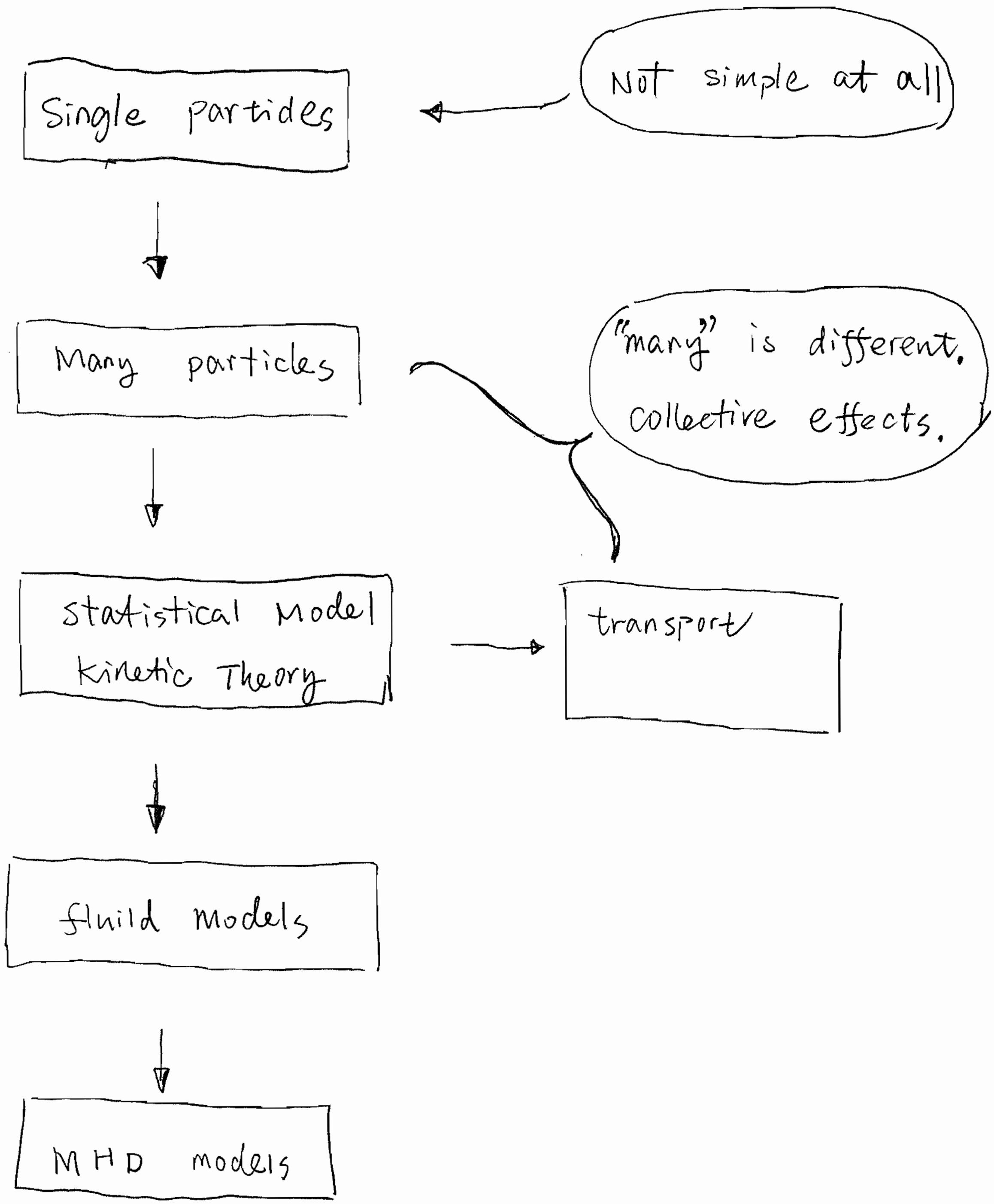
Theoretical Models for plasmas



Kinetic Theory — Statistical Model for plasmas

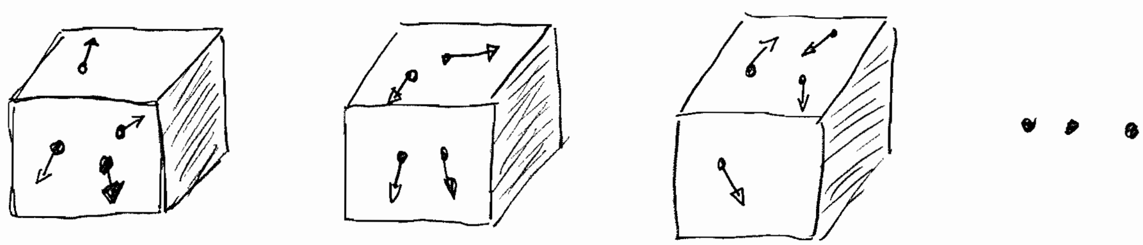
"Many" is different.

But, "many" is too many.

<u>Ensemble</u> Average:

same Macroscopic Conditions

different Microscopic conditions



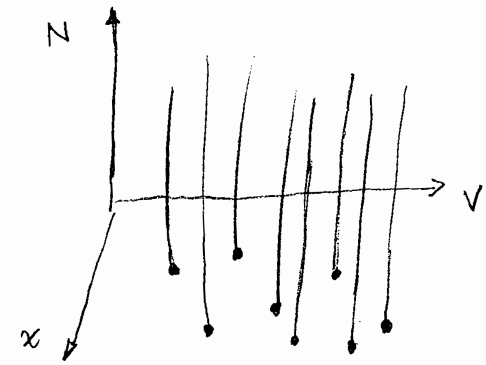
Two approaches:

- BBGKY Hierarchy
- Klimontovich Eq.

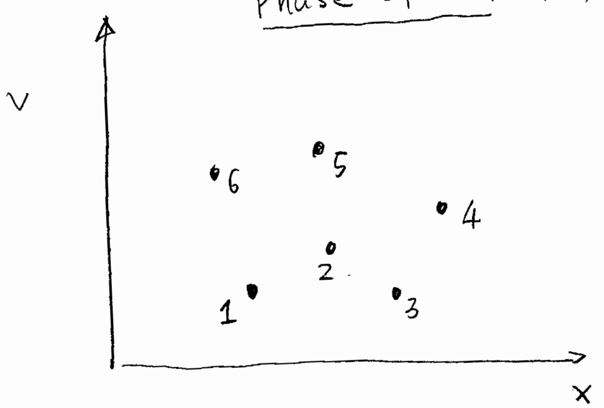
Start by looking one of ensemble ....

plasma Kinetic Theory

Klimontovich Equation



Phase space (x, v)



$$\boxed{N(x, v, t) = \sum_{i=1}^{N_0} \delta(x - X_i(t))\, \delta(v - V_i(t))}$$

$\frac{1}{L^3 V^3}$ unit

$\delta(x^1 - X_i^1)\,\delta(x^2 - X_i^2)\,\delta(x^3 - X_i^3)$

$$\begin{cases} \dot{X}_i(t) = V_i(t) \\ m \dot{V}_i(t) = q\, E^m(X_i, t) + \frac{q}{c} V_i \times B^m(X_i, t) \end{cases}$$

For each particle

from N

Maxwell's Eqs.

$$\begin{cases} \nabla \cdot E^m(x,t) = 4\pi \rho^m(x,t) \\ \nabla \cdot B^m(x,t) = 0 \\ \nabla \times E^m(x,t) = -\frac{1}{c}\frac{\partial B^m(x,t)}{\partial t} \\ \nabla \times B^m = \frac{4\pi}{c} J^m(x,t) + \frac{1}{c}\frac{\partial E^m(x,t)}{\partial t} \end{cases}$$

from N

$$\rho^{m} \equiv \sum_{s} q \int d^{3}v \; N(x, v, t)$$

$$J^{m} \equiv \sum_{s} q \int d^{3}v \; \mathbf{v} \, N(x, v, t)$$

$$\boxed{\text{What's the governing equation for } N\,?}$$

$$\frac{\partial}{\partial t} N(x, v, t) = \sum_{i=1}^{N_0} \dot{X}_i \cdot \frac{\partial N}{\partial X_i} + \dot{V}_i \cdot \frac{\partial N}{\partial V_i}$$

$\dot{X}_i = V_i$

$\dot{V}_i = q\left(E^{m} + \frac{V_i}{c} \times B^{m}\right)$

$$\boxed{\frac{\partial f(a-b)}{\partial a} = -\frac{\partial f(a-b)}{\partial b}}$$

$$\frac{\partial N}{\partial X_i} = \frac{\partial \delta(x - X_i)}{\partial X_i} \delta(v - V_i) = -\frac{\partial \delta(x - X_i)}{\partial x} \delta(v - V_i)$$

$$\frac{\partial N}{\partial V_i} = \delta(x - X_i) \frac{\partial \delta(v - V_i)}{\partial V_i} = -\delta(x - X_i) \frac{\partial \delta(v - V_i)}{\partial v}$$

$\Longrightarrow$

$$\frac{\partial N(x, v, t)}{\partial t} = -\sum_{i} V_i \cdot \frac{\partial \delta(x - X_i)}{\partial x} \delta(v - V_i)$$

$$- \sum_{i} \frac{q}{m} \left(E^{m} + \frac{V_i \times B^{m}}{c} \right) \delta(x - X_i) \cdot \frac{\partial \delta(v - V_i)}{\partial v}$$

$$V_i \cdot \frac{\partial \delta(x - X_i)}{\partial x}\, \delta(v - V_i) = v \cdot \frac{\partial \delta(x - X_i)}{\partial x}\, \delta(v - V_i)$$

$$f(a)\, \delta(a-b) = f(b)\, \delta(a-b)$$

$$\overset{m}{E}(X_i, t) \cdot \frac{\partial \delta(v - V_i)}{\partial v}\, \delta(x - X_i) = \overset{m}{E}(x, t) \cdot \frac{\partial \delta(v - V_i)}{\partial v}\, \delta(x - X_i)$$

$$V_i \times \overset{m}{B}(X_i, t) \cdot \frac{\partial \delta(v - V_i)}{\partial v}\, \delta(x - X_i)$$

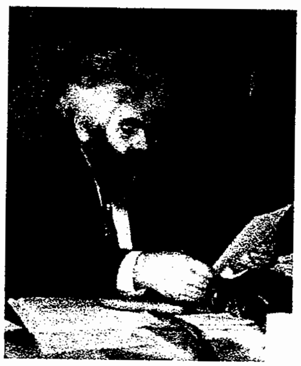
$$= v \times \overset{m}{B}(x, t) \cdot \frac{\partial \delta(v - V_i)}{\partial v}\, \delta(x - X_i)$$

why?

$$\boxed{a \frac{\partial \delta(a-b)}{\partial a} \neq b \frac{\partial \delta(a-b)}{\partial a}}$$

$$\Rightarrow \quad \boxed{\frac{\partial N}{\partial t} + v \cdot \frac{\partial N}{\partial x} + \frac{q}{m}\left(E^{m}(x,t) + \frac{v \times B^{m}(x,t)}{c}\right) \cdot \frac{\partial N}{\partial v} = 0}$$

Klimontovich Eq.

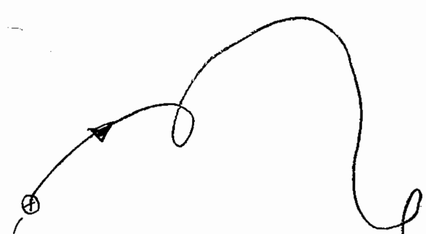


Klimontovich, Yu. L.
(1924 – 2002)

- Equivalent to motion equations of all the particles

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \left.\frac{dx}{dt}\right|_{orbit} \cdot \frac{\partial}{\partial x} + \left.\frac{dv}{dt}\right|_{orbit} \cdot \frac{\partial}{\partial v}$$

$$\Rightarrow \quad \boxed{\frac{d}{dt} N(x, v, t) = 0}$$



density is constant following a particle's orbit

Conservative form:

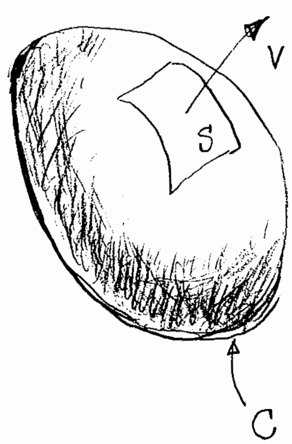
$$\boxed{\frac{\partial N}{\partial t} + \frac{\partial}{\partial x}\cdot(vN) + \frac{\partial}{\partial v}\cdot\left[\frac{q}{m}\left(E^m + \frac{v\times B^m}{c}\right)N\right] = 0}$$

move v inside $\frac{\partial}{\partial x}$

$\frac{q}{m}\left(E^m + \frac{v\times B^m}{c}\right)$ inside $\frac{\partial}{\partial v}$

Can you really do that?

Another proof!



$$\frac{\partial \int_C N\, d^6 v}{\partial t} = -\int_S vN\cdot ds$$

$$= -\int_C \nabla\cdot(vN)\, d^6 v$$

$$\Rightarrow \quad \frac{\partial N}{\partial t} + \nabla\cdot(vN) = 0$$

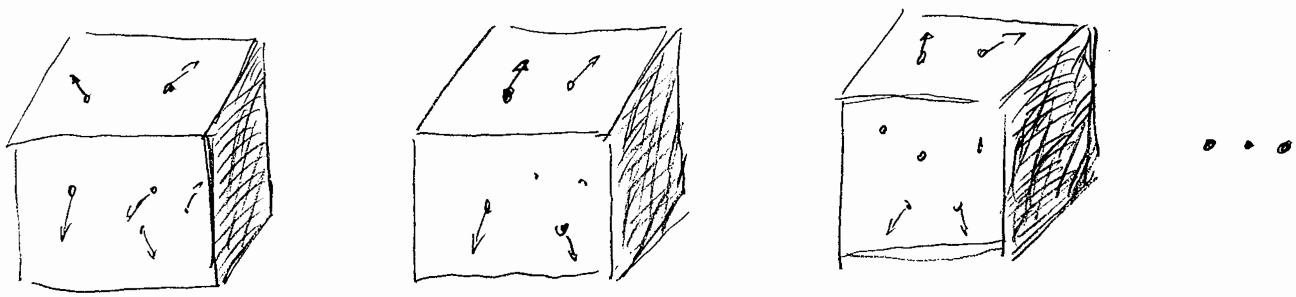
Divergence in 6D phase space

use of Klimontovich Eq.

(Statistical)

① starting point of plasma kinetic theory

② Theory for large scale particle simulations.
(won't cover here)



<u>Ensemble</u> : same Macroscopic conditions
different Microscopic conditions

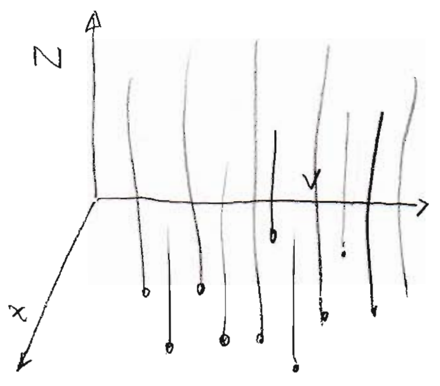
Ensemble average

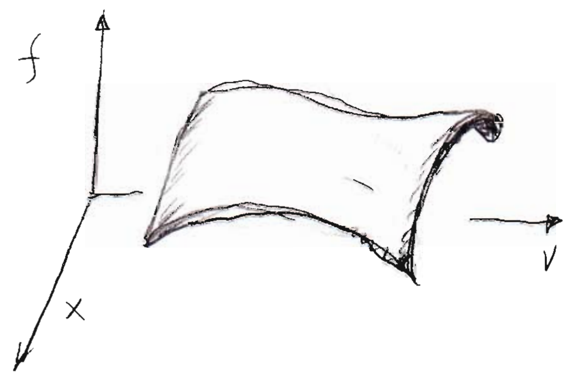
fluctuation

$$f(x,v,t) \equiv \langle N(x,v,t) \rangle , \qquad \delta N \equiv N - f$$

$$E(x,t) \equiv \langle E^{m}(x,t) \rangle , \qquad \delta E \equiv E - E^{m}$$

$$B(x,t) \equiv \langle B^{m}(x,t) \rangle , \qquad \delta B \equiv B - B^{m}$$

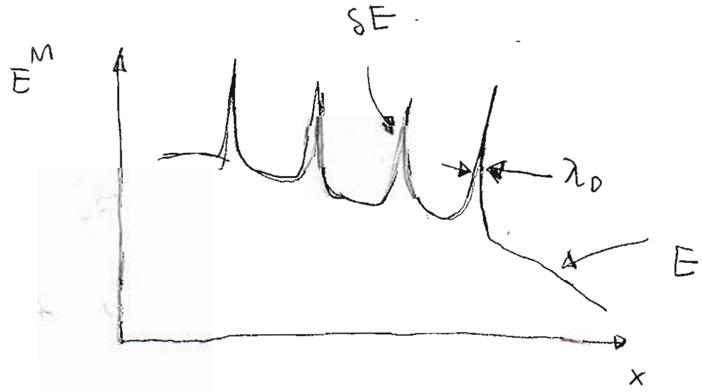




$$\Longrightarrow \quad \boxed{\begin{aligned} &\frac{\partial f}{\partial t} + v\cdot\nabla_x f + \frac{q}{m}\left(E+\frac{v\times B}{c}\right)\cdot\nabla_v f \\ &= -\frac{q}{m}\left\langle \left(\delta E + \frac{v\times\delta B}{c}\right)\cdot\nabla_v \delta N\right\rangle \end{aligned}}$$

2-point correlation $\sim \dfrac{1}{n\lambda_D^3} \ll 1$

(Advanced Kinetic Theory)



Why is the 2-point correlation small?

Heuristic illustration:

LHS: $f \sim N$

RHS:

$$\lambda_e \sim \frac{1}{\sqrt{N}}$$

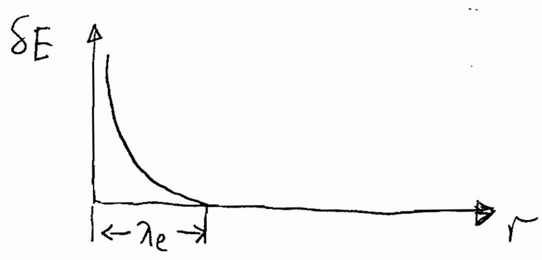
Statistical fluctuation $\delta N \sim \sqrt{N}$

$$\delta E \sim \lambda_e e\,\delta n \sim \delta N / \sqrt{N} \sim 1$$

$$\delta E \cdot \delta N \sim N^{1/2}$$

However: δE is shielded by Deby shielding.

$$\langle \delta E \cdot \delta N \rangle \sim \delta E \cdot \delta N \, \frac{\lambda^3}{L^3} \sim \frac{1}{N}$$



$$\Rightarrow \quad \frac{\text{RHS}}{\text{LHS}} \sim \frac{1}{N^2} \ll 1$$

Heuristic illustration from particle simulation point of view

Chopping particles $\Longleftrightarrow$ Increase simulation particles

$N \to \infty$

$n_0 \to \infty$, $m_e \to 0$, $e \to 0$, $n_0 e = \text{const}$

$e/m = \text{const}$

$v_e = \text{const}$

$$\omega_e^2 = \frac{4\pi n e^2}{m} = \text{const}, \qquad \lambda_e = \frac{v_e}{\omega_e} = \text{const}$$

Statistical Fluctuation

$\delta N \sim N^{1/2}$

LHS: $f \sim N$

RHS: $\delta E \sim \lambda_e e\, \delta n \sim \dfrac{\delta N}{N}$

$$\delta E \cdot \nabla_v \delta N \sim \frac{(\delta N)^2}{N} \sim \text{const}$$

$$\Rightarrow \quad \frac{\text{RHS}}{\text{LHS}} \sim \frac{1}{N} \sim \frac{1}{n \lambda_D^3} \ll 1$$

Collision operator

RHS: Discret particle effect, (collisional effect)

LHS: Ensemble averaged dynamics (Collective field)

Ensemble averaged field Eqs are trivial:

$$\nabla \cdot E = 4\pi \rho$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$$\nabla \times B = \frac{4\pi}{c} J + \frac{1}{c}\frac{\partial E}{\partial t}$$

$$\rho \equiv \langle \rho^m \rangle = \sum_s q \int dv\, f$$

$$J \equiv \langle J^m \rangle = \sum_s q \int dv\, f v$$