③

Electrostatic wave

- Vlasov – Maxwell eqs predicts electrostatic waves.
- Linear waves in a unmagnetized, homogeneous plasma.

$$f_s = f_{s0} + f_{s1}, \qquad n_{s0} = \int f_{s0}\, d^3v = \text{const.}$$

$$E = \underset{\substack{\downarrow \\ 0}}{E_0} + E_1, \qquad f_{s1} \ll f_{s0}$$

homogeneous equilibrium

$$f_{s1},\ E_1 \sim \exp(i\vec{k}\cdot\vec{x} - i\omega t)$$

- Why $B_1 \approx 0$?

IF $\vec{k} \parallel \vec{E}$, Then $\nabla \times E_1 = -\frac{1}{c}\frac{\partial B_1}{\partial t}$

$$\Rightarrow B_1 = 0$$

But,

$$\nabla \times B_1 = \frac{1}{c}\frac{\partial E_1}{\partial t} + \frac{4\pi}{c} j_1$$

$$\left\{ \begin{array}{l} j_1 \sim n_1 q V_t \\ \nabla \cdot E_1 = 4\pi n_1 q \end{array} \right.$$

$$\Longrightarrow \quad k B_1 \sim \frac{1}{c} \omega E_1 + \frac{V_t k E_1}{c}$$

$$B_1 \sim \frac{1}{c}\frac{\omega}{k} E_1 + \frac{V_t}{c} E_1$$

Look at the terms in the vlasov Eq.

$$E + \frac{V \times B}{c}$$

$$\frac{\frac{V \times B}{c}}{E} \sim \frac{1}{c^2}\frac{V_t \omega}{k} + \frac{V_t^2}{c^2} \ll 1$$

$\boxed{\text{Assume } \frac{\omega}{k},\ V_t \ll c}$

$\boxed{\text{Therefore, if } \vec{k} \parallel \vec{E}_1,\ \frac{\omega}{k} \ll c, \text{ and } V_t \ll c, \text{ we can drop the } B_1 \text{ in the vlasov-Maxwell Eqs.}}$

$$\left\{ \begin{array}{l} \dfrac{\partial f_s}{\partial t} + v \cdot \dfrac{\partial f}{\partial x} + \dfrac{q_s}{m_s} E_1 \cdot \dfrac{\partial f}{\partial v} = 0 \\[2ex] \nabla \cdot E = 4\pi \sum_s q_s \int f_s \, d^3 v \end{array} \right.$$

- $\boxed{\text{1D, in } x}$

Equilibrium, 0th order: trivial

perturbation, 1st order:

$$\frac{\partial f_{s1}}{\partial t} + v \cdot \frac{\partial f_{s1}}{\partial x} + \frac{q_s}{m_s} E_1 \frac{\partial f_{s0}}{\partial v_x} = 0$$

$$\nabla \cdot E_1 = 4\pi \sum_s q_s \int f_{s1} \, d^3v$$

Look at a single harmonic (k, ω)

$$f_{s1} = \hat{f}_{s1} \exp(ikx - i\omega t)$$

$$E_1 = \hat{E}_1 \exp(ikx - i\omega t)$$

$$\Rightarrow \begin{cases} -i\omega \hat{f}_{s1} + ikv \hat{f}_{s1} + \frac{q_s}{m_s} \hat{E}_1 \frac{\partial f_{s0}}{\partial v_x} = 0 \\ ik \hat{E}_1 = 4\pi \sum_s q_s \int \hat{f}_{s1} \, d^3v \end{cases}$$

$$\Rightarrow \quad \hat{f}_{s1} = \frac{\frac{q_s}{m_s}\hat{E}_1 \frac{\partial f_{s0}}{\partial v_x}}{i\,(kv - \omega)}$$

$$ik\hat{E}_1 = 4\pi \sum_s \frac{q_s^2 \hat{E}_1}{m_s\, i} \int \frac{\frac{\partial f_0}{\partial v_x}}{i(kv-\omega)}\, d^3v$$

$$\Rightarrow \quad \boxed{1 + \frac{1}{k^2} \sum \omega_{ps}^2 \int_{-\infty}^{+\infty} \frac{\frac{\partial g_s}{\partial v_x}}{\omega/k - v_x}\, dv_x = 0}$$

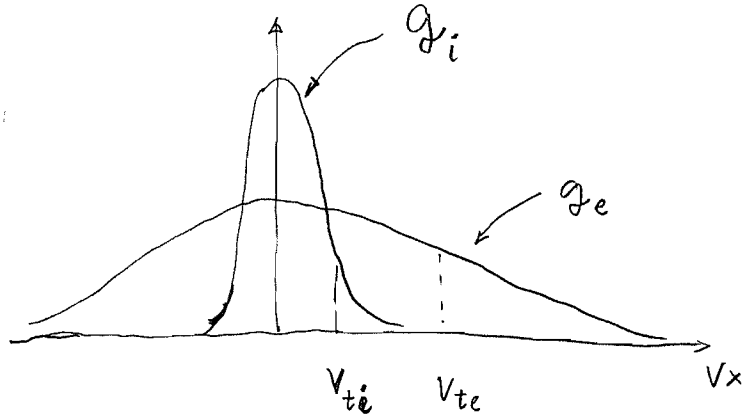
$$g(v_x) \equiv \frac{1}{n_{s0}} \int dv_y\, dv_z\, f_{s0}(v)$$

Dispersion Relation

- Pole at $v_x = \frac{\omega}{k}$,

need Landau's treatment, (later)

Langmuir wave



- Assume $T_e \cong T_i$

$\Longrightarrow$ $V_{ti} \ll V_{te}$

thermal velocity

- Assume: $\frac{\omega}{k} \gg V_{te} \gg V_{ti}$

For every species,

$$\frac{\omega_{ps}^2}{k^2} \int \frac{\frac{\partial g_s}{\partial V_x}}{\frac{\omega}{k} - V_x} dV_x = 0$$

pole $\to \infty$, I. B. P.

$$\Downarrow$$

$$-\frac{\omega_{ps}^2}{k^2} \int \frac{g_s}{\left(\frac{\omega}{k} - V_x\right)^2} dV_x = 0$$

$$\frac{1}{\left(\frac{\omega}{k} - V_x\right)^2} = \frac{k^2}{\omega^2}\left(1 + \frac{2V_x k}{\omega} + \frac{3V_x^2 k^2}{\omega^2} + \cdots\right)$$

Taylor expansion

$$\int_{-\infty}^{+\infty} v_x g_e \, dv_x = 0$$

Definition of V_{te}

$$\int_{-\infty}^{+\infty} v_x^2 g_e \, dv_x = V_{te}^2$$

$$\Rightarrow \quad 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \underbrace{\frac{3 V_{te}^2 k^2}{\omega^2}}_{\text{small}}\right) - \frac{\omega_{pi}^2}{\omega^2}\left(1 + \frac{3 V_{ti}^2 k^2}{\omega^2}\right) = 0$$

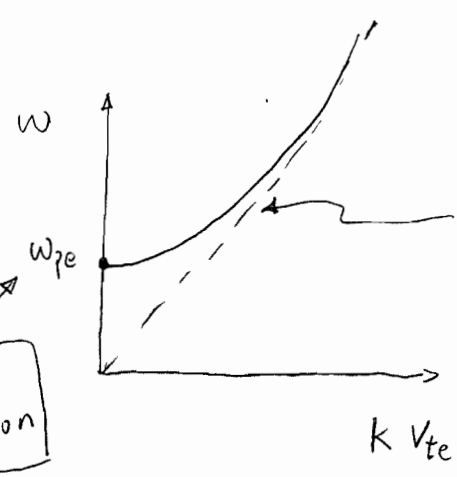
drop because $\omega_{pi}^2 \ll \omega_{pe}^2$

Leading order: $\omega^2 = \omega_{pe}^2$

next order:

$$\boxed{\omega^2 = \omega_{pe}^2 + 3 k^2 V_{te}^2}$$

Langmuir wave



$\boxed{\omega = \sqrt{3}\, k V_{te}, \text{ electron sound wave}}$

plasma oscillation

Ion Acoustic wave

- Assume: $V_{ti} \ll \frac{\omega}{k} \ll V_{te}$, $k^2 \lambda_D^2 \ll 1$

Ion's response, $-\frac{\omega_{pi}^2}{\omega^2}\left(1 + \frac{3 V_{ti}^2 k^2}{\omega^2}\right)$

Electron's response:

$$\frac{\omega_{pe}^2}{k^2} \int \frac{\frac{\partial g_e}{\partial V_x}}{\frac{\omega}{k} - V_x} dV_x$$

drop

Assume $\frac{\partial g_e}{\partial V_x} = 0$ at $V_x = 0$

$$\frac{\omega_{pe}^2}{k^2} \int \frac{\frac{\partial g_e}{\partial V_x}}{-V_x} dV_x$$

To complete the integral, let's assume:

$$g_e = \frac{1}{\left(2\pi \frac{T_e}{m_e}\right)^{1/2}} \exp\left(-\frac{\frac{1}{2} m_e V_x^2}{T_e}\right)$$

Thermal distribution

$$\int_{-\infty}^{+\infty} \frac{\frac{\partial g_e}{\partial v_x}}{-v_x}\, dv_x = \frac{1}{\left(2\pi \frac{T_e}{m_e}\right)^{1/2}} \int_{-\infty}^{+\infty} +\frac{m_e}{T_e}\, e^{-\frac{\frac{1}{2} m_e v_x^2}{T}}\, dv_x$$

$$= \frac{m}{T_e} = \frac{1}{v_{te}^2}$$

putting together electron & ion's response:

Ion plasma oscillation

$$\underbrace{1}_{\text{drop}} - \frac{\omega_{pi}^2}{\omega^2} - \underbrace{\frac{3\,\omega_{pi}^2\, v_{ti}^2\, k^2}{\omega^4}}_{\text{small}} + \frac{\omega_{pe}^2}{k^2 v_{te}^2} = 0$$

$\frac{1}{k^2 \lambda_D^2} \gg 1$, electron response

$$\times \frac{\omega^2 v_{te}^2}{\omega_{pe}^2} \Downarrow$$

$$\frac{\omega^2}{k^2} - \frac{\omega_{pi}^2\, v_{te}^2}{\omega_{pe}^2} - \underbrace{3\,\frac{\omega_{pi}^2}{\omega_{pe}^2}\, v_{ti}^2\, v_{te}^2\, \frac{k^2}{\omega^2}}_{\text{small}} = 0$$

Leading order:

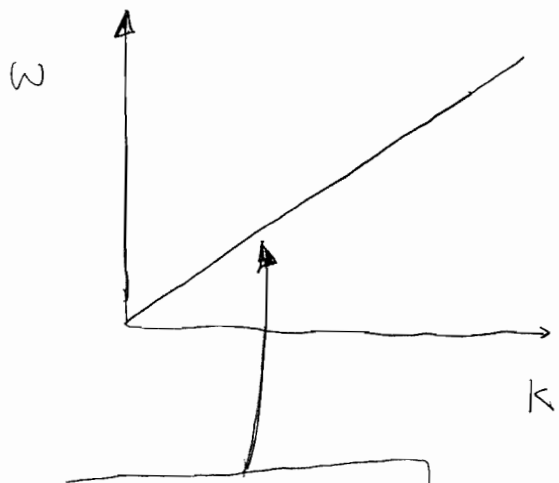
$$\frac{\omega^2}{k^2} = \frac{\omega_{pi}^2}{\omega_{pe}^2}\, v_{te}^2$$

$$\boxed{\sum_s q_s n_s = 0}$$

next order:

$$\frac{\omega^2}{k^2} = \frac{\omega_{pi}^2}{\omega_{pe}^2}\, v_{te}^2 + 3\, v_{ti}^2 = \frac{q_i\, m_e}{|q_e|\, m_i}\, v_{te}^2 + 3\, v_{ti}^2$$

Compare Ion Acoustic Wave & Langmuir Wave

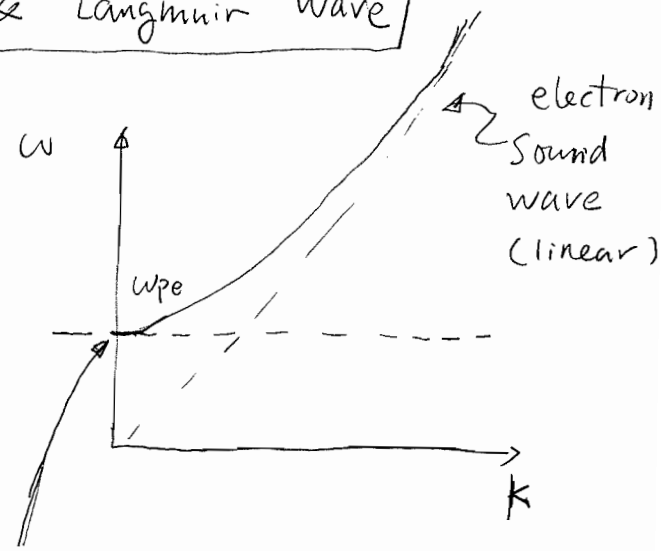


ion sound wave
+
electron dynamics

- electron's response to the wave dominates over ion's plasma oscillation

- will bend if do not drop the ion plasma oscillation term (1)

ion plasma wave

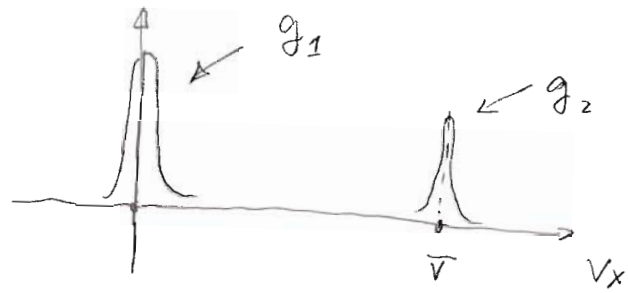


electron
plasma oscillation (flat)

- No ion dynamics

Two-stream Instability

Two species with relative velocity



$$g_1(V_x) = \delta(V_x)$$

for

$$g_2(V_x) = \delta(V_x - \bar{V})$$

Background charge and current neutrality is provided by a third species of heavy background ions.

$$\left.\begin{aligned} &E_0 = 0, \quad B_0 = 0, \\ &\frac{dg_1}{dt} = \frac{\partial g_1}{\partial t} + v \cdot \frac{\partial g_1}{\partial x} = 0 \\ &\frac{dg_2}{dt} = 0 \end{aligned}\right\} \text{ Equilibrium.}$$

- Consider 1D electrostatic perturbation with $k = k\,\hat{e}_x$

$$E = E\,\hat{e}_x$$

Two Streams

Kurokawa 2001

Dispersion relation

$$D(k,\omega) \equiv 1 + \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{+\infty} \frac{\frac{\partial g_s}{\partial v_x}}{\frac{\omega}{k} - v_x} \, dv_x = 0$$

ignore the background ions,

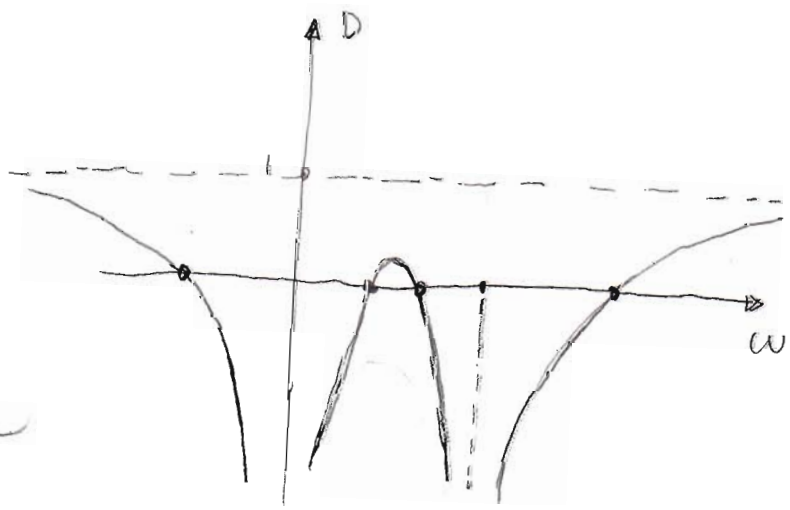
$$D(k,\omega) \equiv 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{+\infty} \frac{g_s}{\left(\frac{\omega}{k} - v_x\right)^2} \, dv_x$$

I.B.P

$$= 1 - \frac{\omega_{p1}^2}{\omega^2} - \frac{\omega_{p2}^2}{(\omega - kV)^2} = 0$$

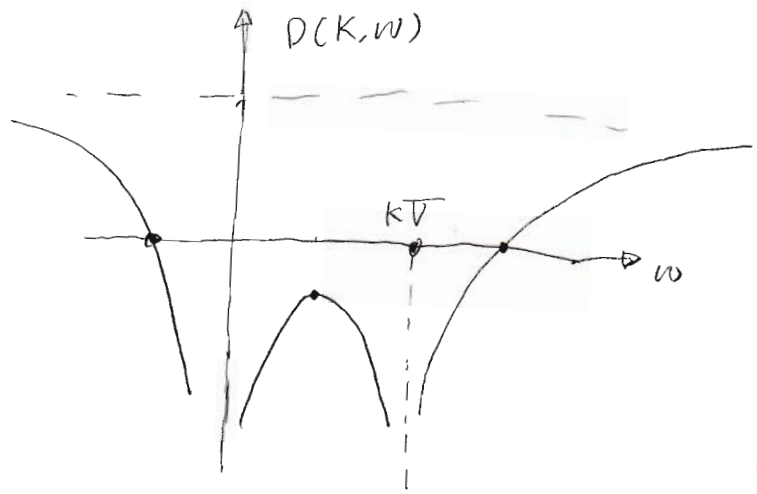
$\boxed{\text{4th order algebraic equation for } \omega}$

Graphic Method for algebraic equation.



$D(k,\omega) = 1$

has four Real Roots.



2 real roots

2 conjugate complex roots

$\Longrightarrow$ <u>unstable</u>,

$D(\omega, k) = 0$

$$\Rightarrow \quad (\omega - kV)^2 \omega^2 = \omega_{p1}^2 (\omega - kV)^2 + \omega_{p2}^2 \omega^2$$

$$(\omega - kV)^2 (\omega^2 - \omega_{p1}^2) = \omega_{p2}^2 \omega^2$$

$\boxed{\text{Asymptotic Analysis for small } \omega_{p2}^2 \ll \omega_{p1}^2}$.

<u>Leading order</u>

$\omega = kV$ $\longleftarrow$ "Doppler shifted" plasma oscillation of g_2"

$\omega = \pm \omega_{p1}$ $\longleftarrow$ plasma oscillation of g_1

<u>Next order, ω_{p2}^2 induces coupling,</u>

$\boxed{\text{Look at } \omega = kV + \Delta\omega, \quad \Delta\omega \ll kV}$

$$\Delta\omega^2 (k^2 V^2 + 2kV\Delta\omega + \Delta\omega^2 - \omega_{p1}^2) = \omega_{p2}^2 k^2 V^2$$

$$\Delta\omega^4 + 2kV\Delta\omega^3 + (k^2 V^2 - \omega_{p1}^2)\Delta\omega^2 = \omega_{p2}^2 k^2 V^2$$

IF $\left|k^2 v^2 - \omega_{p1}^2\right| \gg \Delta\omega$, keep only the $\Delta\omega^2$ term

$$\Delta\omega^2 = \frac{\omega_{p2}^2 \, k^2 v^2}{k^2 v^2 - \omega_{p1}^2}$$

unstable, if $k^2 v^2 < \omega_{p1}^2$

$$\Delta\omega = \pm i \, \frac{\omega_{p2} \, k v}{\sqrt{\omega_{p1}^2 - k^2 v^2}}$$

$$\gamma = \Delta\omega_i \sim \omega_{p2} \, k v$$

γ increases as $k^2 v^2 \to \omega_{p1}^2$

when $\left|k^2 v^2 - \omega_{p1}^2\right| \ll \Delta\omega$, keep only the $\Delta\omega^3$ term,

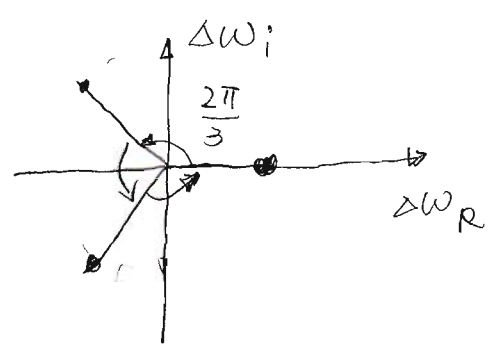
$\boxed{\text{resonant condition}}$

$$2 k v \, \Delta\omega^3 = \omega_{p2}^2 \, k^2 v^2 = \omega_{p2}^2 \, \omega_{p1}^2$$

$$\Delta\omega^3 = \frac{1}{2} \, \omega_{p2}^2 \, \omega_{p1}$$

$$\Delta\omega = \left(\frac{1}{2}\right)^{1/3} \omega_{p2}^{2/3} \, \omega_{p1}^{1/3} \, e^{i \frac{2n\pi}{3}} \qquad n = 0, 1, 2$$

$$\gamma \sim \omega_{p2}^{2/3} \, \omega_{p1}^{1/3}$$



$\boxed{\text{Look at the root near } \omega = +\omega_{p1}}$

$$\omega = \omega_{p1} + \Delta\omega$$

$$(\Delta\omega + \omega_{p1} - kV)^2 (2\omega_{p1} + \underbrace{\Delta\omega}_{\text{drop}}) \Delta\omega = \omega_{p2}^2 \omega_{p1}^2$$

$$\left[\Delta\omega^2 + 2\Delta\omega(\omega_{p1} - kV) + (\omega_{p1} - kV)^2\right] 2\Delta\omega = \omega_{p2}^2 \omega_{p1}$$

when $\quad |\omega_{p1} - kV| \ll \Delta\omega$

only keep $\quad \Delta\omega^3$ term

$$2\Delta\omega^3 = \omega_{p2}^2 \omega_{p1}$$

unstable, the same as the root near $\omega = kV$.

$\Rightarrow$ $\boxed{\text{The two-stream instability is the result of the resonant between the plasma oscillations of the two streams through the Doppler shift.}}$