Vlasov - Maxwell Equations:

Vlasov

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + V \cdot \nabla_x f + \frac{q}{m}\left(E + \frac{V \times B}{c}\right) \cdot \nabla_v f = 0$$

Maxwell

$$\nabla \cdot E = 4\pi \sum_s q \int f \, d^3v$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$$\nabla \times B = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}\sum_s q \int f v \, d^3v$$

- Nonlinear, partial differential-integral Eqs.
- describe collisionless collective effects.
- starting point for many other simplified models.

Pauli writing the Maxwell equations in his ETH Zurich office in 1953. The **E** term in the second equation has the wrong sign.

General properties :

(1) Conserve energy - momentum :

$$\mathcal{E} = \int d^3x \left[\frac{E^2 + B^2}{8\pi} + \int d^3v \, \frac{m}{2} v^2 f \right]$$

$$\mathbb{P} = \int d^3x \left[\frac{c\,(E \times B)}{4\pi} + \int d^3v \, m v f \right]$$

(2) Functions of constants of motion for single particles are solutions of the Vlasov Eq.

$$f = f(C_1, C_2, \cdots)$$

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial C_i} \underbrace{\frac{dC_i}{dt}}_{\nearrow 0} = 0$$

(3) Vlasov - Maxwell equations have many solutions.

$$\boxed{E = B = 0}$$

$$\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} = 0$$

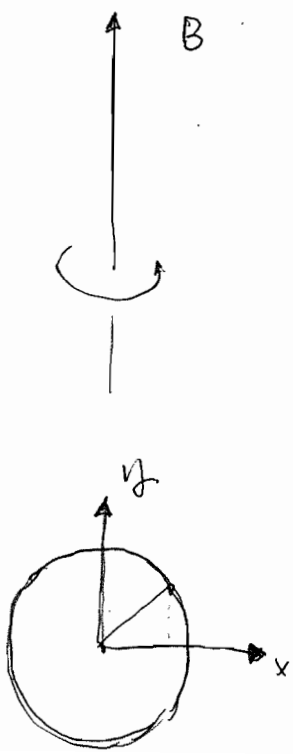
$$f = f(x - v_x t,\ y - v_y t,\ z - v_z t,\ v_x,\ v_y,\ v_z)$$

in general: ① $\frac{\partial f}{\partial t} \neq 0$ ⟵ Not an equilibrium.

② f may not satisfy the Maxwell's eqs.

IF $\sum_i q_i f_i = 0$, then the Maxwell's eq are satisfied.

$$\boxed{E = 0,\ B = \text{Const}}$$



particle dynamics is integrable

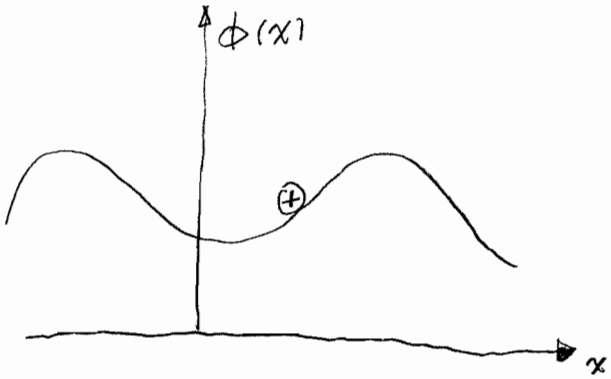
Constants of motion

$$= \left\{ v_\perp,\ v_z,\ x^2 + y^2,\ z - v_z t,\ \cos^{-1}\frac{x}{\sqrt{x^2+y^2}} - \Omega t,\ \sin^{-1}\frac{v_x}{v_\perp} - \Omega t \right\}$$

Many ways for the Maxwell equations to be satisfied.

For example: if $\sum_{q_i} q_i f_i = 0$

$$\boxed{B = 0, \quad E = E(x) \neq 0}$$



Again, particle dynamics is integrable.

Constants of motion $= \{ y, z, v_y, v_z, q\phi(x) + \frac{1}{2} m v_x^2, __ \}$

need to solve for

Look for solution of the form

$$f_s(x, v) = f_s\big(\underbrace{v_x^2 + 2q\phi(x)/m + v_y^2 + v_z^2}_{\text{Energy}} \big)$$

$$\frac{\partial^2 \phi(x)}{\partial x^2} = -4\pi \sum_s q_s \int d^3v \, f_s \left(v_x^2 + v_y^2 + v_z^2 + \frac{2q\phi(x)}{m} \right)$$

For any functional form, this is a well defined ODE for $\phi(x)$.

$\boxed{\text{Deby shielding}}$

Take thermal equilibrium

$$f_s = \frac{n_{s0}}{(2\pi T/m)^{3/2}} \exp\left(- \frac{v_x^2 + v_y^2 + v_z^2 + \frac{2q\phi(x)}{m}}{2T/m} \right)$$

$$n_s = \int d^3v \, f_s = n_{s0} \exp\left[- \frac{q\phi(x)}{T} \right]$$

$$\int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\Rightarrow \quad \frac{\partial^2 \phi}{\partial x^2} = -4\pi \sum_s q_s n_{s0} \exp\left[- \frac{q_s \phi(x)}{T_s} \right]$$

Assume: $\dfrac{\phi q_s}{T_s} \ll 1$, and $\sum q_s n_{s0} = 0$

$$\exp\left(-\frac{q_s \phi}{T_s}\right) = 1 - \frac{q_s \phi}{T_s} + \cdots$$

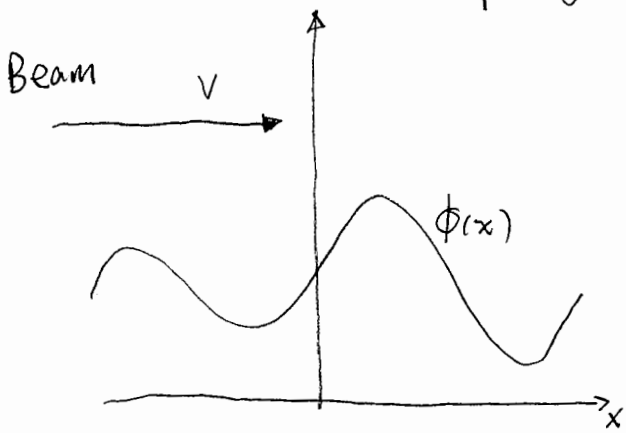
$$\Rightarrow \quad \frac{\partial^2 \phi}{\partial x^2} = 4\pi \sum_s \frac{q_s^2 n_{s0}}{T_s} \phi = \underbrace{\sum_s \frac{1}{\lambda_s^2}}_{\frac{1}{\lambda_D^2}} \phi$$

$\boxed{\text{Deby shielding}}$

$\boxed{\text{BGK mode}}$

Take:

$$f_s(V_x, x) = \begin{cases} \overset{\text{Const}}{2 n_{s0} V_s} \, \delta\left[V_x^2 + \frac{2q\phi(x)}{m} - V_s^2 \right], & V \geq 0 \\ 0, & V < 0 \end{cases}$$



all the particles move in the $\hat{x}$ direction with $V_x = u_s(x)$ at every point x

A beam travelling through a self-generated low potential

$$f_s = n_{s0} \frac{V_s}{V} \delta(V_x - u_s(x))$$

$$u_s(x) \equiv \sqrt{V_s^2 - \frac{2q\phi(x)}{m}}$$

$$\delta(f(y)) = \sum_{y_0} \frac{\delta(y - y_0)}{\left|\frac{df}{dy}\right|_{y_0}}$$

$f(y_0) = 0$

$$n(x) = n_{s0} \frac{V_s}{u_s(x)}$$

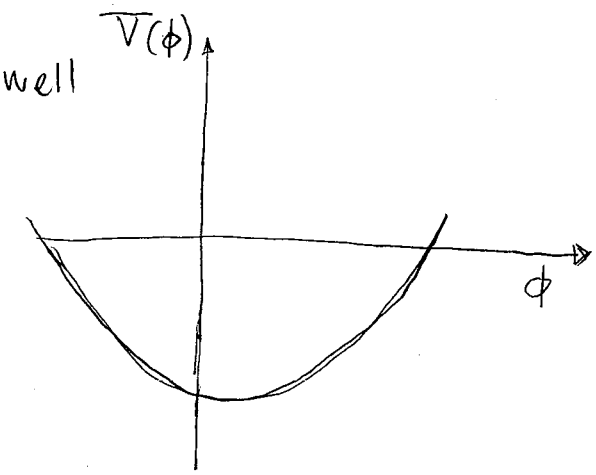
$$\frac{d^2\phi(x)}{dx^2} = -\sum 4\pi n_{s0} q_s v_s \frac{1}{\sqrt{v_s^2 - \frac{2q_s\phi(x)}{m}}}$$

$$= -\frac{\partial V(\phi)}{\partial \phi}$$

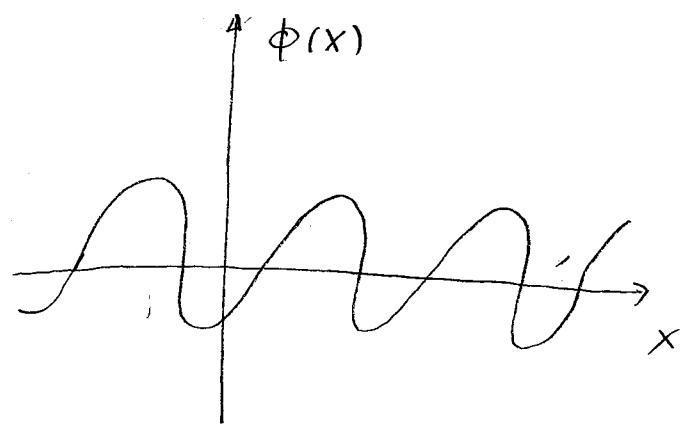
think x as a time coordinate

$$V(\phi) = -\sum 4\pi n_{s0} m_2 v_s \sqrt{v_s^2 - \frac{2q_s\phi}{m}}$$

potential well for ϕ



$\Rightarrow$ ϕ is periodic in x



Assume Low potential:

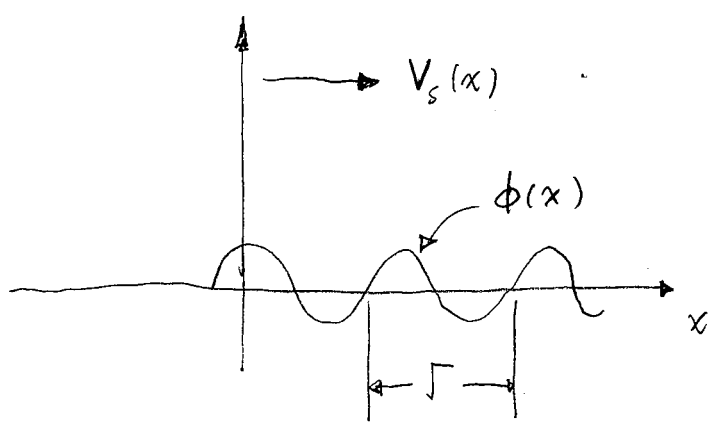
$$v_s^2 \gg \frac{2q_s\phi}{m}$$

$$V(\phi) = -\sum 4\pi n_{s0} m_s v_s^2 \left(1 - \frac{q_s\phi}{m_s v_s^2} - \frac{1}{4}\frac{q_s^2\phi^2}{m_s^2 v_s^4}\right)$$

$$V(\phi) = -\sum 4\pi n_{s0} m_s V_s^2 \rightarrow 0 \text{ (assumption)}$$

$$+ \left(\sum_s 4\pi n_{s0} q_s\right)\phi$$

$$+ \underbrace{\sum_s \frac{4\pi n_{s0} q_s^2}{4 m_i V_s^2}}_{\frac{\Gamma^2}{2}} \phi^2$$

$$\Rightarrow \quad \frac{d^2\phi(x)}{dx^2} + \Gamma^2 \phi = 0$$

$$\phi \sim A \sin \Gamma x + B \cos \Gamma x$$