GPPI Homework Set 5

Due on December 17, 2007

Problem 1. Energy conservation

For ideal MHD, prove the energy conservation in the form of

$$\frac{\partial w}{\partial t} + \nabla \cdot \boldsymbol{s} = 0,$$

where the energy density w and flux $\boldsymbol{s}$ are defined as

$$w \equiv \frac{1}{2}\rho v^2 + \frac{B^2}{8\pi} + \frac{p}{\gamma - 1},$$

$$\boldsymbol{s} \equiv \left(\frac{1}{2}\rho v^2 + \frac{p\gamma}{\gamma - 1}\right)\boldsymbol{v} + \frac{\boldsymbol{B} \times (\boldsymbol{v} \times \boldsymbol{B})}{4\pi}.$$

Problem 2. Frozen field lines from the perspective of Clebsch coordinates

Suppose that the magnetic field can be expressed in terms of two Clebsch coordinates $\alpha(\boldsymbol{x}, t)$ and $\beta(\boldsymbol{x}, t)$ as

$$\boldsymbol{B} = \nabla\alpha \times \nabla\beta.$$

Show that if

$$\frac{d\alpha}{dt} = \frac{d\beta}{dt} = 0,$$

then $\boldsymbol{B}$ satisfies the magnetic field equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}).$$

Problem 3. Bennet pinch condition

Show that for a z-pinch the relation

$$\int_0^R p(r) 2\pi r dr = \frac{I^2}{2c^2}$$

holds for arbitrary $p(r)$ with $p(R) = 0$ on the bounding wall at $r = R$. I is the total axial current inside the bounding wall.