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MHD Equations

Magneto- Hydro - Dynamics:

- collisions are important for equilibrium.
- One - fluid model with collisions.

start from:

$$\frac{\partial f_s}{\partial t} + v \cdot \nabla f_s + \frac{q}{m}\left(E + \frac{v \times B}{c}\right) \cdot \frac{\partial f}{\partial v} = \left(\frac{\partial f_s}{\partial t}\right)_c$$

with: $\int \left(\frac{\partial f_s}{\partial t}\right)_c d^3v = 0$: particle conservation

$\sum_s \int v\, m_s \left(\frac{\partial f_s}{\partial t}\right)_c d^3v = 0$: momentum conservation

Define:

$$n_s \equiv \int f_s \, d^3v$$

$$V_s \equiv \frac{1}{n_s} \int v f_s \, d^3v$$

$$\overleftrightarrow{P}_s \equiv m_s \int f_s (v - \overline{V}_s)(v - \overline{V}_s)\, d^3v$$

For each species:

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \vec{V}_s) = 0$$

$$\frac{\partial}{\partial t}(\rho_s \vec{V}_s) + \nabla \cdot (\rho_s \vec{V}_s \vec{V}_s) = \rho_{qs} E + \frac{J_s \times B}{c} - \nabla (\overleftrightarrow{P}_s) + \vec{K}_s$$

Charge density

$$\rho_{qs} \equiv n_s q_s, \qquad \rho_s \equiv n_s m_s$$

$$J_s \equiv n_s V_s q_s,$$

mass density

$$\vec{K}_s \equiv \int m_s v \left(\frac{\partial f_s}{\partial t}\right)_c d^3v$$

change of momentum due to collisions with other species.

Define one-fluid fields:

$$\rho \equiv \sum_s \rho_s \quad , \quad \rho_q \equiv \sum_s \rho_{qs}$$

mass density charge density

$$\vec{u} \equiv \frac{\sum \rho_s V_s}{\rho} \; ; \quad \vec{J} \equiv \sum \vec{J}_s$$

$$\overleftrightarrow{P} \equiv \sum_s m_s \left[\int f_s (\vec{v}-\vec{U})(\vec{v}-\vec{U}) d^3v + (\vec{V}_s-\vec{u})\vec{u} + \vec{u}(\vec{V}_s-\vec{u}) \right]$$

$$= \sum_s \overleftrightarrow{P}_s' \neq \sum_s \overleftrightarrow{P}_s$$

$\boxed{\text{relative to } \vec{U}}$ $\boxed{\text{relative to } \vec{V}_s}$

$\boxed{\text{one-fluid Continuity Eq}}$

$$\sum_s \frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \vec{V}_s) = 0$$

$$\Downarrow$$

Mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

Also: $$\frac{\partial \rho_q}{\partial t} + \nabla \cdot J = 0$$ ↙ Charge conservation

$\boxed{\text{one-fluid momentum Eq}}$.

$$\sum_s \left[\overset{①}{\frac{\partial}{\partial t}(\rho_s \vec{V}_s)} + \overbrace{\nabla\cdot(\rho_s \vec{V}_s \vec{V}_s) + \nabla\cdot \overleftrightarrow{P}_s}^{②} - \underbrace{\left(\rho_{qs} E + \frac{J_s \times B}{c}\right)}_{③} \right] = \sum_s \vec{K}_s = 0$$

②: $$\nabla\cdot(\rho \vec{V}_s \vec{V}_s) + \nabla\cdot \overleftrightarrow{P}_s = \nabla\cdot\left[m_s \int \vec{v}\vec{v} f_s \, d^3v \right]$$

$$= \nabla\cdot(\rho_s \vec{U}\vec{U} + \overleftrightarrow{P}_s')$$

where: $$\overleftrightarrow{P}_s' \equiv m_s \left[\int f_s (\vec{v}-\vec{u})(\vec{v}-\vec{u}) d^3v + (\vec{V}_s - \vec{u})\vec{u} + \vec{u}(\vec{V}-\vec{u}) \right]$$

$$\Rightarrow \quad \underbrace{\frac{\partial}{\partial t}(\rho \vec{u}) + \nabla\cdot(\rho \vec{U}\vec{u})} + \left(\rho_q E + \frac{J\times B}{c}\right) - \nabla\cdot \overleftrightarrow{P}' = 0$$

$$\rho \frac{\partial \vec{U}}{\partial t} + \rho \vec{u}\cdot\nabla\vec{u}$$

$\boxed{\text{Energy Eq.}}$

$$\vec{\vec{P}}' = p\,\delta_{ij} \qquad \boxed{\text{As before}}$$

$$\frac{d}{dt}\left(P/\rho^{\gamma} \right) = 0$$

ρ,	ρ_s	$\vec{u}$	$\vec{J}$	p	$\vec{E}$, $\vec{B}$
mass conservation	charge conservation	momentum conservation	?	Energy	Maxwell's Eq.

$\boxed{\text{Need one more (vector) equation: Ohm's Law}}$

$$\sum_s \left[\frac{\partial}{\partial t}\left(\vec{J}_s\right) + \nabla\cdot\left(\rho_{qs}\vec{V}_s\vec{V}_s + \vec{\vec{P}}_s \cdot \frac{q_s}{m_s} \right) - \left(\frac{q_s^2 n_s E}{m_s} + \frac{q_s^2 \vec{V}_s n_s}{c\, m_s} \times B \right) - \frac{K_s q_s}{m_s} \right] = 0$$

$$\sum_s k_s = 0 \implies \left|\frac{k_e q_e}{m_e}\right| \gg \left|\frac{k_i q_i}{m_i}\right|$$

$$\underbrace{\sum_s q_s n_s \doteq 0}_{\text{Quasi-neutrality}} \implies \left|\frac{q_e^2 n_e E}{m_e}\right| \gg \left|\frac{q_i^2 n_i E}{m_i}\right|$$

$$\vec{V}_e \gtrsim \vec{V}_i \implies \left|\frac{q_e^2 V_e n_e}{c m_e}\right| \gg \left|\frac{q_i^2 V_i n_i}{c m_i}\right|$$

$$\implies E + \frac{\vec{V}_e \times B}{c} + \frac{\vec{k}_e}{n_e q_e} = \frac{m_e}{q_e^2 n_e}\frac{\partial J}{\partial t} + \frac{m_e}{q_e^2 n_e}\sum_s \nabla\cdot\left(\rho_{q_s}\vec{V}_s\vec{V}_s + \vec{\vec{P}}_s \frac{q_s}{m_s}\right)$$

$$\vec{V}_e = \frac{1}{n_e q_e}\left[n_e V_e q_e + \sum_i n_i V_i q_i - \sum_i n_i V_i q_i\right]$$

$$= \frac{J}{n_e q_e} - \frac{\sum_i n_i q_i V_i}{n_e q_e} = \frac{J}{n_e q_e} + V_i$$

$$\doteq \frac{J}{n_e q_e} + U$$

Assumption

only one ion component with $\sum_s n_s q_s = n_e q_e + n_i q_i = 0$

$$\vec{k}_e = -\nu_{ei} m_e n_e (V_e - V_i) \text{ , for example}$$

$$\frac{\vec{k}_e}{n_e q_e} \simeq -\frac{\nu_{ei} m_e}{q_e}(V_e - V_i) = -\frac{\nu_{ei} m_e}{q_e^2 n_e} J \equiv -\eta J$$

$T_e \approx T_i \Longrightarrow \boxed{V_e \gg V_i}$ $\boxed{\text{resistivity}}$

$$\Longrightarrow \boxed{E + \frac{U \times B}{c} - \eta J = \frac{m_e}{q_e^2 n_e}\frac{\partial J}{\partial t} - \frac{J}{c n_e q_e} \times B + \frac{m_e}{q_e^2 n_e} \nabla \cdot \sum_s (\rho_s \vec{V}_s \vec{V}_s + \overleftrightarrow{P}_s)}$$

Generalized Ohm's Law

(Approximate truth, really)

Drop the RHS

$$\boxed{E + \frac{U \times B}{c} = \eta J,}$$

Putting together:

$$\left\{\begin{array}{l}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{u} \rho) = 0 \\[2ex]
\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} - \left(\rho_q E + \frac{J \times B}{c} \right) + \nabla p = 0 \\[2ex]
\frac{d}{dt} \left(P / \rho^{\gamma} \right) = 0 \\[2ex]
\frac{\partial \rho_q}{\partial t} + \nabla \cdot J = 0 \\[2ex]
\left.\begin{array}{l} \nabla \cdot E = 4\pi \rho_q \\ \nabla \cdot B = 0 \end{array}\right\} \\[2ex]
\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t} \\[2ex]
\nabla \times B = \frac{4\pi J}{c} + \frac{1}{c} \frac{\partial E}{\partial t} \\[2ex]
E + \frac{U \times B}{c} = \eta J
\end{array}\right.$$

Can be viewed as initial conditions

Two more approximations.

(1): Ignore Maxwell's Brilliant contribution, the displacement current: $\frac{1}{c}\frac{\partial E}{\partial t}$.

$$E \sim \frac{V \times B}{c} \Rightarrow \boxed{E \sim \frac{V}{c} B}$$

$$\rho \frac{\partial V}{\partial t} + \rho V \cdot \nabla V \sim -\nabla P + \frac{(\nabla \times B) \times B}{4\pi} \Rightarrow \rho V \omega \sim K \frac{B^2}{4\pi}$$

$$\Rightarrow \boxed{V \sim \frac{K}{\omega} \frac{B^2}{4\pi \rho} \sim \frac{K}{\omega} V_A^2}$$

For most laboratory & space plasmas: $V_A << c$

$$\frac{4\pi}{c} j + \frac{1}{c}\frac{\partial E}{\partial t} = \nabla \times B$$

$$\frac{\frac{1}{c}\frac{\partial E}{\partial t}}{\nabla \times B} \sim \frac{\frac{1}{c}\omega E}{K B} \sim \frac{\omega}{K}\frac{V}{c^2} \sim \frac{V_A^2}{c^2} << 1$$

(2) Poisson's Eq $\Rightarrow$ Quasineutrality condition

Conitnuity Eq: $\Rightarrow$ $i\omega n_1 \sim ikV_1 n_0$

momentum Eq $\Rightarrow$ $i\omega V_1 \sim \frac{q}{m} E_1$

$$\Rightarrow n_1 \sim \frac{k}{\omega^2} \frac{n_0 \cdot q}{m} E_1$$

$$\nabla \cdot E_1 = \sum_s 4\pi q n_1 \sim \sum_s \frac{4\pi q^2 n_0}{m} \frac{k}{\omega^2} E_1$$

$\frac{RHS}{LHS} \sim \frac{\omega_{pe}^2}{\omega^2}$, For $\omega^2 \ll \omega_{pe}^2$, to the leading order of $O(\frac{\omega^2}{\omega_{pe}^2})$, $\rho_q = \sum_s q_s n_{s1} \approx 0$

$\boxed{\text{14 Eqs for 14 fields}}$

$$\Rightarrow \begin{cases} \frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{u}\rho) = 0 \\ \rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} - \frac{J \times B}{c} + \nabla P = 0 \\ \frac{d}{dt}(P/\rho^\gamma) = 0 \\ \nabla \cdot B = 0 \\ \nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t} \\ \nabla \times B = \frac{4\pi}{c} J \\ E + \frac{u \times B}{c} = \eta J \end{cases}$$

$\boxed{\text{initial condition}}$

- $\eta = 0$: Ideal MHD

  $\eta \neq 0$: resistive MHD

- MHD eqs do not include $\nabla \cdot E = 0$

  Actually in MHD, $\nabla \cdot E \neq 0$, this is because of the concept of "Quasi-neutrality".

- Eliminate $\vec{j}$ and $\vec{E}$

$\boxed{\text{8 Eqs for 8 fields}}$

$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \\[2ex]
\rho \dfrac{\partial u}{\partial t} + \rho\, u \cdot \nabla u = -\nabla p + \dfrac{(\nabla \times B) \times B}{4\pi} \\[2ex]
\dfrac{d}{dt}\left(\dfrac{p}{\rho^{\gamma}}\right) = 0 \qquad \text{(sometimes replaced by } \nabla \cdot u = 0\text{)} \\[2ex]
\dfrac{\partial B}{\partial t} = \nabla \times (u \times B) + \dfrac{\eta c^2}{4\pi} \nabla^2 B
\end{cases}
$$

$\boxed{\text{Momentum Eq.}}$

$$\begin{cases} \rho \dfrac{dv}{dt} = -\nabla p + \dfrac{j \times B}{c} \\[2ex] \dfrac{4\pi}{c} j = \nabla \times B \end{cases}$$

$\left(\text{No Displacement current } \dfrac{1}{c}\dfrac{\partial E}{\partial t} \right)$

$$\Rightarrow \quad \rho \frac{dv}{dt} = -\nabla p + \frac{(\nabla \times B) \times B}{4\pi}$$

?

$$= -\nabla p - \nabla \frac{B^2}{8\pi} + \frac{B \cdot \nabla B}{4\pi}$$

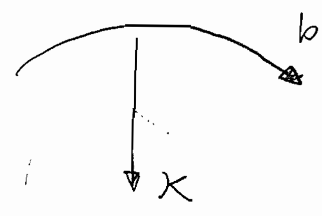
$$\nabla (B \cdot B) = 2 \left[B \times (\nabla \times B) + B \cdot \nabla B \right]$$

magnetic pressure ($\parallel$ & $\perp$)

Not really

$$\vec{B} \cdot \nabla \vec{B} = B \vec{b} \cdot \nabla (\vec{b} B) = B^2 \underbrace{\vec{b} \cdot \nabla \vec{b}}_{\kappa} + \vec{b}\, \vec{b} \cdot \nabla \frac{B^2}{2}$$

Curvature



$$\frac{j \times B}{c} = \frac{(\nabla \times B) \times B}{4\pi} = \kappa \frac{B^2}{4\pi} - \nabla_\perp \frac{B^2}{8\pi}$$

Bending force

pressure force

Momentum Eq: in Conservative form

$$\frac{\partial(\rho V)}{\partial t} + \nabla\cdot T = 0$$

magnetic pressure

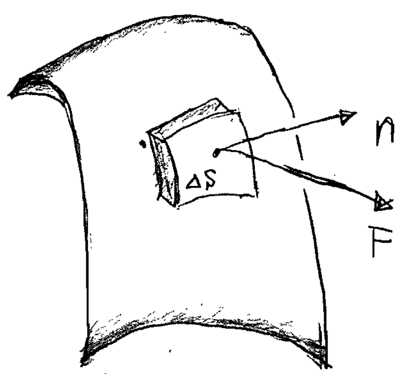
$$T = \rho\vec{V}\vec{V} + \left(p + \frac{B^2}{8\pi}\right)\vec{\vec{I}} - \frac{\vec{B}\vec{B}}{4\pi}$$

stress tensor

Reynolds stress

Magnetic stress

Why the name "stress"?



Force

$$\frac{\vec{F}}{\Delta S} = T\cdot n$$

Area

$\boxed{\text{proof:}}$

$$\nabla\cdot T = \frac{\partial}{\partial x_i}\left[\rho V_i V_j + \left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \frac{B_i B_j}{4\pi}\right]$$

$$\frac{\partial}{\partial x_i}\left[\rho V_i V_j\right] = \rho_{,i} V_i V_j + \rho V_{i,i} V_j + \rho V_i V_{j,i}$$

$$= V\,(V\cdot\nabla)\rho + \rho V\,\nabla\cdot V + \rho\, V\cdot\nabla V$$

$$\frac{\partial}{\partial x_i}\left[\left(p+\frac{B^2}{8\pi}\right)\delta_{ij}\right] = \nabla\left(p+\frac{B^2}{8\pi}\right)$$

$$\frac{\partial}{\partial x_i}\,\frac{B_i B_j}{4\pi} = \frac{1}{4\pi}\left[B_{i,i} B_j + B_i B_{j,i}\right]$$

$$= \underbrace{\frac{(\nabla\cdot B)B}{4\pi}}_{0} + \frac{1}{4\pi}(B\cdot\nabla)B$$

0, continuity Eq.

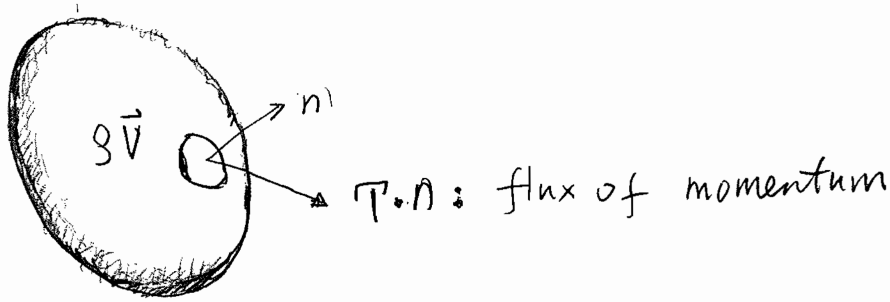
$$\frac{\partial(\rho V)}{\partial t} + \nabla\cdot T = \underline{V\rho_{,t}} + \rho V_{,t} + \underline{V(V\cdot\nabla)\rho} + \underline{\rho V\,\nabla\cdot V}$$

$$+ \rho\, V\cdot\nabla V + \nabla\left(p+\frac{B^2}{8\pi}\right) - \frac{1}{4\pi}(B\cdot\nabla)B$$

$$= \rho\frac{\partial V}{\partial t} + \rho\, V\cdot\nabla V + \nabla\left(p+\frac{B^2}{8\pi}\right) - \frac{1}{4\pi}(B\cdot\nabla)B$$

$$= \rho\frac{\partial V}{\partial t} + \rho\, V\cdot\nabla V + \nabla p - \frac{(\nabla\times B)\times B}{4\pi}$$

$$= 0 \qquad \blacksquare$$



global conservative form

$$\frac{\partial}{\partial t} \int_{V} \rho \vec{v} \, d^3x = - \int_{S} \mathbb{T} \cdot n \, ds$$

$$= - \int_{V} \nabla \cdot \mathbb{T} \, d^3x$$

Stokes's Theorem

$$\Updownarrow$$

$$\frac{\partial (\rho \vec{v})}{\partial t} + \nabla \cdot \mathbb{T} = 0$$

local conservative form

Stokes's Theorem in general form:

$$\int_{V} d\omega = \int_{\partial V} \omega$$

$d\omega$: (P+1)-form

ω: P-form

Magnetic Field Eq.

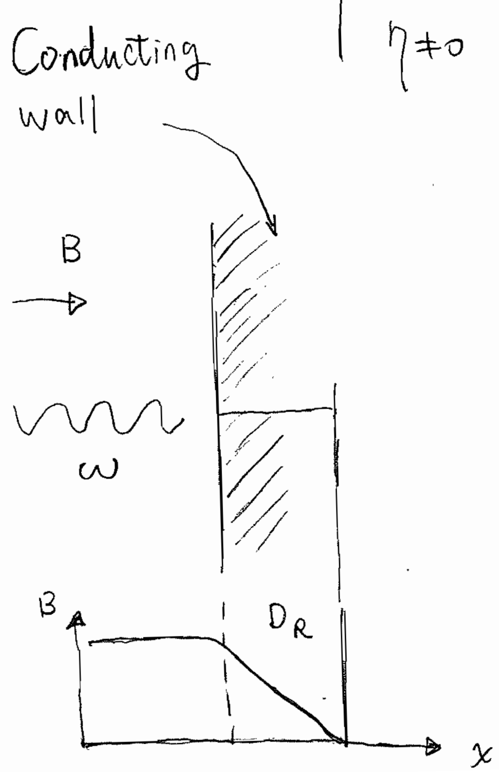
$$\begin{cases} \dfrac{\partial B}{\partial t} = -c\nabla\times E \\[2ex] E + \dfrac{V\times B}{c} = \eta j \end{cases} \Rightarrow \quad \frac{\partial B}{\partial t} = \nabla\times(V\times B) - c\eta\nabla\times j$$

$$\nabla\times j = \frac{c}{4\pi}\nabla\times(\nabla\times B) = \frac{c}{4\pi}\left[\nabla(\nabla\cdot B) - \nabla^2 B\right]$$

$$\Rightarrow \quad \frac{\partial \vec{B}}{\partial t} = \nabla\times(V\times B) + \boxed{\frac{c^2\eta}{4\pi}\nabla^2\vec{B}}$$

Diffusion

$$\begin{cases} \eta = 0 \quad \text{— ideal MHD} \\ \eta \neq 0 \quad \text{— resistive MHD} \end{cases}$$

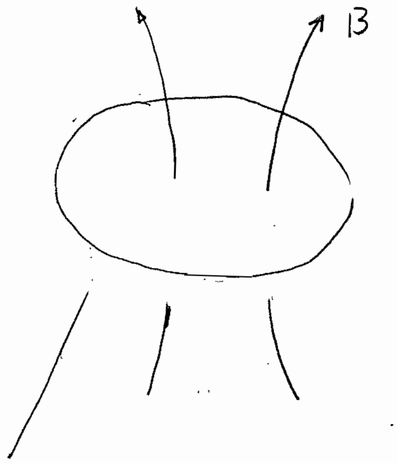


Skin Depth

$$\frac{\partial \vec{B}}{\partial t} \sim \frac{c^2\eta}{4\pi}\nabla^2\vec{B}$$

$$\omega B \sim \frac{c^2\eta}{4\pi}\frac{B}{D_R^2} \Rightarrow D_R = \sqrt{\frac{c^2\eta}{4\pi\omega}}$$

Frozen-in Law: For ideal MHD

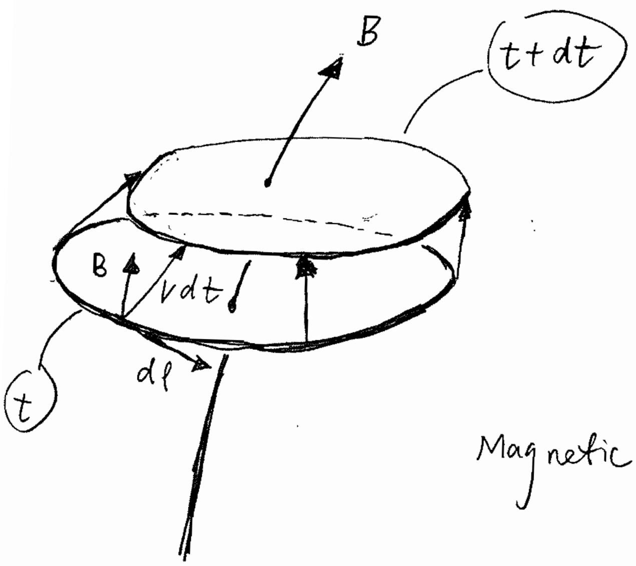


$$\int_S B \cdot ds \equiv \psi$$

$$\boxed{\frac{d\psi}{dt} = 0}$$

Fluid-element ring

$\boxed{\text{proof:}}$



$$\frac{d\psi}{dt} = \int_S \frac{\partial B}{\partial t} \cdot d\vec{s} + \int_{\partial S} B \cdot [V \times dl]$$

$$= \int_S \nabla \times (V \times B) \cdot d\vec{s} + \int_{\partial S} (B \times V) \cdot dl$$

Magnetic eq.

$$= \int_{\partial S} (V \times B) \cdot dl + \int_{\partial S} (B \times V) \cdot dl$$

$$= 0$$