6 Geometry of Magnetic Field

Hamiltonian structure

$\boxed{\vec{B}}$: vector field in $\mathbb{R}^3$

$\boxed{\text{field line } x(\tau)}$: integral curve of $\vec{B}$

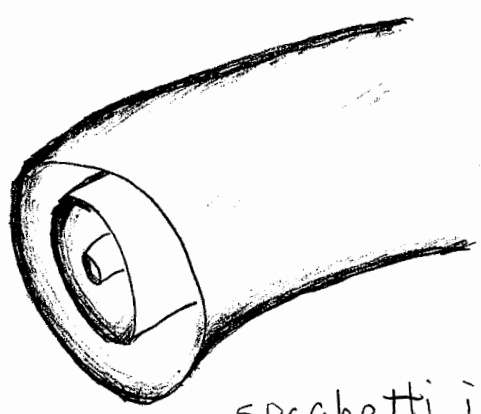
$$\frac{d\,x(\tau)}{d\tau} = \vec{B}(x)$$

Given $\vec{B}$, what does $x(\tau)$ look like?

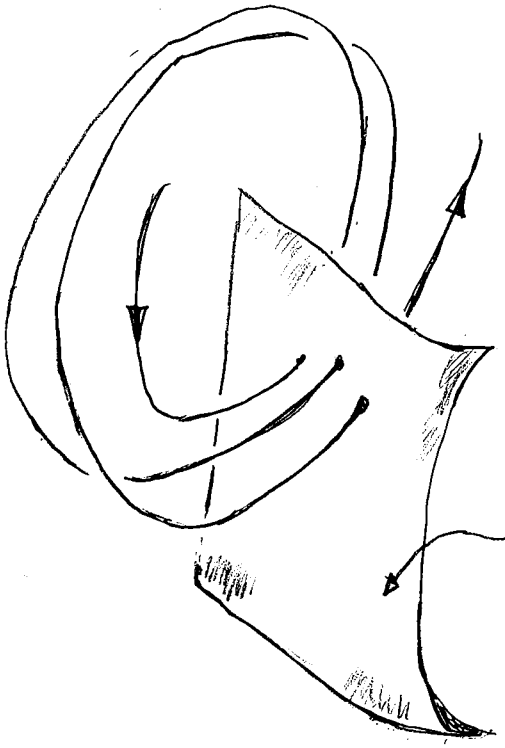
Chaotic? :

cooked spaghetti

integrable surface forming ?

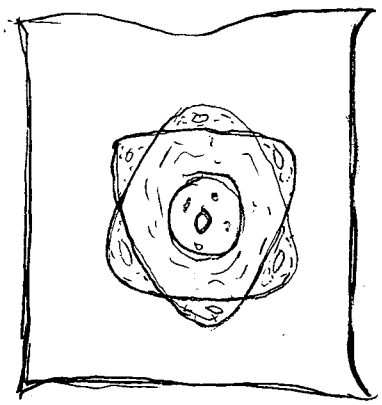


spaghetti in a box

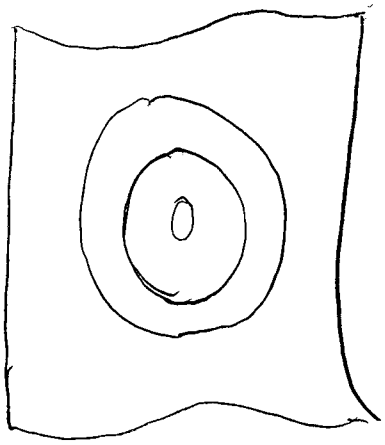


poincáre section

Chaotic



integrable.



The dynamics of the field line is given by

$$\underset{\substack{\parallel\!\!\mid \\ \frac{dx}{d\tau}}}{V} \times B = 0 \quad \Longleftrightarrow \quad i_V \, \overbrace{dA}^{\text{2-form}} = 0$$

inner product

vector potential 1-form

Equivalent to $\frac{dx}{d\tau} = \vec{B}$ by re-parameterization

Hamilton's Eq in the most general form

$$\boxed{\text{symmetry} \iff \text{Integrable (surface-forming)}}$$

symmetry-vector field η:

$$L_\eta A = ds \iff i_\eta dA = df \iff \eta \times B = -\nabla f$$

$$\| $$

$$i_\eta (dA) + d(i_\eta A)$$

Cartan's formula

$$\Downarrow$$

$B \cdot \nabla f = 0$,

$f = \text{const}$ is a flux surface

flux coordinates

$$\boxed{\text{Integrable (surface forming)} \iff B = \nabla f \times \nabla g \text{ for two global functions}}$$

proof:

(i) $\underline{B = \nabla f \times \nabla g} \Rightarrow \underline{B \text{ surface forming}}$

obviously, $B \cdot \nabla f = 0$

$B \cdot \nabla g = 0$

(ii) $\underline{\text{Integrable} \Rightarrow B = \nabla f \times \nabla g}$. $\boxed{\text{Not Easy at all}}$.

$$B \cdot \nabla f = 0 \Rightarrow \nabla \times A \cdot \nabla f = 0 \Rightarrow \underline{A \times \nabla f = \nabla h \times \nabla f}$$

Poincáre lemma on $f(x) = \text{Const}$ Next page

$\boxed{\text{Partition of Unit}}$

$$A = \frac{A \cdot \nabla f}{(\nabla f)^2} \nabla f + \frac{\nabla f \times A \times \nabla f}{(\nabla f)^2} = \frac{A \cdot \nabla f}{(\nabla f)^2} \nabla f + \nabla h + \frac{\nabla f \cdot \nabla h}{(\nabla f)^2} \nabla f$$

$$= g \nabla f + \nabla h$$

$$\Rightarrow B = \nabla \times A = \nabla g \times \nabla f$$

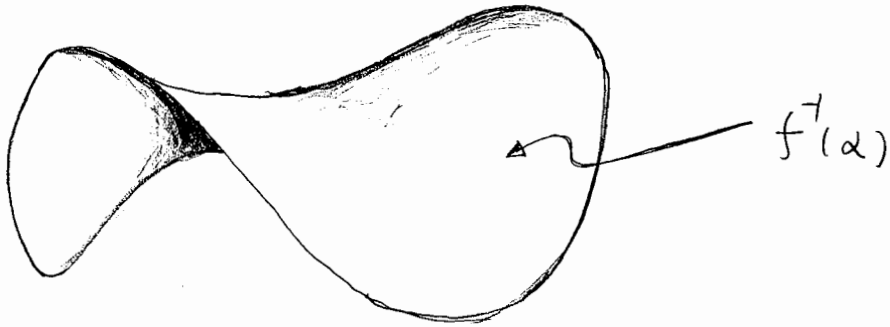
$$g \equiv \frac{(A + \nabla h) \cdot \nabla f}{(\nabla f)^2}$$

$\blacksquare$

$\boxed{\text{Poincare Lemma on a surface}}$

potato chip

simply connect manifold



$$\boxed{\nabla \times A = 0 \quad \text{on } f^{-1}(\alpha)} \Longrightarrow \boxed{A = \nabla h \quad \text{on } f^{-1}(\alpha)}$$

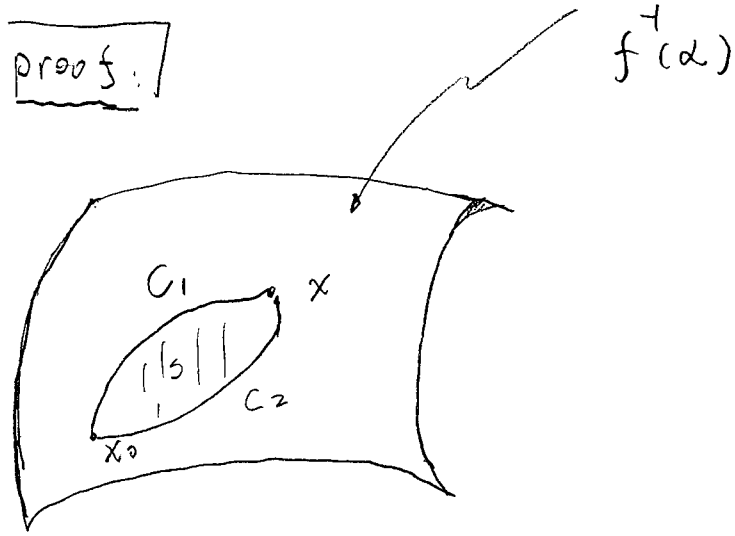
$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$(\nabla \times A) \cdot \nabla f = 0 \qquad\qquad A \times \nabla f = \nabla h \times \nabla f$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$d\tilde{A} = 0 \qquad\qquad \tilde{A} = dh$$

$\boxed{\text{proof:}}$



$$h(x) \equiv \int_{x_0}^{x} A(x)\cdot dl$$

along C_1 on $f^{-1}(\alpha)$

★ $h(x)$ is independent from path: $\int_{x_0}^{x}{}_{C_1} A(x)\cdot dl = \int_{x_0}^{x}{}_{C_2} A(x)\cdot dl$

because

$$\oint A(x)\cdot dl = \int_S \nabla\times A\cdot ds = 0 .$$

any close path on $f^{-1}(\alpha)$

For simply-connected surface

From the definition of $h(x)$,

$\vec{\Delta l}\cdot\nabla h = A(x)\cdot\vec{\Delta l}$ for any vector at x on $f^{-1}(\alpha)$

$$\Rightarrow \quad \nabla h \times \nabla f = A\times\nabla f .$$

∎

$$\boxed{\text{Darboux Theorem} \implies \text{Clebsch coordinates}}$$

- Integrable $\iff$ Global flux coordinates $B = \nabla f \times \nabla g$.

- Locally, $A = d\gamma + \alpha\, d\beta$ ← $\boxed{\text{Darboux Theorem}}$

  $(A = \nabla\gamma + \alpha \nabla\beta)$

  $\implies \quad B = \nabla\alpha \times \nabla\beta$ ← Clebsch coordinates

True, even when $\vec{B}$ is chaotic