kappa_{fq} =$ const. is defined in Eq. (9). It should be emphasized that the nonlinear Vlasov–Maxwell Eqs. (15) and (16) in the slow variables $(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}')$, when supplemented by the coordinate transformation in Eqs. (12) and (13), are fully equivalent to the nonlinear Vlasov–Maxwell Eqs. (1) and (2) in the laboratory-frame variables (x,y,x',y'), correct to order ϵ^3. Furthermore, because the coordinate transformation is canonical, the laboratory-frame distribution function $f_b(x,y,x',y',s)$ is related to the transformed distribution function $F_b(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}',s)$ by $f_b(x,y,x',y',s)\ dxdydx'dy' = F_b(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}',s)\ d\tilde{X}d\tilde{Y}d\tilde{X}'d\tilde{Y}'$, and the Jacobian of the transformation is equal to unity, $\partial(x,y,x',y')/\partial(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}') = 1$, which can also be verified by direct calculation[14] from Eqs. (12) and (13) correct to order ϵ^3. Therefore, once the distribution function $F_b(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}',s)$ in the transformed variables is calculated from Eqs. (18) and (19), the laboratory-frame distribution function
$f_b(x,y,x',y',s)$ is given by

$$f_b(x,y,x',y',s) = F_b(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}',s). \tag{17}$$

Here, $\tilde{X}(x,y,x',y',s)$, $\tilde{Y}(x,y,x',y',s)$, etc., denotes the *inverse* coordinate transformation [14] to Eqs. (12) and (13) given by

$$\tilde{X}(x,y,x',y',s) = [1+\beta_q(s)]\,x - 2\left(\int_0^s ds\beta_q(s)\right)x',$$

$$\tilde{Y}(x,y,x',y',s) = [1-\beta_q(s)]\,y + 2\left(\int_0^s ds\beta_q(s)\right)y', \qquad (18)$$

and

$$\begin{aligned}\tilde{X}'(x,y,x',y',s) &= [1-\beta_q(s)]\,x' - \Big\{-[\alpha_q(s)-\langle\alpha_q\rangle]\\ &\quad\times[1+\beta_q(s)] - \left(\int_0^s ds[\delta_q(s)-\langle\delta_q\rangle]\right)\Big\}x\\ &\quad-\left(\int_0^s ds\beta_q(s)\right)\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}-y\frac{\partial\psi}{\partial y}\right),\end{aligned}$$

$$\begin{aligned}\tilde{Y}'(x,y,x',y',s) &= [1+\beta_q(s)]\,y' - \Big\{[\alpha_q(s)-\langle\alpha_q\rangle]\\ &\quad\times[1-\beta_q(s)] - \left(\int_0^s ds[\delta_q(s)-\langle\delta_q\rangle]\right)\Big\}y\\ &\quad+\left(\int_0^s ds\beta_q(s)\right)\frac{\partial}{\partial y}\left(y\frac{\partial\psi}{\partial y}-x\frac{\partial\psi}{\partial x}\right),\end{aligned} \qquad (19)$$

correct to order ϵ^3. In obtaining Eqs. (18) and (19), we have made use of the fact that the self-field contributions in Eq. (13) are proportional to $(\int_0^s ds\beta_q(s))$, which is of order ϵ^3 [see Eq. (14)]. Therefore, to leading order, we approximate $(\partial/\partial\tilde{X})(\tilde{X}\partial/\partial\tilde{X} - \tilde{Y}\partial/\partial\tilde{Y})\psi(\tilde{X},\tilde{Y},s)$ by $(\partial/\partial x)(x\partial/\partial x - y\partial/\partial y)\psi(x,y,s)$, etc., in obtaining Eqs. (18) and (19).

Because of the simple form of the Vlasov–Maxwell Eqs. (15) and (16), with constant focusing coefficient $\kappa_{\rm fq} = $ const., a wide range of literature developed for the constant focusing case can be applied virtually intact in the transformed variables. Furthermore, the present analysis makes accessible for the first time numerous examples (in addition to the KV beam equilibrium) of nonlinear solutions to the Vlasov–Maxwell equations which are *periodically-focused* in the laboratory frame. Detailed examples, including the back-transformation of beam properties to the laboratory frame, will be presented elsewhere [14], and we summarize here several key results. For present purposes, it is assumed for simplicity that the conducting wall is infinitely far removed from the beam ($r_{\rm b} \ll r_{\rm w} \to \infty$).

Because $\kappa_{\rm fq} = $ const., Eqs. (15) and (16) support a wide range of axisymmetric equilibrium solutions [5,9]. Here, we introduce cylindrical polar coordinates $(\tilde{R},\Theta)$ with $\tilde{X} = \tilde{R}\cos\theta$ and $\tilde{Y} = \tilde{R}\sin\Theta$, where $\tilde{R} = (\tilde{X}^2 + \tilde{Y}^2)^{1/2}$. Setting $\partial/\partial s = 0 = \partial/\partial\Theta$ in Eqs. (15) and (16), it is readily shown that any distribution function of the form

$$F_{\rm b}^0(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}') = F_{\rm b}^0(\mathscr{H}^0), \qquad (20)$$

where $\mathscr{H}^0 = \frac{1}{2}(\tilde{X}'^2 + \tilde{Y}'^2) + \frac{1}{2}\kappa_{\rm fq}\tilde{R}^2 + \psi^0(\tilde{R})$ is the single-particle Hamiltonian, is an exact nonrotating equilibrium solution to the nonlinear Vlasov Eq. (15). Here, $\psi^0(\tilde{R})$ is determined self-consistently from

$$\frac{1}{\tilde{R}}\frac{\partial}{\partial\tilde{R}}\tilde{R}\frac{\partial}{\partial\tilde{R}}\psi^0(\tilde{R}) = -\frac{2\pi K_{\rm b}}{N_{\rm b}}\int d\tilde{X}'d\tilde{Y}'F_{\rm b}^0(\mathscr{H}^0), \qquad (21)$$

where $n_{\rm b}^0(\tilde{R}) = \int d\tilde{X}'d\tilde{Y}'F_{\rm b}^0(\mathscr{H}^0)$ is the equilibrium density profile in the transformed variables. Note that Eq. (21) is generally a nonlinear differential equation for the self-field potential $\psi^0(\tilde{R})$.

There is enormous latitude [5,9] in specifying the functional form of $F_{\rm b}^0(\mathscr{H}^0)$ in the transformed variables. Once the functional form of $F_{\rm b}^0(\mathscr{H}^0)$ is specified, however, and $\psi^0(\tilde{R})$ is calculated self-consistently from Eq. (21), other equilibrium properties in the transformed variables can be readily determined, such as the density profile, the transverse temperature profile, etc. One important example is the thermal equilibrium distribution function [5,8]

$$F_{\rm b}^0(\mathscr{H}^0) = \hat{n}_{\rm b}\left(\frac{\gamma_{\rm b}m_{\rm b}\beta_{\rm b}^2c^2}{2\pi\hat{T}_{\perp b}}\right)\times\exp\left\{-\frac{\gamma_{\rm b}m_{\rm b}\beta_{\rm b}^2c^2}{\hat{T}_{\perp b}}\mathscr{H}^0\right\}, \qquad (22)$$

where $\hat{n}_{\rm b}$ and $\hat{T}_{\perp b}$ are positive constants with dimensions of density and temperature (energy units), respectively. Taking the zero of potential to be $\psi^0(\tilde{R}=0)=0$, Eq. (22) readily gives the density profile $n_b^0(\tilde{R}) = \hat{n}_b \exp\{-(\gamma_b m_b \beta_b^2 c^2/2\hat{T}_{\perp b})[\kappa_{fq}\tilde{R}^2 + 2\psi^0(\tilde{R})]\}$. Solving Eq. (21) numerically for $\psi^0(\tilde{R})$ then gives a bell-shaped density profile for $n_b^0(\tilde{R})$, which assumes a maximum value $(\hat{n}_b)$ at $\tilde{R}=0$ and decreases monotonically to zero as $\tilde{R}\to\infty$ provided the applied focusing field is sufficiently strong that $\kappa_{fq}\beta_b^2 c^2 > (1/2\gamma_b^2)\hat{\omega}_{pb}^2$ [14]. Here, $\hat{\omega}_{pb}^2 = 4\pi\hat{n}_b Z_b^2 e^2/\gamma_b m_b$ is the on-axis plasma frequency-squared.

Eq. (22) is an important example of an equilibrium distribution function that is known to be *stable*. Making use of global conservation constraints satisfied by the nonlinear Vlasov–Maxwell Eqs. (15) and (16) in the transformed variables, it can be shown that

$$\frac{\partial}{\partial\mathscr{H}^0} F_b^0(\mathscr{H}^0) \le 0 \tag{23}$$

is a sufficient condition for stability [11]. Whenever Eq. (23) is satisfied, the system is stable, and perturbations, $\delta F_b(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}',s)$ and $\delta\psi(\tilde{X},\tilde{Y},s)$, about equilibrium do not amplify. The stability theorem in Eq. (23) is a very powerful result, and is valid nonlinearly (finite-amplitude perturbations) as well as for small-amplitude perturbations.

Statistical averages are also readily calculated for the general class of equilibrium distribution functions in Eq. (20). Here, the statistical average of a phase function $\chi(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}',s)$ in the transformed variables is defined in the usual manner by $\langle\chi\rangle_0 = N_b^{-1}\int d\tilde{X}d\tilde{Y}d\tilde{X}'d\tilde{Y}'\chi F_b^0(\mathscr{H}^0)$, where $N_b = \int d\tilde{X}d\tilde{Y}d\tilde{X}'d\tilde{Y}'F_b^0(\mathscr{H}^0)$ is the number of particles per unit axial length. For example, because $\mathscr{H}^0$ is an *even* function of $\tilde{X},\tilde{Y},\tilde{X}'$ and $\tilde{Y}'$, it follows trivially that $\langle\tilde{X}\rangle_0 = 0 = \langle\tilde{Y}\rangle_0$, $\langle\tilde{X}'\rangle_0 = 0 = \langle\tilde{Y}'\rangle_0$, and $\langle\tilde{X}\tilde{X}'\rangle_0 = 0 = \langle\tilde{Y}\tilde{Y}'\rangle_0$. Furthermore, the mean-square beam radius is $R_{b0}^2 = \langle\tilde{X}^2+\tilde{Y}^2\rangle_0 = \partial\langle\tilde{X}^2\rangle_0 = \partial\langle\tilde{Y}^2\rangle_0$. Finally, for the general class of beam equilibria $F_b^0(\mathscr{H}^0)$, the global radial force balance equation can be expressed as [10,14].

$$\left(\kappa_{fq} - \frac{K_b}{2R_{b0}^2}\right)R_{b0} = \frac{\epsilon_0^2}{4R_{b0}^3}, \tag{24}$$

where $\epsilon_0^2 = 4\langle\tilde{X}^2+\tilde{Y}^2\rangle_0\langle\tilde{X}'^2+\tilde{Y}'^2\rangle_0$ is the total unnormalized transverse emittance-squared.

Properties of the *periodically-focused* beam distribution function $f_b(x,y,x',y',s)$ in the laboratory frame are readily calculated for the entire class of beam equilibria $F_b^0(\mathscr{H}^0)$ by making use of Eq. (17) and the coordinate transformations in Eqs. (18) and (19). From Eq. (17) it follows that $f_b(x,y,x',y',s) = F_b^0(\mathscr{H}^0)$, where $\mathscr{H}^0$ is defined by

$$\begin{aligned}\mathscr{H}^0&\left(\tilde{X},\tilde{Y},\tilde{X}',\tilde{Y}'\right)\\ &= \tfrac{1}{2}\left[\tilde{X}'^2(x,y,x',y',s) + \tilde{Y}'^2(x,y,x',y',s)\right]\\ &\quad + \tfrac{1}{2}\kappa_{fq}\tilde{R}^2(x,y,x',y',s)\\ &\quad + \psi^0\left(\tilde{R}(x,y,x',y',s)\right).\end{aligned} \tag{25}$$

Here, $\tilde{R}^2(x,y,x',y',s) = \tilde{X}^2(x,y,x',y',s) + \tilde{Y}^2 \times (x,y,x',y',s)$. Because the s-dependent coefficients of $\alpha_q(s)$, $\beta_q(s)$, etc., have axial periodicity length $S=$ const., the laboratory-frame distribution function also satisfies $f_b(x,y,x',y',s+S) = f_b(x,y,x',y',s)$.

A wide range of beam properties in the laboratory frame can be calculated from Eq. (25) and $f_b(x,y,x',y',s) = F_b^0(\mathscr{H}^0)$. Because $\mathscr{H}^0$ is an even function of $\tilde{X}(x,y,x',y',s)$, $\tilde{X}'(x,y,x',y',s)$, etc., there is enormous simplification in calculating statistical averages and macroscopic moments. For example, defining the statistical average of a phase function $\chi(x,y,x',y',s)$ in the laboratory frame by $\langle\chi\rangle = N_b^{-1}\int dxdydx'dy'\chi f_b(x,y,x',y',s)$, it is readily shown that $\langle x\rangle = 0 = \langle y\rangle$ and $\langle x'\rangle = 0 = \langle y'\rangle$ correct to order ϵ^3, which corresponds to a beam equilibrium that remains centered in the laboratory frame. Proceeding to higher-order moments, and making use of Eq. (14), it can be shown that

$$\begin{aligned}\langle x^2\rangle(s) &= \left[1-\beta_q(s)\right]^2\langle\tilde{X}^2\rangle_0 \equiv \tfrac{1}{2}a^2(s),\\ \langle y^2\rangle(s) &= \left[1+\beta_q(s)\right]^2\langle\tilde{Y}^2\rangle_0 \equiv \tfrac{1}{2}b^2(s),\end{aligned} \tag{26}$$

and that $\langle x'^2\rangle(s) = [1-\beta_q(s)]^2\langle\tilde{X}'^2\rangle_0 + [\alpha_q(s) - \langle\alpha_q\rangle]^2\langle\tilde{X}^2\rangle_0$, $\langle y'^2\rangle(s) = [1+\beta_q(s)]^2\langle\tilde{Y}'^2\rangle_0 + [\alpha_q(s) - \langle\alpha_q\rangle]^2\langle\tilde{Y}^2\rangle_0$, $\langle xx'\rangle^2 = [\alpha_q(s) - \langle\alpha_q\rangle]^2\langle\tilde{X}^2\rangle_0^2$, $\langle yy'\rangle^2 = [\alpha_q(s) - \langle\alpha_q\rangle]^2\langle\tilde{Y}^2\rangle_0^2$, correct to order ϵ^3. Here, $\langle\tilde{X}^2\rangle_0 = \langle\tilde{Y}^2\rangle_0 = R_{b0}^2/2$, and as expected, the circular cross-section beam equilibrium $F_b^0(\mathscr{H}^0)$, which has constant mean-square radius $R_{b0}^2 = \langle \tilde{X}^2 + \tilde{Y}^2 \rangle_0$ in the transformed variables, when mapped back to the laboratory frame has a pulsating elliptical cross-section with $\langle x^2 \rangle / a^2(s) + \langle y^2 \rangle / b^2(s) = 1$. A further striking result is evident from the present analysis. Even though the kinetic energy components, $\frac{1}{2}\langle x'^2 \rangle(s)$ and $\frac{1}{2}\langle y'^2 \rangle(s)$, are oscillatory functions of s, the transverse emittances are conserved quantities (independent of s). This follows because

$$\begin{aligned}
\epsilon_x^2(s) &= 4\left[\langle x^2\rangle(s)\langle x'^2\rangle(s) - \langle xx'\rangle^2(s)\right] \\
&= 4\langle \tilde{X}^2\rangle_0 \langle \tilde{X}'^2\rangle_0 = \epsilon_{x0}^2 = \text{const.}, \\
\epsilon_y^2(s) &= 4\left[\langle y^2\rangle(s)\langle y'^2\rangle(s) - \langle yy'\rangle^2(s)\right] \\
&= 4\langle \tilde{Y}^2\rangle_0 \langle \tilde{Y}'^2\rangle_0 = \epsilon_{y0}^2 = \text{const.},
\end{aligned} \tag{27}$$

correct to order ϵ^3.

Finally, for specified equilibrium distribution in the transformed variables, various macroscopic properties in the laboratory frame can also be calculated from Eq. (25) and $f_{\mathrm{b}}(x,y,x',y',s) = F_{\mathrm{b}}^0(\mathscr{H}^0)$. For example, if the equilibrium density profile in the transformed variables is determined to be $n_{\mathrm{b}}^0(\tilde{R}/R_{b0})$ (here we scale $\tilde{R}$ by R_{b0} without loss of generality), then the density profile in the laboratory frame, $n_{\mathrm{b}}(x,y,s) = \int dx'dy' f_{\mathrm{b}}(x,y,x',y',s)$, is found to be [14]

$$n_{\mathrm{b}}(x,y,s) = \frac{1}{1-\beta_q^2(s)} n_{\mathrm{b}}^0\left[\tilde{R}(x,y,s)/R_{b0}\right], \tag{28}$$

correct to order ϵ^3, where $\tilde{R}^2(x,y,s)/R_{b0}^2 \equiv x^2/a^2(s) + y^2/b^2(s)$.

Eq. (26)–(28) are important results for the general class of equilibrium distribution functions $F_{\mathrm{b}}^0(\mathscr{H}^0)$. Not only are the transverse emittances conserved, the constant-density contours in the laboratory frame correspond to elliptical surfaces with $x^2/a^2(s) + y^2/b^2(s) = \text{const.}$ In Eq. (28), note that the factor $[1-\beta_q^2(s)]^{-1} \simeq 1$ correct to order ϵ^3 [Eq. (14)].

Important in the present analysis is the definition and size of the dimensionless small parameter ϵ. This is best determined by examination of the relative size of the *correction terms* in Eqs. (12) and (13) to the identity transformation $(x,y,x',y') = (X,Y,X',Y')$. As an example, we take $\kappa_q(s) = \hat{\kappa}_q \sin(2\pi s/S)$, where $\hat{\kappa}_q = \text{const.}$ In this case, $\kappa_{\mathrm{fq}} = (3/2)\lambda_q^2/S^2$ follows from Eq. (9), where $\lambda_q \equiv \hat{\kappa}_q S^2/2\pi$. Careful examination of the correction terms shows that the key dimensionless parameter is $\epsilon = \lambda_q/2\pi < 1$ [14]. In addition, the vacuum phase advance σ_{ov} over one lattice period S, estimated from Eq. (24) and $\sigma_{\mathrm{ov}} = \lim_{K_{\mathrm{b}}\to 0}[\epsilon_0 \int_0^S ds/2R_{b0}^2]$, is given by $\sigma_{\mathrm{ov}} = \sqrt{\kappa_{\mathrm{fq}}}\,S = (3/2)^{1/2}\lambda_q$. Therefore, $\sigma_{\mathrm{ov}} < \pi/3 = 60°$ corresponds to $\lambda_q < (2/3)^{1/2}$ and $\epsilon = \lambda_q/2\pi < 0.13$. This is the reason for the conjecture that $\sigma_{\mathrm{ov}} < \pi/3$ should be adequate to assure validity of the Hamiltonian averaging technique developed here. A more detailed discussion of the range of validity of the asymptotic expansion procedure used here is presented in Ref. [14]. In this regard, it is important to recognize that the expansion parameter ϵ is proportional to $\kappa_q(s)$, the strength of the applied focusing field. Hence, the analysis is restricted to moderate values of phase advance, which we estimate to be $\sigma_{\mathrm{ov}} < \pi/3$. In future research, it is planned to carry out a detailed assessment of the range of validity of the Hamiltonian averaging technique developed here by systematic comparison with numerical simulations for various choices of beam distribution function $F_{\mathrm{b}}^0(\mathscr{H}^0)$.

Referring to Eq. (9), it is evident that all of the oscillatory coefficients $\alpha_q(s)$, $\beta_q(s)$ and $\delta_q(s)$ are directly related to the integral over the lattice function $\kappa_q(s)$ defined by $\alpha_q(s) = \int_0^s ds \kappa_q(s)$. In turn, the quantity $\alpha_q(s)$ can be related to the familiar Courant–Snyder amplitude function $\hat{\beta}(s)$ [1,2] defined by $\hat{\beta}(s) = w^2(s)$, where $w(s)$ solves $w'' + \kappa_q(s) w = 1/w^3$. Expressing $\hat{\beta}(s) = \bar{\beta}[1 + f_q(s)]$, where $\bar{\beta} = \text{const.}$ and $|f_q| \ll 1$ is assumed for small $\kappa_q(s)$, some straightforward algebra [1] shows that the Courant–Snyder amplitude function $\hat{\beta}(s)$ and the integral $\alpha_q(s) = \int_0^s ds \kappa_q(s)$ defined in Eq. (9) are related to leading order by the simple expression $(d/ds)\hat{\beta}(s) = -2\bar{\beta}\alpha_q(s)$.

To summarize, the present formalism represents a powerful framework for investigating the equilibrium and stability properties of an intense beam propagating through an alternating-gradient quadrupole field. First, the analysis applies to a broad class of distributions $F_{\mathrm{b}}^0(\mathscr{H}^0)$ in the transformed variables. Second, the determination of (periodically-focused) beam properties in the laboratory frame is straightforward. Third, the analysis applies to beams with arbitrary space-charge intensity, consistent only with requirement for radial confinement of the beam particles by the applied focusing field ($\kappa_{fq}\beta_b^2c^2 > \hat{\omega}_{pb}^2/2\gamma_b^2$). Finally, the formalism can be extended in a straightforward manner to the case of a periodic-focusing solenoidal field $\boldsymbol{B}_{sol}(\boldsymbol{x}) = B_z(s)\hat{\boldsymbol{e}}_z - \frac{1}{2}B_z'(s)(x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y)$ [14], and to the case where weak nonlinear corrections to the transverse focusing force are retained in the analysis.

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