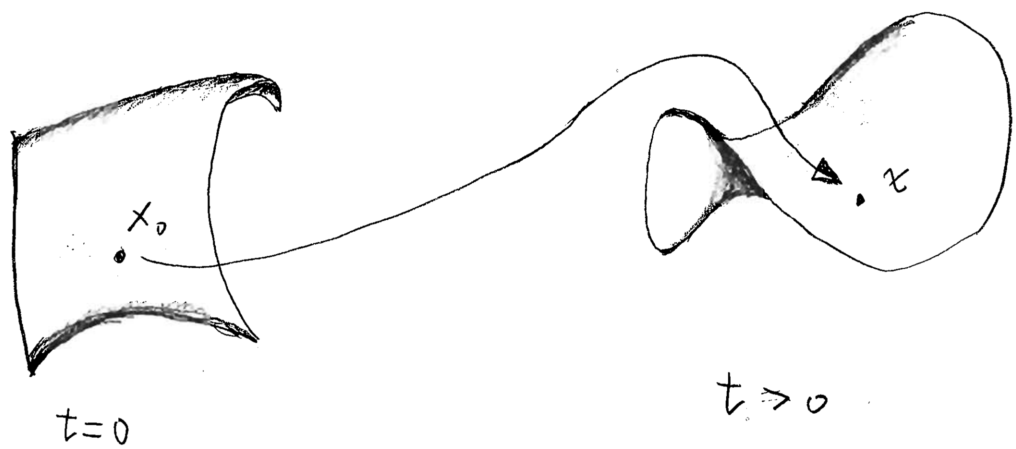
3 Variational Principles for MHD (*)

Why?

- It's a common wisdom that a real physics theory has to be a variational one.
- Variational principles enable geometric methods, e.g. Hamiltonian structure, symmetry, gauge ...
- Very Recently, geometric algorithm with exact global Conservation properties are being developed from variational principles
- It turns out the variational principles for plasma physics are more challenging that quantum ones

starting point:

Lagrangian coordinate



Use x_0 as a label for each fluid element.

The position of the same element at t is

$$x = r(x_0, t),$$

Knowing $x = r(x_0, t)$, we should know everything else.

What are $\rho(x,t)$, $\vec{v}(x,t)$, and $B(x,t)$?
$p(x,t)$,

$$A_{ij} \equiv (-1)^{i+j} \det(M_{ij})$$

Deleting $\begin{pmatrix} i\text{th row} \\ j\text{th column} \end{pmatrix}$ of (a)

$$a \cdot a_{cof} = \det(a)\, I$$

$$\Rightarrow \quad a_{ik} (a_{cof})_{kj} = \det(a)\, \delta_{ij}$$

$$\| $$

$$a_{ik} A_{jk}$$

$$\Rightarrow \quad \boxed{\frac{\partial \det(a)}{\partial a_{ik}} = A_{ik}}$$

$$J \equiv \det(a) = \epsilon_{ijk}\, a_{1i}\, a_{2j}\, a_{3k}$$

$$\frac{dJ}{dt} = \sum_{i,j=1}^{3} A_{ij}\, \dot{a}_{ij}$$

Obviously:

$$V(x,t) = \left.\frac{\partial r(x_0,t)}{\partial t}\right|_{x_0}, \text{ where } r(x_0,t) = x.$$

Not obviously:

$$\rho(x,t) = \rho(x_0, t=0)/J$$

$$J \equiv \mathrm{Det}\,|r_{ij}|, \quad r_{ij} \equiv \frac{\partial r_j(x_0,t)}{\partial x_{0i}}$$

$\boxed{\text{proof}}$:

$$\underbrace{\frac{\partial x_i}{\partial x_{0j}}}\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{0j}}$$

$$r_{ji} \equiv a_{ji}$$

$$\Rightarrow \quad \frac{\partial}{\partial x_i} = \frac{(a_{cof})_{ij}}{J}\frac{\partial}{\partial x_{0j}}$$

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Cramer's rule $a^{-1} = \dfrac{(a_{cof})}{J}$

a_{cof}: Co-factor Matrix of a

$(a_{cof})_{ij} \equiv A_{ji}$

cofactor

$$\nabla \cdot v = \frac{\partial}{\partial x_i} \dot{r}_i(x_0, t) = \frac{A_{ji}}{J} \frac{\partial}{\partial x_{0j}} \dot{r}_i(x_0, t)$$

$$= \frac{A_{ji}}{J} \dot{a}_{ji} = \frac{\dot{J}}{J}$$

$$\left.\begin{array}{l} \\ \dfrac{d \ln \rho}{dt} = -\nabla \cdot v \end{array}\right\} \Rightarrow \frac{d \ln \rho}{dt} = -\frac{dJ}{J\,dt}$$

$$\Rightarrow \rho J = \text{const} \Rightarrow \rho(x,t) = \overset{\rho_0(x_0)}{\overset{\equiv}{\rho(x_0, t=0)}} / J$$

$$\boxed{\nabla \cdot v = \dot{J}/J \text{ is a measure of the compression}}$$

Also, recall $d^3x = J\, d^3x_0$

$$\Rightarrow p = \underset{p_0(x_0)}{\underset{\equiv}{p(x_0, t=0)}} / J^{\gamma}$$

$\vec{B}(x,t)$ in terms of $\quad x = r(x_0, t)$,

$$B_i = \frac{B_{om}}{J} \frac{\partial x_i}{\partial x_{om}}$$

$$\vec{B}_0 = \vec{B}(x_0, t=0)$$

$\boxed{\text{proof}}$:

magnetic equation $\Longrightarrow \quad \frac{dB}{dt} = -B\, \nabla\cdot V + (B\cdot\nabla) V$

$$\Longrightarrow \quad \frac{d\, B/\rho}{dt} = \frac{1}{\rho}\frac{dB}{dt} - \frac{B}{\rho^2}\frac{d\rho}{dt}$$

$$= \left(\frac{B}{\rho}\cdot\nabla\right) V - \underbrace{\frac{B}{\rho}\nabla\cdot V - \frac{B}{\rho^2}\frac{d\rho}{dt}}_{=0} = \left(\frac{B}{\rho}\cdot\nabla\right) V$$

Let $\quad \frac{B_i}{\rho} = C_m \frac{\partial x_i}{\partial x_{om}}$

when $\; J = \det\left(\frac{\partial x_i}{\partial x_{om}}\right) \neq 0 \quad \Longrightarrow \quad C_m$ exist.

(Indeed, $J \neq 0$, because
$x_0 \to x$ is invertable)

$$\frac{d}{dt}\left(\frac{B_i}{\rho}\right) = \dot{C}_m \frac{\partial x_i}{\partial x_{om}} + C_m \frac{\partial \dot{x}_i}{\partial x_{om}}$$

$$\| \qquad\qquad \Longrightarrow \quad \dot{C}_m \frac{\partial x_i}{\partial X_{om}} = 0$$

$$\left(\frac{B}{\rho}\cdot\nabla\right) v_i = C_m \underbrace{\frac{\partial x_j}{\partial x_{om}} \cdot \frac{\partial \dot{x}_i}{\partial x_j}}_{\frac{\partial \dot{x}_i}{\partial x_{om}}}$$

Again, $J \neq 0 \Rightarrow \dot{C}_m = 0$

$$\Rightarrow \quad \frac{B_i}{\rho} = C_m \frac{\partial x_i}{\partial x_{om}} \quad \text{for constant } C_m$$

at $t=0$, $C_m = \dfrac{B_{om}}{\rho_o} \quad \Rightarrow \quad \dfrac{B_i}{\rho} = \dfrac{B_{om}}{\rho_o} \dfrac{\partial x_i}{\partial X_{om}}$

$$\Rightarrow \quad B_i = \frac{B_{om}}{J} \frac{\partial x_i}{\partial X_{om}} \qquad \blacksquare$$

$\rho = \rho_o / J$

There is a better proof using the fact $B = dA$.

2-form $\qquad$ 1-form

$\boxed{\text{But, what determines } r(x_0,t)\text{ ?}}$

$\boxed{\text{Variational Principle}}$

$\boxed{\text{Newcomb 1962}}$

$$\delta \int_0^T dt \int d^3x \left(\underbrace{\frac{1}{2}\rho v^2 - \frac{P}{\gamma-1} - \frac{B^2}{8\pi}}_{L} \right) = 0,$$

where the variation is for

$$r(x_0,t) \longrightarrow r(x_0,t) + \delta r(x_0,t)$$

with

$$\delta r(x_0, 0) = \delta r(x_0, T) = 0,$$

and ρ, $\vec{v}$, P, $\vec{B}$ are considered as functions of $r(x_0,t)$.

Kinetic Energy

$$\text{Lagrangian density: } L \equiv \frac{1}{2}\rho v^2 - \underbrace{\left(\frac{P}{\gamma-1} + \frac{B^2}{8\pi}\right)}_{\text{potential Energy}}$$

$\boxed{\text{proof}}$

$\delta\rho = ?$ in terms of δr

$$\rho(x,t) = \rho_0(x_0)/J \quad \Rightarrow$$

$$\delta\rho(x,t) + \frac{\partial\rho}{\partial x}\cdot\delta r = -\frac{\rho(x_0)}{J^2}\,\delta J = -\rho\,\frac{\delta J}{J}$$

$$\frac{\delta J}{J} = \frac{1}{J}\frac{\partial J}{\partial r_{ij}}\,\delta r_{ij} = \frac{1}{J}\frac{\partial J}{\partial r_{ij}}\frac{\partial \delta r_j}{\partial x_k}\frac{\partial x_k}{\partial x_{0i}}$$

$$= \underbrace{\frac{1}{J}\frac{\partial J}{\partial r_{ij}}\,r_{ik}}_{\delta_{jk}}\frac{\partial \delta r_j}{\partial x_k}$$

$$A_{ij}\,a_{ik} = \delta_{jk}\det(a)$$

$$= \nabla\cdot\delta r \equiv \nabla\cdot\varepsilon$$

$$\delta r(r_0,t) = \delta r(r_0(x,t),t) \equiv \epsilon(x,t)$$

$$\Rightarrow \quad \delta\rho = -\rho\,\nabla\cdot\epsilon - \epsilon\cdot\nabla\rho = -\nabla\cdot(\epsilon\rho)$$

$$\underline{V(x,t) = \left.\frac{\partial r(x_0,t)}{\partial t}\right|_{x_0}}$$

$$\frac{\partial v}{\partial x}\delta r + \delta v(x,t) = \left.\frac{\partial \delta r}{\partial t}\right|_{x_0} = \frac{\partial \epsilon}{\partial t} + \frac{\partial \epsilon}{\partial x}\cdot v$$

$$\Rightarrow \quad \delta v = \frac{\partial \epsilon}{\partial t} + \frac{\partial \epsilon}{\partial x}\cdot v - \epsilon\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial \epsilon}{\partial t} + \nabla\times(\epsilon\times v) - \epsilon\,\nabla\cdot v + v\,\nabla\cdot\varepsilon$$

$$\underline{p = p_0/J^{\gamma}}$$

$$\Rightarrow \quad \delta p + \frac{\partial p}{\partial x}\delta r = -\frac{\gamma p_0}{J^{\gamma}}\frac{\delta J}{J} = -\gamma p \underbrace{\frac{\delta J}{J}}_{=\nabla\cdot\varepsilon}$$

$$\Rightarrow \quad \delta p = -\gamma p\,\nabla\cdot\varepsilon - \varepsilon\cdot\nabla p$$

$$\underline{B_i = x_{ji}\,B_{0j}/J}$$

$$\delta B_i + \frac{\partial B_i}{\partial x}\cdot\delta r = \delta r_{ji}\,B_{0j}/J - \overbrace{\frac{x_{ji}\,B_{0j}}{J}}^{=B_i}\;\overbrace{\frac{\delta J}{J}}^{=\nabla\cdot\varepsilon}$$

$$\frac{\delta r_{ji}\, B_{0j}}{J} = \frac{B_{0j}}{J}\frac{\partial \delta r_i}{\partial x_{0j}} = \underbrace{\frac{B_{0j}}{J}\frac{\partial x_k}{\partial x_{0j}}}_{B_k}\frac{\partial \delta r_i}{\partial x_k}$$

$$= B\cdot\nabla\,\varepsilon$$

$$\Rightarrow \quad \delta B = -\epsilon\cdot\nabla B - B\,\nabla\cdot\varepsilon + B\cdot\nabla\epsilon$$

$$= \nabla\times(\epsilon\times B)$$

$$\delta\int_0^T dt\int d^3x\, L = \int_0^T dt\int d^3x\Big[\rho v\cdot\delta v + \frac{1}{2}v^2\delta\rho - \frac{\delta P}{\gamma-1} - \frac{B\cdot\delta B}{4\pi}\Big]$$

$$= \int_0^T dt\int d^3x\Big[\rho v\cdot\frac{\partial\epsilon}{\partial t} + \rho v\cdot\nabla\times(\epsilon\times v) - \rho v\cdot\epsilon\,\nabla\cdot v + \rho v^2\,\nabla\cdot\epsilon - \frac{1}{2}v^2\,\nabla\cdot(\vec{\epsilon}\rho)$$

$$+ \frac{\gamma P}{\gamma-1}\nabla\cdot\varepsilon + \frac{\epsilon}{\gamma-1}\cdot\nabla P - \frac{B\cdot\nabla\times(\epsilon\times B)}{4\pi}\Big]$$

Integration by part

$$\overset{\uparrow}{=} \int_0^T \int d^3x \Big[\underset{①}{-\epsilon \cdot \frac{\partial \rho V}{\partial t}} + \underset{②}{(\epsilon \times V) \cdot \nabla \times (\rho V)}$$

$$+ \underset{③}{\epsilon \cdot \nabla(\rho V^2)} - \underset{④}{\epsilon \cdot \rho \vec{v} \nabla \cdot \vec{v}} + \underset{⑤}{\epsilon \rho \cdot \nabla \frac{V^2}{2}}$$

$$- \frac{\gamma \nabla \cdot P}{\gamma - 1} \cdot \epsilon + \frac{\epsilon \nabla P}{\gamma - 1} - \frac{(\epsilon \times B) \cdot \nabla \times B}{4\pi} \Big]$$

$$\nabla \cdot (A \times B)$$
$$= B \cdot \nabla \times A - A \cdot \nabla \times B$$

② + ③ + ④ + ⑤

$$= \epsilon \cdot \Big\{ + V \times [\nabla \rho \times V + \rho \nabla \times V] - \rho \nabla V^2 - V^2 \nabla \rho$$
$$- \rho \vec{v} \nabla \cdot \vec{v} + \rho \nabla \frac{V^2}{2} \Big\}$$

$$\overset{\uparrow}{=} \epsilon \Big\{ [V^2 \nabla \rho - V V \cdot \nabla \rho] + \underline{\rho V \times \nabla \times V - \rho \nabla \frac{V^2}{2}}$$
$$- \rho \vec{v} \nabla \cdot V - V^2 \nabla \rho \Big\}$$

$\boxed{B^2 \vec{A} = \vec{B} \times \vec{A} \times \vec{B} + \vec{B}\, \vec{B} \cdot \vec{A} : \text{ partition of Unit}}$

$$= \epsilon \left\{ \underbrace{- v\, v\cdot\nabla\rho - \rho v\, \nabla\cdot v}_{v \frac{\partial \rho}{\partial t}} - \rho\, v\cdot\nabla v \right\}$$

$$\Rightarrow \quad \delta \int dT \int d^3x\, L$$

$$= \int dT \int d^3x\, \epsilon \cdot \left[-\rho \frac{\partial v}{\partial t} - \rho\, v\cdot\nabla v - \nabla P - \frac{B \times \nabla \times B}{4\pi} \right]$$

$$\delta \int dT \int d^3x\, L = 0$$

$$\Longleftrightarrow \quad \rho \frac{\partial v}{\partial t} + \rho\, v\cdot\nabla v + \nabla p - \frac{B \times \nabla \times B}{4\pi} = 0$$

↑ Momentum Eq.

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