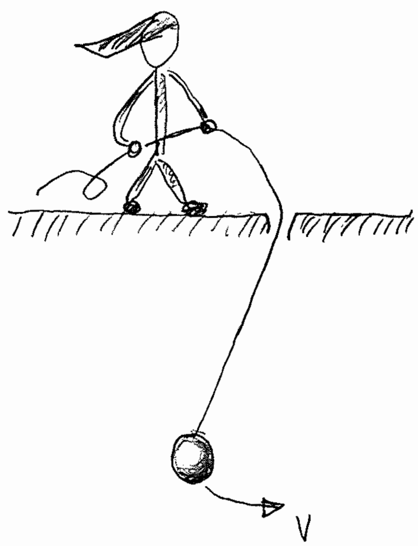
A Short Introduction to Adiabatic Invariant

$$\ddot{x} + \omega^2(t)\, x = 0$$

$$\left| \frac{1}{\omega} \frac{d\omega}{dt} \right| \sim \epsilon \ll 1$$



$$E(t) \equiv \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2(t)\, x^2$$

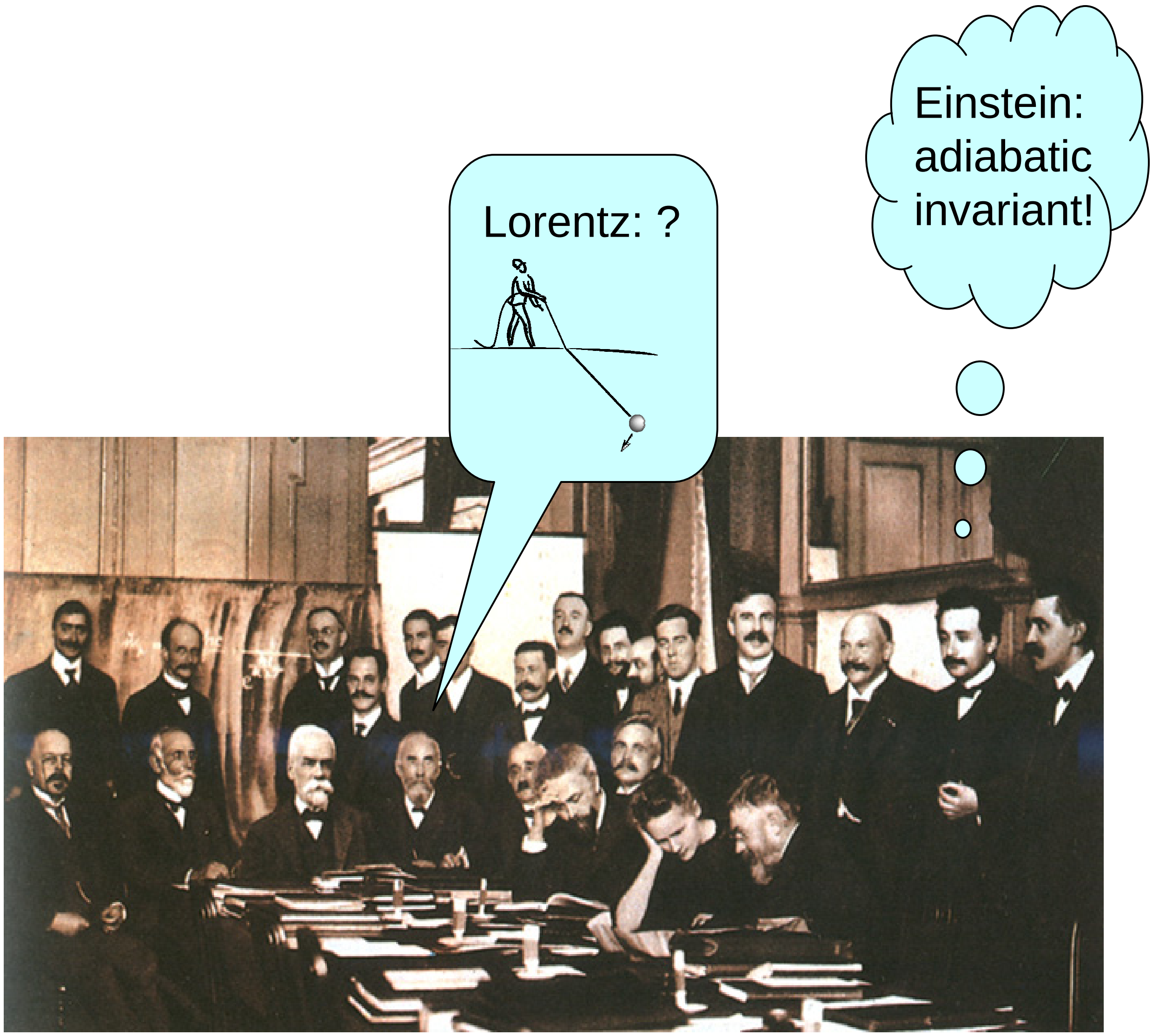
$I \equiv \dfrac{E(t)}{\omega(t)}$ is a constant of motion

- Einstein, 1911, solvay conference
- Initial Quantization scheme.

  Abandoned from 1910 ~ 1980

  Revived from 1980 ~, Geometric Quantization

1911 Solvay Conference

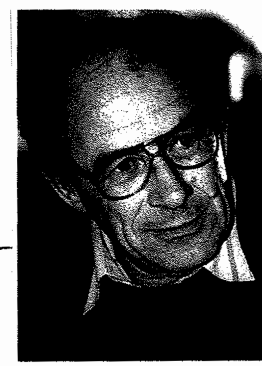


- I is only an <u>Approximation Invariant</u>

in what sense?

$$\frac{1}{I}\frac{dI}{dt} \sim \epsilon$$ ← Asymptotic Invariant

(trivial)



V. I. Arnold
1937 -

Adiabatic Invariant.

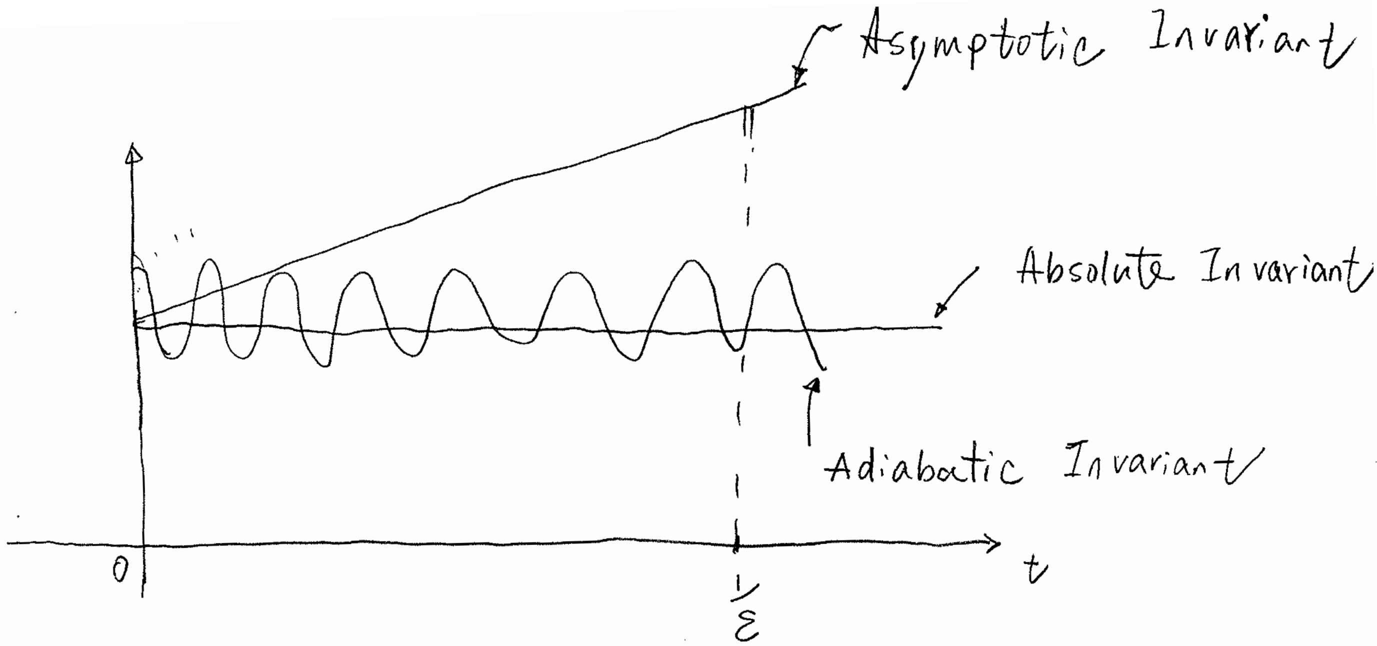
$$\Delta I \equiv I(t) - I(0) \sim O(\epsilon)$$

for $0 < t < \frac{1}{\epsilon^N}$, $N \geq 1$

(stronger than)

why is it different from $\frac{1}{I}\frac{dI}{dt} \sim \epsilon$?

In general, if $\frac{1}{I}\frac{dI}{dt} \sim \epsilon$, then the best we can hope for is $\Delta I \sim 1$ for $t \sim \frac{1}{\epsilon}$, if I is not a adiabatic invariant.


Asymptotic Invariant
Absolute Invariant
Adiabatic Invariant
0
1/ε
t

W. R. Hamilton
1805 - 1865

How to prove $I \equiv \frac{E}{\omega}$ is an adiabatic invariant?

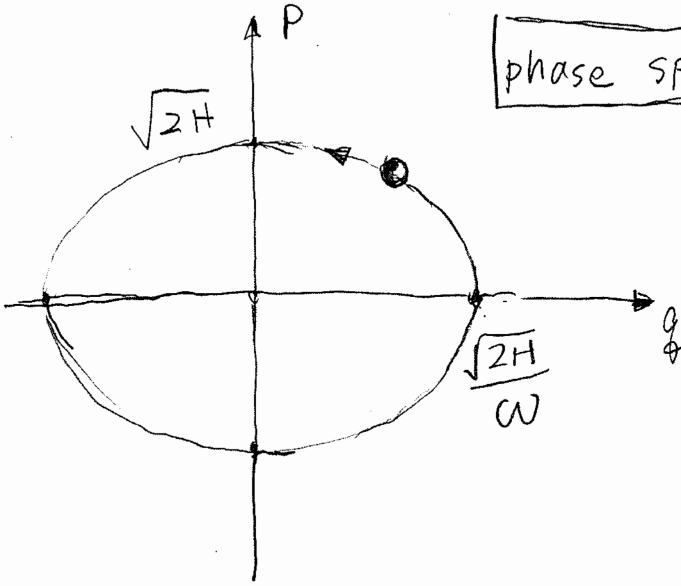
start from Hamiltonian Dynamics.

$$H = \frac{p^2}{2} + \frac{\omega^2(t)}{2} q^2$$

$$\dot{q} = \frac{\partial H}{\partial p} , \quad \dot{p} = - \frac{\partial H}{\partial q}$$

why "−" ?

phase space



If. ω = const

$$A = \oint p(t)\, dq(t)$$

$$= \pi \sqrt{2H} \sqrt{\frac{2H}{\omega}}$$

$$= 2\pi \frac{H}{\omega}$$

When $\frac{1}{\omega}\frac{d\omega}{dt} \neq 0$, $\frac{1}{\omega}\frac{d\omega}{dt} \sim O(\epsilon)$

Then $A = \oint p(t)\, dq(t)$ is an Adiabatic Invariant !

Vange problem: ① In general there is no closed orbit

② A actually depends on t.

Theorem

Let $H(t) = \frac{p^2}{2} + \omega^2(t)\frac{q^2}{2}$, and $\frac{1}{\omega}\frac{d\omega}{dt} \sim O(\epsilon)$.

For any $s > 0$, define $A(s) \equiv \oint_{t=s}^{t=s+\frac{2\pi}{\omega(s)}} p(t)\, dq(t)$

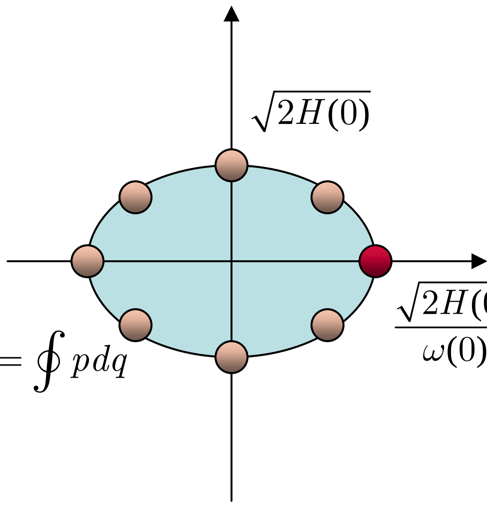
where $p(t)$, $q(t)$ are the dynamics of the time independent

$H_s = \frac{p^2}{2} + \frac{q^2}{2}\omega^2(s)$, where s enters parametrically.
($\omega(s)$ is a constant)

Then $\left[A(s) - A(0)\right] \sim O(\epsilon)$ for $0 \leq s \leq \frac{1}{\epsilon}$

Proof:

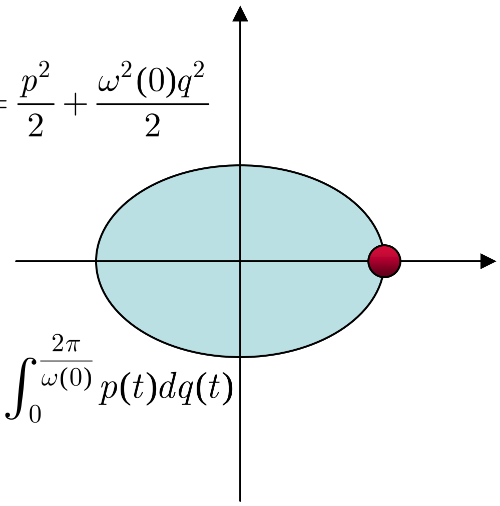
$t = 0$



$$\sqrt{2H(0)}$$

$$\frac{\sqrt{2H(0)}}{\omega(0)}$$

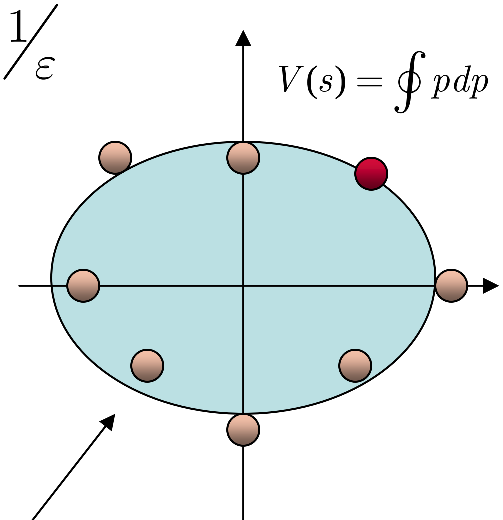
$$V(0) = \oint p\,dq$$



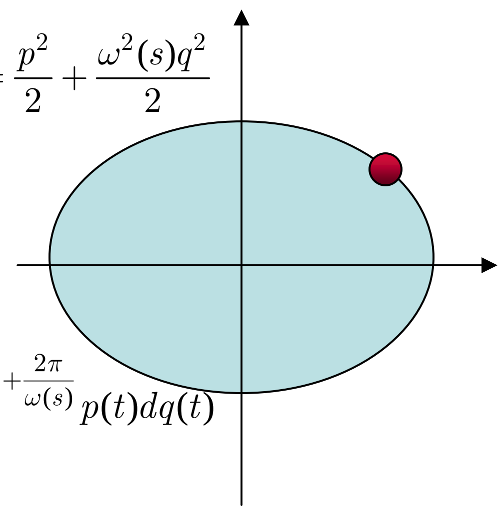
$$H(0) = \frac{p^2}{2} + \frac{\omega^2(0)q^2}{2}$$

$$I(0) = \int_0^{\frac{2\pi}{\omega(0)}} p(t)\,dq(t)$$

$t = s < 1/\varepsilon$



$$V(s) = \oint p\,dp$$



$$H(s) = \frac{p^2}{2} + \frac{\omega^2(s)q^2}{2}$$

$$I(s) = \int_s^{s+\frac{2\pi}{\omega(s)}} p(t)\,dq(t)$$

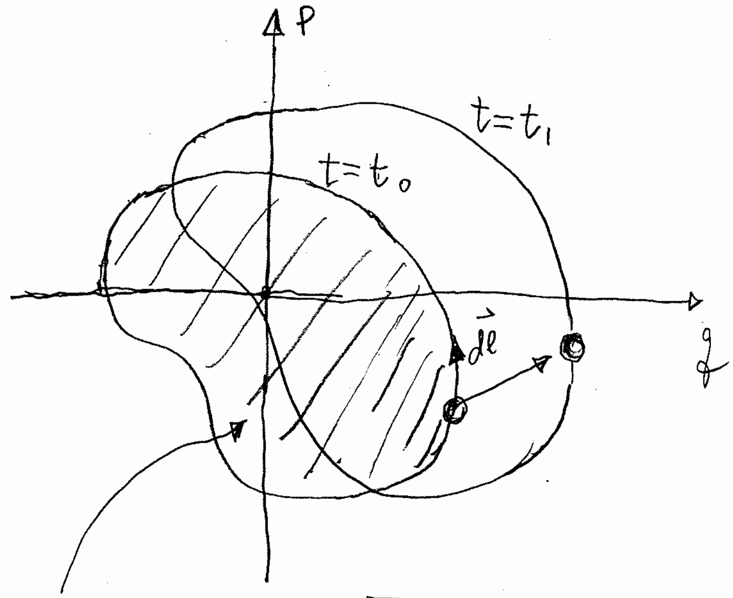
All particles are only $O(\varepsilon)$
away from a constant I surface.
(Average Theorem, Arnold's P294)

$$I(0) = V(0) = V(s) = I(s) + O(\varepsilon)$$

Liouville's Theorem

Josheph Liouville
1809 - 1882

Liouville's Theorem:



$$\mathcal{V} = \oint p\, dq\,, \qquad \boxed{\frac{d\mathcal{V}}{dt} = 0}$$

proof:

$$\frac{d\mathcal{V}}{dt} = \oint \left[(\dot{q}, \dot{p}) \times d\ell \right] \cdot \hat{e}_z$$

$$= \oint \left[\hat{e}_z \times (\dot{q}, \dot{p}) \right] \cdot d\vec{\ell}$$

$$= \oint (-\dot{p}, \dot{q}) \cdot d\vec{\ell}$$

Stokes' Theorem

$$= \int \left[\nabla \times (-\dot{p}, \dot{q}) \right] \cdot d\vec{s}$$

$$\begin{vmatrix} \hat{e}_q, & \hat{e}_p & \hat{e}_z \\ \frac{\partial}{\partial q} & \frac{\partial}{\partial p} & \frac{\partial}{\partial z} \\ -\dot{p} & \dot{q} & 0 \end{vmatrix}$$

$$\boxed{\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned}}$$

$$= \int \left[\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} \right] ds$$

$$= 0$$