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MHD waves and Instabilities

Ideal MHD waves

Equilibrium; $\vec{B}_0$, P_0, ρ_0, const.

No equilibrium flow: $\vec{V}_0 = 0$

Consider very cold plasma for simplicity,

$P_0 = 0, \quad P_1 = 0$

Linearized Ideal MHD Equations

$$\begin{cases} \dfrac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 V_1) = 0 \\[2ex] \rho_0 \dfrac{\partial V_1}{\partial t} = \dfrac{(\nabla \times B_1) \times B_0}{4\pi} \\[2ex] \dfrac{\partial B_1}{\partial t} = \nabla \times (V_1 \times B_0) \end{cases}$$

As before: $\begin{pmatrix} \rho_1 \\ V_1 \\ B_1 \end{pmatrix} \sim \exp(ik\cdot x - i\omega t)$

$$\begin{cases} -i\omega\rho_1 + \rho_0\, ik\cdot V_1 = 0 & (1) \\ -i\omega\rho_0 V_1 = \dfrac{(ik\times B_1)\times B_0}{4\pi} & (2) \\ -i\omega B_1 = ik\times(V_1\times B_0) & (3) \end{cases}$$

① is decoupled:

$$② \times B_0 \Rightarrow \quad \omega\rho_0 (V_1\times B_0) = \frac{(k\times B_1)\times B_0\times B_0}{4\pi}$$

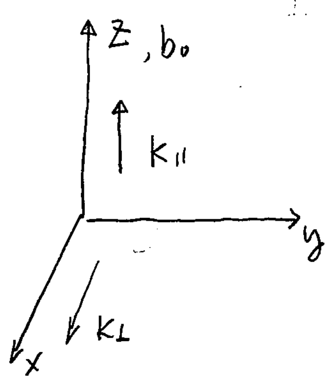
$$\Rightarrow \quad V_1\times B_0 = -\frac{1}{4\pi\omega\rho_0}\left[B_0\,(k\times B_1\cdot B_0) - (k\times B_1)\,B_0^2 \right]$$

$$③ \quad +\vec{B}_1 = \frac{1}{4\pi\omega^2\rho_0}\left[(k\times B_0)\,\underbrace{(k\times\vec{B}_1)\cdot B_0}_{-(k\times B_0)\cdot B_1} - \underbrace{\vec{k}\times(\vec{k}\times\vec{B}_1)}_{\vec{k}\,\vec{k}\cdot\vec{B}_1 - k^2\vec{B}_1}\,B_0^2 \right]$$

$$\Rightarrow \quad \vec{B}_1 = \frac{B_0^2}{4\pi\omega^2\rho_0}\left[-\overrightarrow{(k\times b_0)}\,\overrightarrow{(k\times b_0)} - \vec{k}\vec{k} + k^2\overleftrightarrow{I}\right]\cdot \vec{B}_1$$

$$\boxed{V_A^2 \equiv \frac{B_0^2}{4\pi\rho_0}} \Rightarrow \left[\left(\frac{\omega^2}{V_A^2} - k^2\right)\overleftrightarrow{I} + \overrightarrow{(k\times b_0)}\,\overrightarrow{(k\times b_0)} + \vec{k}\vec{k}\right]\cdot \vec{B}_1 = 0$$

Let $b_0 = \hat{e}_z$, $\quad k = k_\parallel \hat{e}_z + k_\perp \hat{e}_x$, $\quad k\times b_0 = -k_\perp \hat{e}_y$



$$\underbrace{\begin{pmatrix} \frac{\omega^2}{V_A^2} - k_\parallel^2 & 0 & k_\parallel k_\perp \\ 0 & \frac{\omega^2}{V_A^2} - k_\parallel^2 & 0 \\ k_\parallel k_\perp & 0 & \frac{\omega^2}{V_A^2} - k_\perp^2 \end{pmatrix}}_{M} \begin{pmatrix} B_{1x} \\ B_{1y} \\ B_{1z} \end{pmatrix} = 0$$

$$\|M\| = 0$$

$$\Rightarrow$$

$$\left(\frac{\omega^2}{V_A^2} - k_\parallel^2\right)\left[\left(\frac{\omega^2}{V_A^2} - k_\parallel^2\right)\left(\frac{\omega^2}{V_A^2} - k_\perp^2\right) - k_\parallel^2 k_\perp^2\right] = 0$$

$$\left(\frac{\omega^2}{V_A^2} - k_\parallel^2\right)\frac{\omega^2}{V_A^2}\left(\frac{\omega^2}{V_A^2} - k^2\right) = 0$$

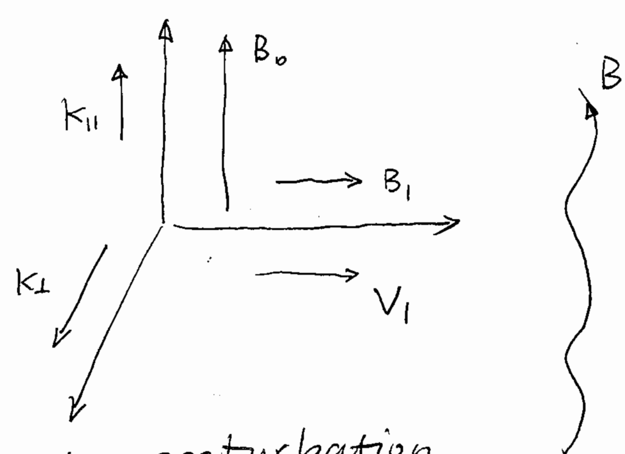
$$
\Rightarrow \begin{cases} \omega^2 = k_\parallel^2 V_A^2 & \text{Shear Alfvén wave} \\ \omega^2 = k^2 V_A^2 & \text{Fast wave} \\ \omega^2 = 0 & \text{Slow wave} \end{cases}
$$

will depend on P_0, if P_0 is considered

Shear Alfvén wave

$$
\begin{pmatrix} 0 & 0 & k_\parallel k_\perp \\ 0 & 0 & 0 \\ k_\perp k_\parallel & 0 & k_\parallel^2 - k_\perp^2 \end{pmatrix} \begin{pmatrix} B_{1x} \\ B_{1y} \\ B_{1z} \end{pmatrix} = 0
$$

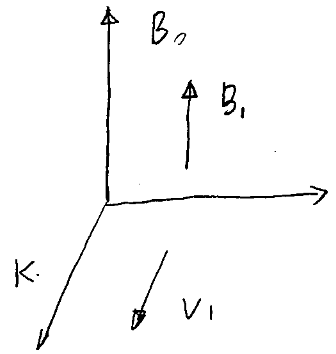
$$
\Rightarrow \quad B_{1x} = 0, \quad B_{1y} \neq 0, \quad B_{1z} = 0
$$



$\rho_1 = 0$, no density perturbation.

<u>Compressional Alfván wave</u>

i.e., Fast wave at $k_{\parallel} = 0$,



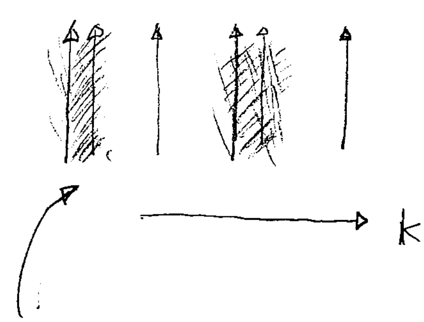
$$\begin{pmatrix} k_\perp^2 & 0 & 0 \\ 0 & k_\perp^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} B_{1x} \\ B_{1y} \\ B_{1z} \end{pmatrix} = 0$$

$$\Rightarrow \quad B_{1x} = 0$$

$$B_{1y} = 0$$

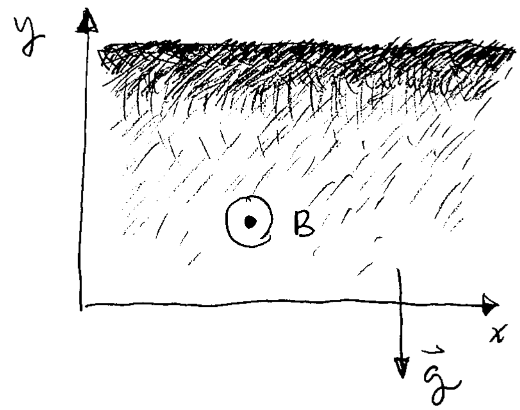
$$B_{1z} \neq 0$$

B



$\rho_1 = \dfrac{\rho_0\, k \cdot v_1}{\omega} = \dfrac{\rho_0 v_1}{v_A}$, density is compressed with B.

Rayleigh - Taylor Instability



Inhomogeneous plasma in gravitational field $\vec{g}$ holding up by a $\vec{B}$ field

Equilibrium: $P_0 = P_0(y)$, $\vec{g} = -g\,\hat{e}_y$

$\rho_0 = \rho_0(y)$, $\vec{B}_0 = B_0(y)\,\hat{e}_z$

$$-\nabla\left(p_0 + \frac{B_0^2}{8\pi}\right) - \rho_0 g = 0$$

Consider a perturbation:

$$\vec{B}_1 = B_1(x,y,t)\,\hat{e}_z$$

$$\vec{V}_1 = V_x(x,y,t)\,\hat{e}_x + V_y(x,y,t)\,\hat{e}_y$$

$$\rho_1 = \rho_1(x,y,t)$$

$$p_1 = p_1(x,y,t)$$

Linearized MHD:

$$\rho_0 \frac{\partial V_1}{\partial t} = -\nabla P_1 - \nabla \left(\frac{B_0 B_1}{4\pi} \right) - \rho_1 g \hat{e}_y \qquad (1)$$

$$\frac{\partial \rho_1}{\partial t} + V_1 \cdot \nabla \rho_0 = 0 \qquad (2)$$

$$\frac{\partial B_1}{\partial t} = \nabla \times (V_1 \times B_0) \qquad (3)$$

$$\nabla \cdot V_1 = 0 \qquad (4)$$

replace $\frac{d}{dt}\left(P/\rho^{\gamma} \right) = 0$

as a different physical model

for the equation of the state: incompressable

Assume:

$$\begin{pmatrix} V_1 \\ B_1 \\ \rho_1 \\ P_1 \end{pmatrix} \sim e^{i(kx - wt)}$$

$$\nabla \times (1) \Rightarrow \quad -i\omega\, \nabla \times (\rho_0 \vec{V}_1) = -g\, \nabla \times (\rho_1 \hat{e}_y)$$

$$\nabla \times (\rho_0 \vec{V}_1) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \rho_0 V_{1x} & \rho_0 V_{1y} & 0 \end{vmatrix} = \hat{e}_z \left[ik\rho_0 V_{1y} - \frac{\partial \rho_0 V_{1x}}{\partial y} \right]$$

$$\nabla \times (\rho_1 \hat{e}_y) = \hat{e}_z\, ik\rho_1$$

$$\Rightarrow \quad ik\rho_0 \omega V_{1y} - \omega \frac{\partial \rho_0 V_{1x}}{\partial y} = k\rho_1 g$$

$$(2) \Rightarrow \quad \rho_1 = \frac{V_{1y}}{i\omega} \frac{\partial \rho_0}{\partial y}$$

$$(4) \Rightarrow \quad V_{1x} = \frac{i}{k} \frac{\partial V_{1y}}{\partial y}$$

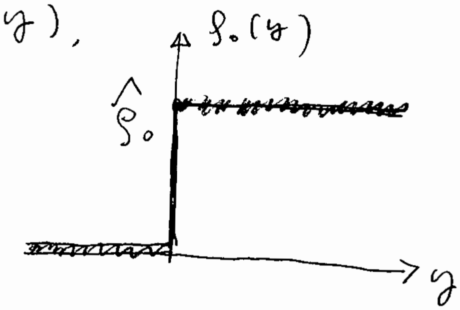
$$-\frac{\omega i}{k} \frac{\partial}{\partial y} \rho_0 \frac{\partial V_{1y}}{\partial y} - \frac{kg}{i\omega} \frac{\partial \rho_0}{\partial y} V_{1y} + ik\rho_0 \omega V_{1y} = 0$$

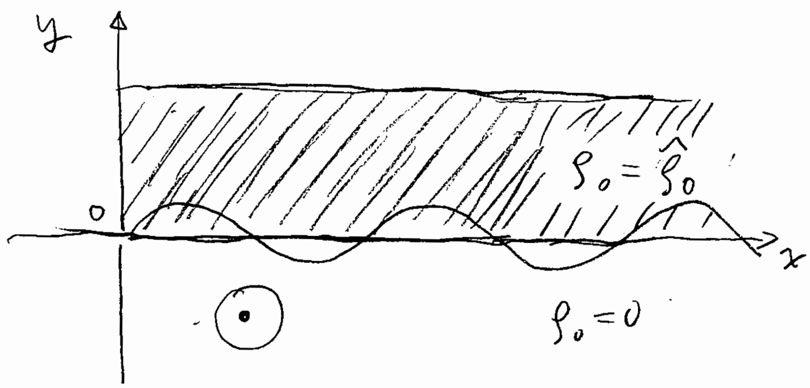
$$\frac{\partial}{\partial y}\left(\rho_0 \frac{\partial V_{1y}}{\partial y}\right) - \frac{k^2 g}{\omega^2} \frac{\partial \rho_0}{\partial y} V_{1y} - k^2 \rho_0 V_{1y} = 0$$

It is an

<u>eigen value problem for an ODE</u>

For simplicity, consider the case of step-function profile for $\rho_0(y)$,





For $y > 0$:

$$\rho_0 \frac{\partial^2 V_{1y}}{\partial y^2} - k^2 \rho_0 V_{1y} = 0$$

$$\Rightarrow \quad V_{1y} = c_1 e^{-ky} + c_2 e^{+ky}$$

$$\lim_{y \to \infty} V_{1y} = 0 \quad \Rightarrow \quad V_{1y} = c_1 e^{-ky}$$

At $y=0$:
$$\frac{\partial \rho_0}{\partial y} = -\delta(y)\,\hat{\rho}_0$$

$$\frac{\partial V_{1y}}{\partial y} - \frac{k^2 g}{\omega^2} V_{1y} = 0$$

$$\Rightarrow \quad V_{1y} = C_2\, e^{\frac{k^2 g}{\omega^2} y}$$

Continuity at $y=0$

$$\Rightarrow \quad C_1 e^{-ky} = C_2\, e^{\frac{k^2}{\omega^2} g y}$$

$$\Rightarrow \quad \omega^2 = -kg$$

$$\omega = i\sqrt{kg} \quad \Rightarrow \quad \text{Unstable.}$$