5 Fluid Equations

- **Basic idea**

For each species, calculate the moments of the distribution function.

$$\boxed{\langle f \rangle \equiv \int f \, d^3v}$$

Density: $n_s(x,t) = \int f_s(x,v,t)\, d^3v = \langle f \rangle$

velocity: $\vec{V}_s(x,t) = \frac{1}{n_s(x,t)} \int v\, f_s(x,v,t)\, d^3v = \frac{1}{n_s(x,t)} \langle v f \rangle$

pressure: $\overleftrightarrow{P}_s = m_s \int \underbrace{(\vec{v} - \vec{V}_s)(\vec{v} - \vec{V}_s)}_{\text{Dyad.}} f_s \, d^3v$

Tensor $= m_s \langle (v - V)(v - V) \rangle$

Dyad: $\vec{v}\vec{v} = v_i v_j \quad (i, j = 1, 2, 3)$

Derive equations for $n_s(x,t)$, $\vec{V}_s(x,t)$, $\overleftrightarrow{P}(x,t)$ from the vlasov Eq.

Continuity Eq.

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + \frac{q}{m}\left(E + \frac{v \times B}{c}\right) \frac{\partial f}{\partial v} = 0$$

conservative form: $\frac{\partial f}{\partial t} + \frac{\partial}{\partial x} \cdot (vf) + \frac{q}{m} \frac{\partial}{\partial v} \cdot \left(\left(E + \frac{v \times B}{c}\right) f\right) = 0$

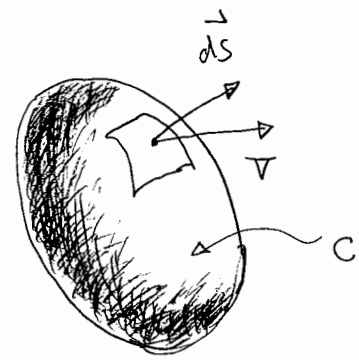
$$\int d^3v \, \cdot$$

$$\Rightarrow \frac{\partial}{\partial t}\Big(\underbrace{\int f \, d^3v}_{n}\Big) + \frac{\partial}{\partial x} \cdot \underbrace{\int v f \, d^3v}_{n\bar{v}} + \frac{q}{m}\int d^3v \, \frac{\partial}{\partial v}\Big(\underbrace{E + \frac{v \times B}{c}}_{0}\Big) f = 0$$

$$\Rightarrow \boxed{\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{v}) = 0}$$ ← Differential form

Global form:

$$\frac{\partial}{\partial t} \int_C n \, d = - \int_{\partial C} n \vec{v} \cdot d\vec{s}$$



$\boxed{\text{Momentum Eq}}$

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\cdot(f v) + \frac{q}{m}\frac{\partial}{\partial v}\cdot\left(E + \frac{v \times B}{c}\right)\cdot f = 0$$

$$\int \vec{v}\cdot d^3v \cdot$$

$$\Rightarrow \quad \underbrace{\frac{\partial}{\partial t}\int f \vec{v}\, d^3v}_{①} + \underbrace{\frac{\partial}{\partial \vec{x}}\cdot\int f \vec{v}\vec{v}\, d^3v}_{②}$$

$$+ \underbrace{\frac{q}{m}\int \vec{v}\,\frac{\partial}{\partial \vec{v}}\cdot\left(E + \frac{v \times B}{c}\right)\cdot f\, d^3v}_{③} = 0$$

$$① = \frac{\partial}{\partial t}(n\bar{v})$$

$$③ = -\frac{q}{m}\int\left(E + \frac{v \times B}{c}\right) f\, d^3v = -\frac{qn}{m}\left(E + \frac{\bar{v} \times B}{c}\right)$$

$$\nabla\cdot(\vec{A}\vec{B}) = (\nabla\cdot\vec{A})B + (\vec{A}\cdot\nabla)\vec{B}$$

$$\nabla_v\cdot\left[\left(E + \frac{v \times B}{c}\right) f\, v\right] = \nabla_v\cdot\left[\left(E + \frac{v \times B}{c}\right) f\right] v$$

$$+ \left(E + \frac{v \times B}{c}\right) f \cdot \underbrace{\nabla_v\, v}_{\delta_{ij}}$$

(2): $$\frac{\partial}{\partial \vec{x}} \cdot \int f \vec{v}\vec{v}\, d^3v = \frac{\partial}{\partial \vec{x}} \cdot \int f (\vec{V} + \vec{v} - \vec{V}) \cdot (\vec{V} + \vec{v} - \vec{V})\, d^3v$$

$$= \frac{\partial}{\partial \vec{x}} \cdot \left[\underbrace{\int f \vec{V}\vec{V}\, d^3v}_{= n\vec{V}\vec{V}} + 2 \underbrace{\int f \vec{V} (v - \vec{V})\, d^3v}_{0} + \underbrace{\int f (v - \vec{V})(v - \vec{V})\, d^3v}_{\frac{\vec{\vec{P}}}{m}} \right]$$

$$= \frac{\partial}{\partial \vec{x}} \cdot \left(n\vec{V}\vec{V} + \frac{\vec{\vec{P}}}{m} \right)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}(mn\vec{V}) + \frac{\partial}{\partial \vec{x}} \cdot (mn\vec{V}\vec{V}) + \frac{\partial}{\partial \vec{x}} \cdot \vec{\vec{P}} + q n \left(\vec{E} + \frac{\vec{V} \times B}{c} \right) = 0$$

$$\frac{\partial}{\partial t}(mn\vec{V}) = m \frac{\partial n}{\partial t} \vec{V} + mn \frac{\partial \vec{V}}{\partial t}$$

$$\frac{\partial}{\partial \vec{x}} \cdot (mn\vec{V}\vec{V}) = m \underbrace{\frac{\partial}{\partial \vec{x}} \cdot (n\vec{V})}_{0} \vec{V} + mn\vec{V} \cdot \frac{\partial}{\partial \vec{x}} \vec{V}$$

$\boxed{\text{Continuity Eq.}}$

$$\Rightarrow \quad \boxed{mn \frac{\partial \vec{V}}{\partial t} + mn\vec{V} \cdot \frac{\partial \vec{V}}{\partial \vec{x}} + \frac{\partial}{\partial \vec{x}} \cdot \vec{\vec{P}} - q n \left(\vec{E} + \frac{\vec{V} \times B}{c} \right) = 0}$$

Energy equation

- Need 3rd moment eq to describe the dynamics of $\overleftrightarrow{P}$.

  ⋮

- Replace the 6D Vlasov equation by infinite number of moment equations in 3D.

- Have to close up the system somewhere in the chain. [closure scheme] by approximation.

$$\overleftrightarrow{P}_{sij} = m_s \langle (v_i - \overline{v}_i)(v_j - \overline{v}_j) \rangle$$

$$= m_s \langle (v_i - \overline{v}_i)(v_j - \overline{v}_j) \rangle \delta_{ij} = \underbrace{nT}_{P} \delta_{ij}$$

Scalar

Assume no off-diagnal terms

NO summation W.R.T i, j

Assume:

$$\langle (v_x - \overline{v}_x)^2 \rangle = \langle (v_y - \overline{v}_y)^2 \rangle = \langle (v_z - \overline{v}_z)^2 \rangle \equiv nT/m$$

$$\frac{\partial}{\partial \vec{x}} \cdot \vec{\vec{p}} = \frac{\partial}{\partial \vec{x}} \cdot \left(T n \, \delta_{ij} \right) = \nabla p$$ in the momentum Eq.

But, we still need an equation to describe the dynamics of p.

Take:

$$\begin{cases} p/\rho^{\gamma} = \text{Const}, & \rho \equiv m n \\ \gamma = 0, & \text{Isobaric} \\ \gamma = 1, & \text{isothermal} \\ \gamma = \frac{(2+D)}{D}, & \text{Adiabatic in Dimension } D \end{cases}$$

"ratio of specific heat"

$$\frac{d}{dt}\left(p/\rho^{\gamma}\right) = 0 \iff \frac{dp}{dt} - \frac{\gamma p}{\rho}\frac{d\rho}{dt} = 0$$

$$\iff T(1-\gamma)\frac{dn}{dt} + n\frac{dT}{dt} = 0$$

$$\boxed{\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v \cdot \nabla_x}$$