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Facing The Singularity, Landau Damping

Electrostatic, 1D

Fourier Transform in x and t:

$$\hat{E}(k,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\, e^{i\omega t} \int_{-\infty}^{+\infty} dx\, e^{-ikx}\, E(x,t)$$

Inverse Fourier

$$E(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t} \int_{-\infty}^{+\infty} dx\, e^{ikx}\, \hat{E}(k,\omega)$$

(k, ω) are real

similar transformations for g_1

$$D(k,\omega) = 1 + \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{+\infty} \frac{\frac{\partial g_{s0}}{\partial v_x}}{\frac{\omega}{k} - v_x}\, dv_x$$

pole at $v_x = \frac{\omega}{k}$

???

Landau: "It's an initial value problem"

Laplace transformation in t,

not Fourier transformation.

Fourier transformation in x is all right.

$$\hat{E}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} E(x,t)\, dx$$

$$\hat{g}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} g(x,t)\, dx$$

Linearized Vlasov Eq.

$$\frac{\partial \hat{g}_{s1}(k,t)}{\partial t} + ikv\hat{g}_{s1} + \frac{q_s}{m_s}\hat{E}_1 \frac{\partial g_{s0}}{\partial v_x} = 0 \quad \cdots (4.1)$$

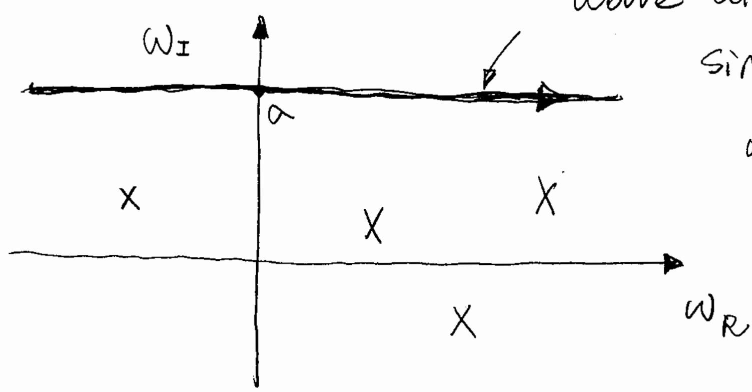
ODE in t

Laplace transformation: in t

$$\hat{E}(k,\omega) = \int_0^\infty dt\, e^{+i\omega t}\, \hat{E}(k,t)$$

Inverse transformation $\rightarrow$ $$\hat{E}(k,t) = \frac{1}{2\pi}\int_{-\infty+i\sigma}^{\infty+i\sigma} d\omega\, e^{-i\omega t}\, \hat{E}(k,\omega)$$

ω = Complex.



above all the singular points of $\hat{E}(k,\omega)$

$$\int_0^\infty dt\, e^{i\omega t}\left[\text{Eq. (4.1)}\right]$$

$$\int_0^\infty dt\, e^{i\omega t}\frac{\partial \hat{g}_{s1}(t)}{\partial t} = e^{i\omega t}\hat{g}_{s1}\Big|_0^\infty - i\omega\int_0^\infty dt\, e^{i\omega t}\,\hat{g}_{s1}(k,t)$$

$$= -\hat{g}_{s1}(k,t=0) - i\omega\,\hat{g}_s(k,\omega)$$

Assume: $|\hat{g}_{s1}(k,t)| < M e^{\gamma t}$, and $\mathrm{Im}\,\omega > \gamma$

So that $e^{i\omega t}\hat{g}_{s1}(k,t) \rightarrow 0$ as $t \rightarrow \infty$

Linearized Vlasov Eq:

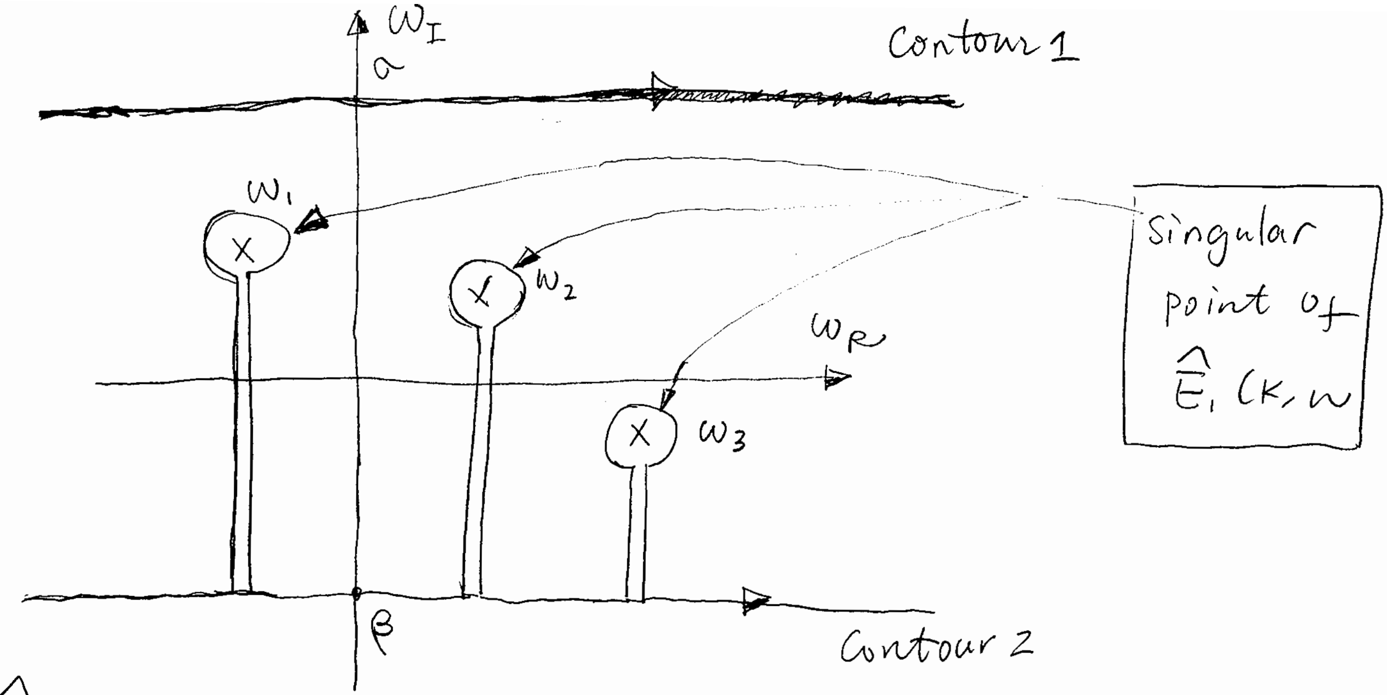
$$(i\omega - ikv)\,\hat{g}_{s1}(k,\omega) = \frac{q_s}{m_s}\hat{E}_1(k,\omega)\frac{\partial g_{s0}}{\partial v_x} - \overbrace{g_{s1}(k,t=0)}^{h_s(k)\ ///}$$

poisson Eq:

$$ik\hat{E}_1 = 4\pi \sum q_s n_{s0}\int_{-\infty}^{+\infty}\hat{g}_{s1}\,dv_x$$

$$\hat{E}_1(k,\omega) = \frac{\frac{4\pi}{k^2}\sum q_s n_{s0}\int_{-\infty}^{+\infty}\frac{h_s(k)}{\omega/k - v_x}dv_x}{\underbrace{1 + \frac{4\pi}{k^2}\sum_s \frac{q_s^2 n_{s0}}{m_s}\int_{-\infty}^{+\infty}\frac{\partial g_{s0}/\partial v_x}{\omega/k - v_x}dv_x}_{D(k,\omega)}} \quad \cdots (4.2)$$

To obtain $\hat{E}_1(k,t)$: Perform the inverse Laplace along a path above all the singular point of $\hat{E}_1(k,\omega)$: Contour 1.



$2\pi \hat{E}_1(k,t)$

$$= \int_{\text{contour 1}} \hat{E}_1 \, d\omega = \underbrace{\int_{\text{contour 2}} \hat{E}_1 \, d\omega}_{\substack{\text{S} \\ e^{-\beta t} \to 0, \text{ as } t \to \infty}} - 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{\omega_k} \hat{E}_1(\omega)$$

As $t \to \infty$, $\hat{E}_1(k,t)$ are determined by the singular points of $\hat{E}_1(k,\omega)$.

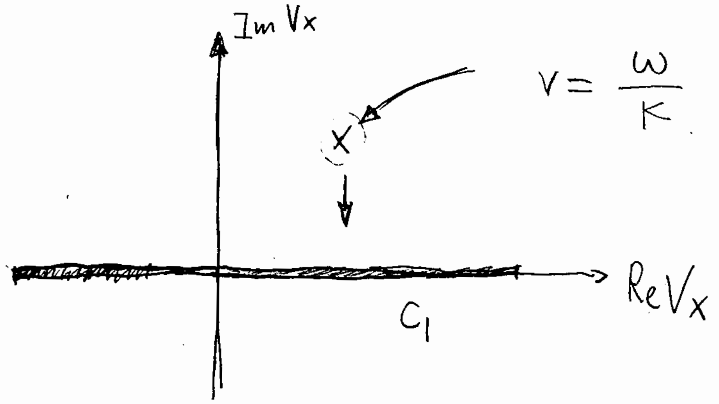
Eigen modes!!

But, $\hat{E}_1(k,\omega)$ is only defined for large $\operatorname{Im} \omega$ so far. We need to analytically continue $\hat{E}_1(k,\omega)$

from the region with large $\mathrm{Im}\,\omega$ to entair complex plan of ω.

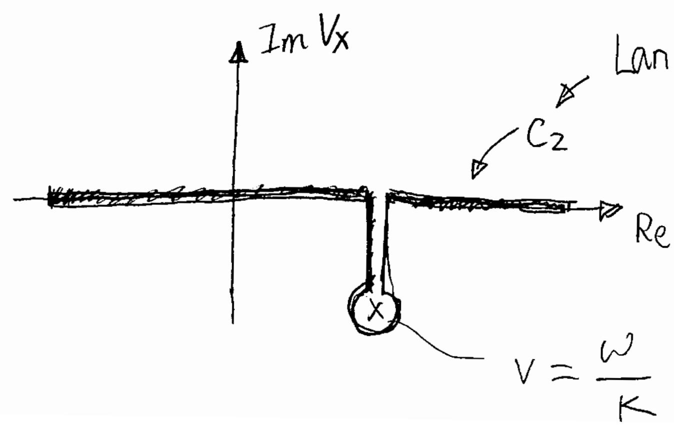
- Analytically continue both the numerator and the denomenator in Eq (4.2).

- For the denomenator $D(k,\omega)$, it is well defined for large $\mathrm{Im}\,\omega$.



$$\int_{C_1} dv_x \frac{\partial g_{s0}}{\frac{\omega}{K} - v_x}$$

As ω moves downward, deform the velocity contour,



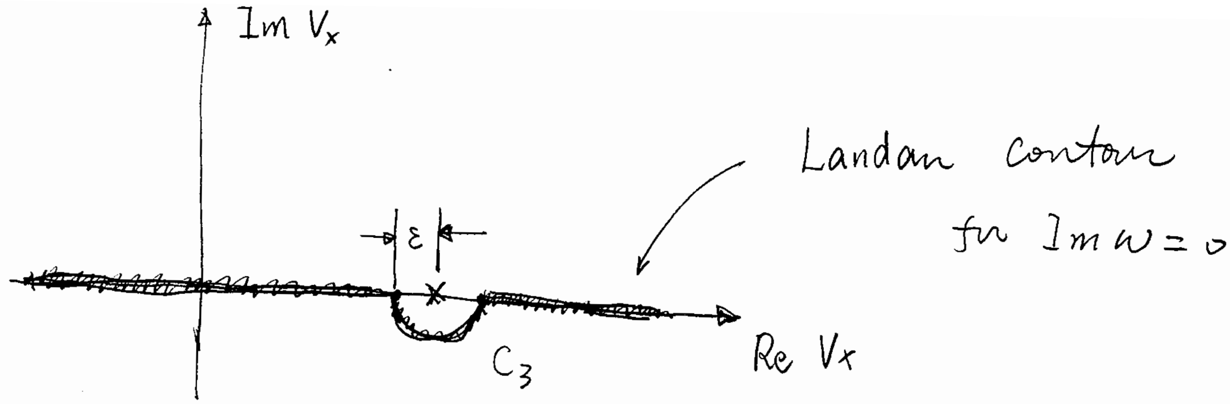
$$\int_{C_2} dv_x \frac{\partial g_{s0}}{\frac{\omega}{K} - v_x}$$

- Use the same method to analytically continue the numerator in Eq.(4.2).

- The singular points of $\hat{E}(K,\omega)$ is given by the zeros of $D(K,\omega)$:

$$\boxed{D(K,\omega)=0} \longleftarrow \left(\begin{array}{l}\text{Analytically}\\ \text{Continued}\end{array}\right)$$

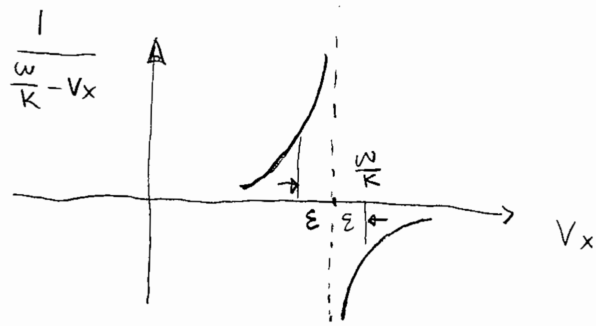
$$D(K,\omega) = \begin{cases} 1+\sum_s \dfrac{\omega_{ps}^2}{K^2}\displaystyle\int_{-\infty}^{+\infty} \dfrac{dV_x}{\frac{\omega}{K}-V_x}\dfrac{\partial g_{s0}}{\partial V_x}, & \mathrm{Im}\,\omega>0 \\[2ex] 1+\sum_s \dfrac{\omega_{ps}^2}{K^2}\left[\displaystyle\int_{-\infty}^{+\infty} \dfrac{dV_x}{\frac{\omega}{K}-V_x}\dfrac{\partial g_{s0}}{\partial V_x} - 2\pi i\left.\dfrac{\partial g_{s0}}{\partial V_x}\right|_{V_x=\frac{\omega}{K}}\right], & \mathrm{Im}\,\omega<0 \\[2ex] 1+\sum_s \dfrac{\omega_{ps}^2}{K^2}\left[\underbrace{P}_{\text{principal value.}}\displaystyle\int_{-\infty}^{+\infty} \dfrac{dV_x}{\frac{\omega}{K}-V_x}\dfrac{\partial g_{s0}}{\partial V_x} - \pi i\left.\dfrac{\partial g_{s0}}{\partial V_x}\right|_{V_x=\frac{\omega}{K}}\right], & \mathrm{Im}\,\omega=0 \end{cases}$$



$$D(k,\omega) = 1 + \sum_s \frac{\omega_{ps}^2}{k^2} \int_{C_3} \frac{\frac{\partial g_{s0}}{\partial v_x}}{\frac{\omega}{k} - v_x}\, dv_x$$

$$= 1 + \sum_s \frac{\omega_{ps}^2}{k^2} \left[\underbrace{\lim_{\varepsilon \to 0} \left(\int_{-\infty}^{\frac{\omega}{k}-\varepsilon} dv_x + \int_{\frac{\omega}{k}+\varepsilon}^{\infty} dv_x \right) \frac{\frac{\partial g_{s0}}{\partial v_x}}{\frac{\omega}{k} - v_x}}_{\equiv} - \pi i \left. \frac{\partial g_{s0}}{\partial v_x} \right|_{v_x = \frac{\omega}{k}} \right]$$

$$\mathbb{P} \int_{-\infty}^{+\infty} \frac{\frac{\partial g_{s0}}{\partial v_x}}{\frac{\omega}{k} - v_x}\, dv_x$$



singularity at the pole is removed by the principal value construction.

moving in from both side at the same rate.

$\Rightarrow$ Contributions from left and right cancel each other.

Langmuir Wave, Revisit

$$D(k,\omega) = D_r + iD_i = 0$$

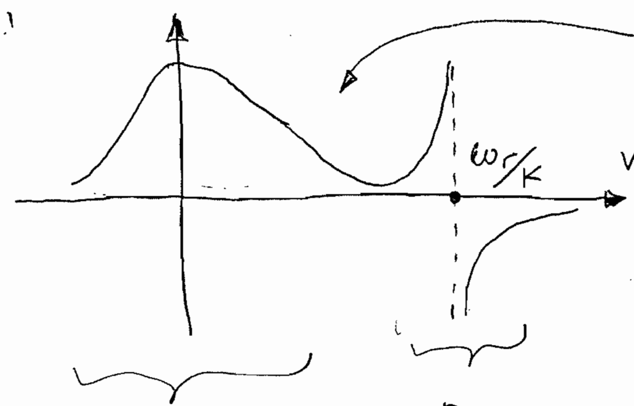
$\omega = \omega_r + \omega_i$, Assume $|\omega_i| \ll |\omega_r|$, $|D_i| \ll |D_r|$

$$D_r(k,\omega_r) + i D_i(k,\omega_r) + i\omega_i \frac{\partial D_r}{\partial \omega_r} = 0$$

Leading order:

$$D_r(k,\omega_r) = 0$$

$$D_r = 1 + \frac{\omega_e^2}{k^2} \, P.V.\int_{-\infty}^{+\infty} dv_x \, \frac{\frac{\partial g_{e0}}{\partial v_x}}{\frac{\omega}{k} - v_x} = 0$$



P.V. $\to 0$

$$D_r = 1 - \frac{\omega_{pe}^2}{\omega_r^2} - 3\frac{k^2 v_{the}^2 \omega_{pe}^2}{\omega_r^4} \implies \omega_r^2 = \omega_{pe}^2 + 3k^2 v_{the}^2$$

Next order: $$\omega_i = -\frac{D_i(k,\omega_r)}{\frac{\partial D_r}{\partial \omega_r}}$$

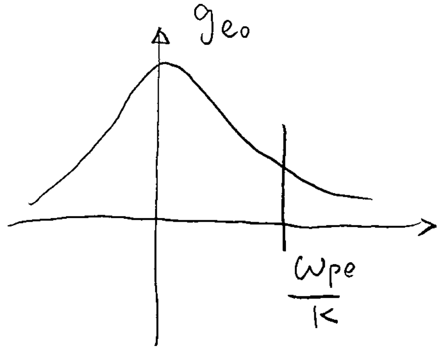
$$\frac{\partial D_r(k,\omega_r)}{\partial \omega_r} \doteq +2\frac{\omega_{pe}^2}{\omega_r^3} \approx \frac{2}{\omega_{pe}}$$

$$\omega_i = + \frac{\frac{\omega_{pe}^2}{K^2}\pi}{\frac{2}{\omega_{pe}}} \left.\frac{\partial g_{eo}}{\partial v_x}\right|_{v_x=\frac{\omega_r}{k}}$$

$$= \frac{\pi\,\omega_{pe}^2}{2k^2}\left.\frac{\partial g_{eo}}{\partial v_x}\right|_{v_x=\frac{\omega_{pe}}{k}}$$

putting together:

$$\omega = \omega_{pe}\left(1 + 3k\frac{v_{te}^2}{\omega_{pe}^2}\right)^{1/2} + \frac{i\pi\,\omega_{pe}^2}{2k^2}\left.\frac{\partial g_{eo}}{\partial v_x}\right|_{v_x=\frac{\omega_{pe}}{k}}$$



$$\left.\frac{\partial g_{eo}}{\partial v_x}\right|_{v_x=\frac{\omega_{pe}}{k}} < 0$$

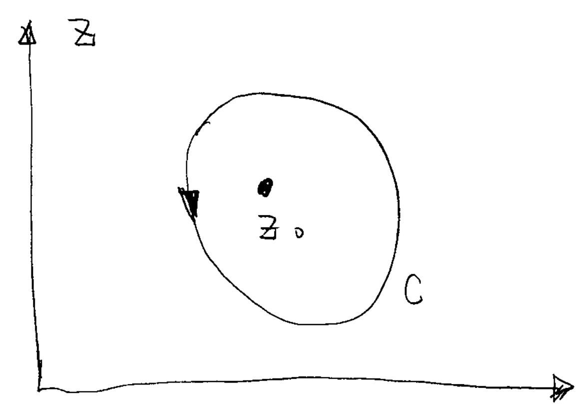
$\Rightarrow$ Landau Damping.

$\boxed{\text{complex analysis}}$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots \frac{b_n}{(z-z_0)^n} + \cdots$$

Laurent series : $\quad 0 < |z - z_0| < R$.

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}} \qquad (n = 1, 2, \cdots)$$



$$\int_C f(z)\,dz = 2\pi i\, b_1$$

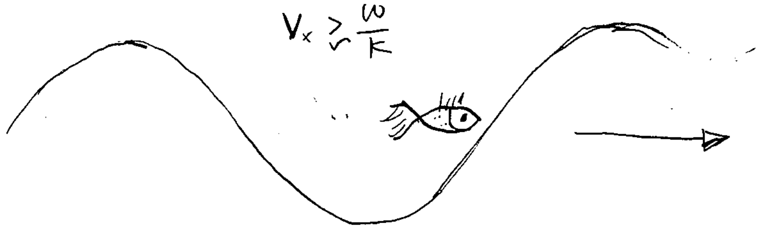
$$b_1 \equiv \operatorname*{Res}_{z=z_0} f(z)$$ residue.

physical picture of Landau Damping

"Every body has his/her own vesion of Landau Damping!"

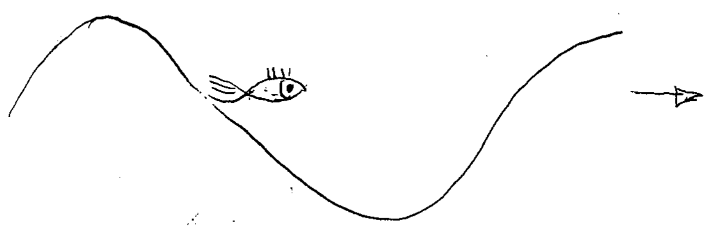
— R. C. Davidson

$V_x \gtrsim \frac{\omega}{k}$



$\frac{\omega}{k}$: loss energy

$V_x \lesssim \frac{\omega}{k}$



$\frac{\omega}{k}$, gain energy

IF, $\left.\frac{\partial g}{\partial V_x}\right|_{V_x = \frac{\omega}{k}} < 0$ more particles gaining energy

$\Rightarrow$ Wave is damped.