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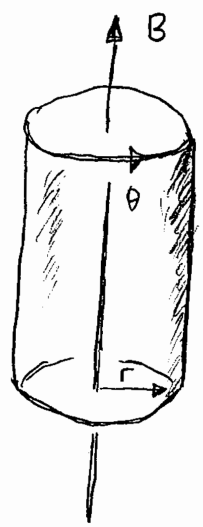
cross field transport by collision, random walk, and Fokker-Planck equation

consider a θ-pinch: (a simple tokamak)



By ideal MHD

$$\frac{j \times B}{c} = \nabla P.$$

No flow, even though $P = n(r)\,T(r)$ is a function of r.

However, if we consider collision, there will be a flow in the r-direction.

$$\frac{j \times B}{c} = \nabla P \qquad (1)$$

$$E + \frac{V \times B}{c} = \eta j \qquad (2)$$

$$② \times B \Rightarrow \quad E \times B - \frac{V_\perp B^2}{c} = \eta\, j \times B$$

$$V_\perp = \frac{c\, E \times B}{B^2} - \frac{c^2 \eta}{B^2} \underbrace{\nabla P}_{= \sum_s T_s \nabla n}$$

Assume $T_s =$ const

$$\Gamma_\perp = n V_\perp = \frac{c\, n\, \vec{E} \times \vec{B}}{B^2} - \underbrace{\frac{c^2 \eta\, n}{B^2} \sum_s T_s}_{D_\perp} \nabla n$$

Diffusion coefficient

$\boxed{\text{Fick's Law} \quad \Gamma = - D \nabla n}$

$\boxed{D_\perp \text{ has the dimension of } \frac{(\Delta x)^2}{\Delta t}}$

In the derivation of MHD, η is derived as

Collision operator

$$\eta = \frac{\nu_{ei}\, m_e}{q_e^2\, n_e} \quad \text{by assuming} \quad \vec{k}_e \equiv \int n_e m_e \left(\frac{\partial f_e}{\partial t}\right)_c V\, d^3V$$

$$= - \nu_{ei}\, m_e\, n_e \left(\vec{V}_e - \vec{V}_i\right)$$

need a kinetic model for

the collisional operator

There are two approches for the collision operator:

(1) BBGKY hierarchy:

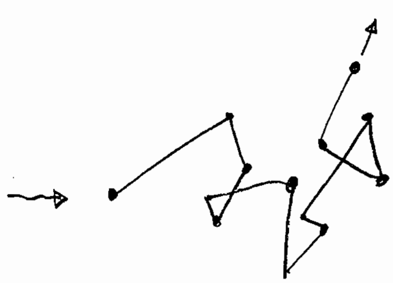
More rigorious

✓ (2) Random walk → Fokker-Plank Eq:

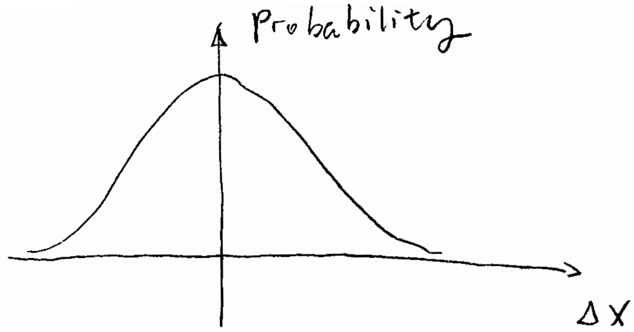
more physical.

<u>We will choose the Random Walk approach here.</u>

collisions are random walks.



A particle moves $\Delta\vec{x}$ at every time step with fixed interval Δt. $\Delta\vec{x}$ is random variabl



Assume: the random walk is Markov.

i.e., $\underbrace{\Delta x_i}_{\boxed{\Delta x \text{ at the } i\text{th step}}}$ and $\underbrace{\Delta x_j}_{\boxed{\Delta x \text{ at the } j\text{th step}}}$ are independent $(i \neq j)$

i.e. $$\langle \Delta x_i \, \Delta x_j \rangle = \begin{cases} \langle \Delta x_i \rangle \langle \Delta x_j \rangle & i \neq j \\ \langle (\Delta x_i)^2 \rangle & i = j \end{cases}$$

After N steps: $T = N \Delta t$

$$x = \sum_{i=1}^{N} \Delta x_i , \qquad \langle x \rangle = \sum_{i=1}^{N} \langle \Delta x_i \rangle = N \langle \Delta x \rangle$$

$$\left\langle (x - \langle x\rangle)^2 \right\rangle = \left\langle \left(\sum_{i=1}^{N} \Delta x_i - \langle x\rangle\right)\left(\sum_{j=1}^{N} \Delta x_j - \langle x\rangle\right)\right\rangle$$

$$= \left\langle \sum_{i,j=1}^{N} \Delta x_i \Delta x_j - 2\langle x\rangle \sum_{i=1}^{N} \Delta x_i + \langle x\rangle^2 \right\rangle$$

$$= \left\langle \sum_{i,j=1}^{N} \Delta x_i \Delta x_j \right\rangle - \langle x\rangle^2$$

$$= \left\langle \sum_{i=1}^{N} (\Delta x_i)^2 \right\rangle + \left\langle \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \Delta x_i \Delta x_j \right\rangle - N^2 \langle \Delta x\rangle^2$$

$$\left\langle \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \Delta x_i \Delta x_j \right\rangle = \sum_{\substack{i,j=1 \\ i\neq j}} \langle \Delta x_i\rangle \langle \Delta x_j\rangle = N(N-1)\langle \Delta x\rangle^2$$

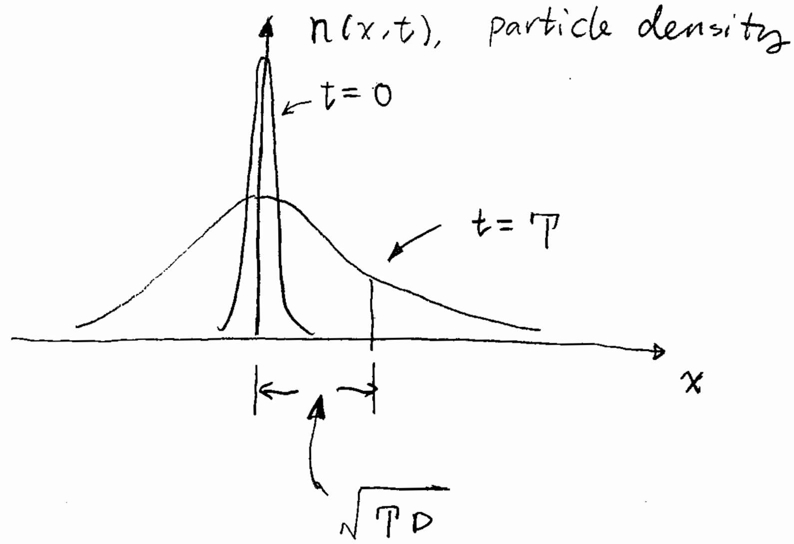
$\Delta x_i, \Delta x_j$ are independent

$$\Rightarrow \left\langle (x - \langle x\rangle)^2 \right\rangle = N\left[\langle \Delta x^2\rangle - \langle \Delta x\rangle^2\right]$$

$$= T \frac{\langle \Delta x^2\rangle - \langle \Delta x\rangle^2}{\Delta t}$$

$$D \equiv \frac{\langle \Delta x^2\rangle - \langle \Delta x\rangle^2}{\Delta t}$$

For example. $\langle \Delta x \rangle = 0$



Fokker - Planck Equation

Particle distribution: $n(x,t)$

What's the governing eq for $n(x,t)$ if particles walk randomly?

$$n(x,t) = \int_{-\infty}^{+\infty} n(x-\Delta x, t-\Delta t)\, P(x-\Delta x; \Delta x)\, d\Delta x$$

probability of a particle at $x-\Delta x$ will move Δx in Δt.

$$n(x,t) = \int_{-\infty}^{+\infty} \Big[n(x, t-\Delta t)\, P(x, \Delta x) +$$

$$- \frac{\partial}{\partial x}\left[n(x, t-\Delta t)\, P(x, \Delta x) \right] \Delta x$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[n(x, t-\Delta t)\, P(x, \Delta x) \right] \Delta x^2 \Big]\, d\Delta x$$

$$n(x,t) = n(x, t-\Delta t) - \frac{\partial}{\partial x}\left[n(x, t-\Delta t) \int_{-\infty}^{+\infty} P(x, \Delta x)\, \Delta x \, d\Delta x \right]$$

$$+ \frac{1}{2} \frac{\partial^2}{\partial x^2}\left[n(x, t-\Delta t) \int_{-\infty}^{+\infty} P(x, \Delta x)\, \Delta x^2 \, d\Delta x \right]$$

$$\frac{\partial n(x,t)}{\partial t} = \lim_{\Delta t \to 0} \frac{n(x,t) - n(x, t-\Delta t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left\{ -\frac{\partial}{\partial x}\left[n(x, t-\Delta t) \frac{\langle \Delta x \rangle}{\Delta t} \right] + \frac{1}{2} \frac{\partial^2}{\partial x^2}\left[n(x, t-\Delta t) \langle \Delta x^2 \rangle \right] \right\}$$

$$\Rightarrow \boxed{\frac{\partial n(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[n(x,t)\, \mu(x) \right] + \frac{\partial^2}{\partial x^2}\left[n(x,t)\, D(x) \right]}$$

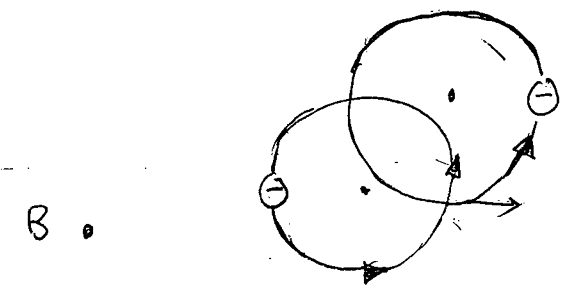
$$\mu \equiv \lim_{\Delta t \to 0} \frac{\langle \Delta x \rangle}{\Delta t}$$

Fokker- Planck Eq

$$D \equiv \lim_{\Delta t \to 0} \frac{\langle \Delta x^2 \rangle}{2 \Delta t}$$

Random walk model for cross field Transport

electrons



$$D_e \sim \frac{\langle \Delta x^2 \rangle}{\Delta t} \sim \rho_e^2 \nu_{ei} = \frac{V_{te}^2}{\Omega_e^2}\nu_{ei} = \frac{V_{te}^2 m_e^2 c^2}{q_e^2 B^2}\nu_{ei}$$

In $\Delta t \sim \frac{1}{\nu_{ei}}$, an electron will suffer a 90° pitch angle scattering, which will shift the gyrocenter by a distance of the order of gyroradius.

$\nu_{ee} \sim \nu_{ei}$, but like-particle collision does not create flow.

$\boxed{\text{ions}}$:

$$D_i \sim \rho_i^2 \nu_{ie} \sim \frac{v_{ti}^2 m_i^2 c^2}{q_i B^2} \nu_{ie} \sim \frac{T_i m_i c^2 \nu_{ie}}{q_i B^2}$$

$$\boxed{\begin{array}{c} \dfrac{\nu_{ei}}{\nu_{ie}} = \dfrac{m_i}{m_e} \\[2ex] T_i \sim T_e \end{array}}$$

$$\sim \frac{T_e m_e c^2 \nu_{ei}}{q_e B^2}$$

$$\sim D_e$$